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Properties of Weakly Precontinuous Multifunctions

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Abstract

In [15], the authors defined a multifunction $F : X \to Y$ to be weakly precontinuous if for each point $x \in X$ and any open sets G_1, G_2 of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$, there exists a preopen set U of X containing x such that $F(U) \subset \mathrm{Cl}(G_1)$ and $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. In this paper, we obtain further characterizations and several properties concerning weakly precontinuous multifunctions.

1 Introduction

In 1982, Mashhour et al. [10] introduced the notions of preopen sets and precontinuity in topological spaces. Przemski [26] and the present authors [14] have independently defined the notion of precontinuity in the setting of multifunctions. Quite recently, in [25], the authors have shown that these notions are equivalent of each other and obtained several characterizations of precontinuous multifunctions. On the other hand, in [15], the present authors have introduced the notion of weakly precontinuous multifunctions.

The purpose of this paper is to obtain several characterizations and some properties of weakly precontinuous multifunctions.

2 Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *semi-open* [9] (resp. *preopen* [10], *α-open* [12], *semi-preopen* [1]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The family of all preopen sets of X containing a point $x \in X$ is denoted by $\mathrm{PO}(X, x)$. The family of all semi-open (resp. preopen, semi-preopen) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\mathrm{SPO}(X)$). The complement of a semi-open (resp. preopen) set is said to be *semi-closed* (resp. *preclosed*). The intersection of all semi-closed (resp. preclosed) sets of X containing A is called the *semi-closure* [3] (resp. *preclosure*) [4] of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$). The union of all preopen sets of X contained in A is called the *preinterior* of A and is denoted by $\mathrm{pInt}(A)$. The *θ-closure* [27] of A, denoted by $\mathrm{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \mathrm{Cl}(U) \neq \emptyset$ for every open neighborhood U of x. If $A = \mathrm{Cl}_\theta(A)$ then A is said to be *θ-closed*. The complement of a θ-closed set is said to be *θ-open*. It is shown in [27] that $\mathrm{Cl}_\theta(A)$ is closed in X for each subset A of X and that $\mathrm{Cl}(U) = \mathrm{Cl}_\theta(U)$ for each open set U of X. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$).

Throughout the present paper, spaces X and Y always mean topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y. For an open V of Y, we denote $V^+ = \{A \in \mathcal{P}(Y) : A \subset V\}$ and $V^- = \{A \in \mathcal{P}(Y) : A \cap V \neq \emptyset\}$ [26].

Definition 1 A multifunction $F : X \to Y$ is said to be *precontinuous* [22] (resp. *almost precontinuous continuous* [15]) at a point $x \in X$ if for each open (resp. regular open) sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. A multifunction $F : X \to Y$ is said to be *precontinuous* (*almost precontinuous*) if it has this property at each point of X.

Definition 2 A multifunction $F : X \to Y$ is said to be *weakly precontinuous* [15] at a point $x \in X$ if for each open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(G_1)$ and $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. A multifunction $F : X \to Y$ is said to be *weakly precontinuous* if it has this property at each point of X.

Remark 1 For the properties of a multifunction, it is pointed out in [15] that the following implications hold: precontinuity ⇒ almost precontinuity ⇒ weak precontinuity.

3 Characterizations

Lemma 1 (Andrijević [1]) *Let A be a subset of a topological space X. The following properties hold: (1)* $\mathrm{pCl}(A) = A \cup \mathrm{Cl}(\mathrm{Int}(A))$ *and (2)* $\mathrm{pInt}(A) = A \cap \mathrm{Int}(\mathrm{Cl}(A))$.

Theorem 1 *The following properties are equivalent for a multifunction $F : X \to Y$:*
(1) F is weakly precontinuous at a point $x \in X$;
(2) $x \in \mathrm{pInt}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))$ for every open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$;
(3) $x \in \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2))))$ for every open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$.

Proof. (1) $\Rightarrow$ (2): Let G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then there exists $U \in PO(X, x)$ such that $F(U) \subset Cl(G_1)$ and $F(u) \cap Cl(G_2) \neq \emptyset$ for every $u \in U$. Thus we have $x \in U \subset F^+(Cl(G_1)) \cap F^-(Cl(G_2))$. Since $U \in PO(X)$, we have $x \in U = pInt(U) \subset pInt(F^+(Cl(G_1)) \cap F^-(Cl(G_2)))$.

(2) $\Rightarrow$ (3): Let G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Now put $U = pInt(F^+(Cl(G_1)) \cap F^-(Cl(G_2)))$. Then $U \in PO(X)$ and $x \in U \subset F^+(Cl(G_1)) \cap F^-(Cl(G_2)))$. Thus $x \in U \subset Int(Cl(U)) \subset Int(Cl(F^+(Cl(G_1)) \cap F^-(Cl(G_2))))$.

(3) $\Rightarrow$ (1): For any open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$. We have $x \in F^+(G_1) \cap F^-(G_2) \subset F^+(Cl(G_1)) \cap F^-(Cl(G_2))$. Put $U = pInt(F^+(Cl(G_1)) \cap F^-(Cl(G_2)))$. Then by (3) and Lemma 1 $U \in PO(X, x), F(U) \subset Cl(G_1)$ and $F(u) \cap Cl(G_2) \neq \emptyset$ for every $u \in U$. Therefore, F is weakly precontinuous at $x \in X$.

Theorem 2 *The following properties are equivalent for a multifunction $F : X \to Y$:*
(1) F is weakly precontinuous;
(2) $F^+(G_1) \cap F^-(G_2) \subset Int(Cl(F^+(Cl(G_1)) \cap F^-(Cl(G_2))))$ for every open sets G_1, G_2 of Y;
(3) $Cl(Int(F^+(G_1) \cup F^-(G_2))) \subset F^+(Cl(G_1)) \cup F^-(Cl(G_2))$ for every open sets G_1, G_2 of Y;
(4) $Cl(Int(F^-(Int(K_1)) \cup F^+(Int(K_2)))) \subset F^-(K_1) \cup F^+(K_2)$ for every closed sets K_1, K_2 of Y;
(5) $pCl(F^-(Int(K_1)) \cup F^+(Int(K_2)))) \subset F^-(K_1) \cup F^+(K_2)$ for every closed sets K_1, K_2 of Y;
(6) $pCl(F^-(Int(Cl(B_1))) \cup F^+(Int(Cl(B_2)))) \subset F^-(Cl(B_1)) \cup F^+(Cl(B_2))$ for every subsets B_1, B_2 of Y;
(7) $F^+(Int(B_1)) \cap F^-(Int(B_2)) \subset pInt(F^+(Cl(Int(B_1))) \cap F^-(Cl(Int(B_2))))$ for every subsets B_1, B_2 of Y;
(8) $F^+(G_1) \cap F^-(G_2) \subset pInt(F^+(Cl(G_1)) \cap F^-(Cl(G_2)))$ for every open sets G_1, G_2 of Y;
(9) $pCl(F^-(G_1) \cup F^+(G_2)) \subset F^-(Cl(G_1)) \cup F^+(Cl(G_2))$ for every open sets G_1, G_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let G_1, G_2 be any open sets in Y and $x \in F^+(G_1) \cap F^-(G_2)$. Then $F(x) \in G_1^+ \cap G_2^-$ and hence there exists $U \in PO(X, x)$ such that $F(U) \subset Cl(G_1)$ and $F(u) \cap Cl(G_2) \neq \emptyset$ for every $u \in U$. Then $U \subset F^+(Cl(G_1)) \cap F^-(Cl(G_2))$. Since $U \in PO(X)$ we have $x \in U \subset Int(Cl(U)) \subset Int(Cl(F^+(Cl(G_1)) \cap F^-(Cl(G_2))))$. Therefore, we obtain $F^+(G_1) \cap F^-(G_2) \subset Int(Cl(F^+(Cl(G_1)) \cap F^-(Cl(G_2))))$.

(2) $\Rightarrow$ (3): Let G_1, G_2 be any open sets in Y. Then, we have

$$X - [F^+(Cl(G_1)) \cup F^-(Cl(G_2))] = (X - F^+(Cl(G_1))) \cap (X - F^-(Cl(G_2))) =$$
$$F^-(Y - Cl(G_1)) \cap F^+(Y - Cl(G_2)) \subset Int(Cl(F^-(Cl(Y - Cl(G_1))) \cap F^+(Cl(Y - Cl(G_2)))))$$
$$= Int(Cl(F^-(Y - Int(Cl(G_1))) \cap F^+(Y - Int(Cl(G_2))))) \subset$$
$$Int(Cl(F^-(Y - G_1) \cap F^+(Y - G_2))) = Int(Cl(X - (F^+(G_1) \cup F^-(G_2)))) =$$
$$X - Cl(Int(F^+(G_1) \cup F^-(G_2))).$$

Therefore, we obtain $Cl(Int(F^+(G_1) \cup F^-(G_2))) \subset F^+(Cl(G_1)) \cup F^-(Cl(G_2))$.

(3) $\Rightarrow$ (4): Let K_1, K_2 be any closed sets in Y, then $Int(K_1)$, $Int(K_2)$ are open sets of Y and thus $Cl(Int(F^+(Int(K_1)) \cup F^-(Int(K_2)))) \subset F^+(Cl(Int(K_1))) \cup F^-(Cl(Int(K_2))) \subset$

$F^+(K_1) \cup F^-(K_2)$.

(4) $\Rightarrow$ (5): Let K_1, K_2 be any closed sets in Y. Then we have $\mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2)))) \subset F^-(K_1) \cup F^+(K_2)$ and $F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2)) \subset F^-(K_1) \cup F^+(K_2)$. Therefore, by Lemma 1 we obtain $\mathrm{pCl}(F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2))) \subset F^-(K_1) \cup F^+(K_2)$.

(5) $\Rightarrow$ (6): Let B_1, B_2 be any subsets of Y, then $\mathrm{Cl}(B_1)$ and $\mathrm{Cl}(B_2)$ are closed sets in Y. Thus, we obtain $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(B_2)))) \subset F^-(\mathrm{Cl}(B_1)) \cup F^+(\mathrm{Cl}(B_2))$.

(6) $\Rightarrow$ (7): Let B_1, B_2 be any subsets of Y. We have $F^+(\mathrm{Int}(B_1)) \cap F^-(\mathrm{Int}(B_2)) = X - [F^-(\mathrm{Cl}(Y - B_1)) \cup F^+(\mathrm{Cl}(Y - B_2))] \subset X - \mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(Y - B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(Y - B_2)))) = \mathrm{pInt}(F^+(\mathrm{Cl}(\mathrm{Int}(B_1))) \cap F^-(\mathrm{Cl}(\mathrm{Int}(B_2))))$.

(7) $\Rightarrow$ (8): This is obvious.

(8) $\Rightarrow$ (1): Let G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then $x \in F^+(G_1) \cap F^-(G_2) \subset \mathrm{pInt}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))$. Set $U = \mathrm{pInt}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))$. Then $U \in \mathrm{PO}(X, x)$, $F(U) \subset \mathrm{Cl}(G_1)$ and $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. Therefore, F is weakly precontinuous.

(6) $\Rightarrow$ (9): Let G_1, G_2 be any open sets of Y. Then we obtain $\mathrm{pCl}(F^-(G_1) \cup F^+(G_2)) \subset \mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$.

(9) $\Rightarrow$ (8): Let G_1, G_2 be any open sets of Y. Then we have $F^+(G_1) \cap F^-(G_2) \subset F^+(\mathrm{Int}(\mathrm{Cl}(G_1))) \cap F^-(\mathrm{Int}(\mathrm{Cl}(G_2))) = X - [F^-(\mathrm{Cl}(Y - \mathrm{Cl}(G_1))) \cup F^+(\mathrm{Cl}(Y - \mathrm{Cl}(G_2)))] \subset X - \mathrm{pCl}[F^-(Y - \mathrm{Cl}(G_1)) \cup F^+(Y - \mathrm{Cl}(G_2))] = \mathrm{pInt}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))$. Therefore, we obtain $F^+(G_1) \cap F^-(G_2) \subset \mathrm{pInt}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))$.

A function $f : X \to Y$ is said to be *almost weakly continuous* [6], *weakly precontinuous*, or *quasi precontinuous* [16] if for each point $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \mathrm{PO}(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$.

Corollary 1 (Noiri [13], Popa-Noiri [21], Paul-Bhattacharyya [17]) *The following properties are equivalent for a function $f : X \to Y$:*
(1) f is almost weakly continuous;
(2) $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$ *for every open set V of Y;*
(3) $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset f^{-1}(\mathrm{Cl}(V))$ *for every open set V of Y;*
(4) $\mathrm{Cl}(\mathrm{Int}(f^{-1}(\mathrm{Int}(K)))) \subset f^{-1}(K)$ *for every closed set K of Y;*
(5) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(K))) \subset f^{-1}(K)$ *for every closed set K of Y;*
(6) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(B))) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset B of Y;*
(7) $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{pInt}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(B))))$ *for every subset B of Y;*
(8) $f^{-1}(V) \subset \mathrm{pInt}(f^{-1}(\mathrm{Cl}(V)))$ *for every open set V of Y;*
(9) $\mathrm{pCl}(f^{-1}(V) \subset f^{-1}(\mathrm{Cl}(V))$ *for every open set V of Y.*

Theorem 3 *The following are equivalent for a multifunction $F : X \to Y$:*
(1) F is weakly precontinuous;
(2) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B_2)))) \subset F^-(\mathrm{Cl}_\theta(B_1)) \cup F^+(\mathrm{Cl}_\theta(B_2))$ *for every subsets B_1, B_2 of Y;*
(3) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(B_2)))) \subset F^-(\mathrm{Cl}_\theta(B_1)) \cup F^+(\mathrm{Cl}_\theta(B_2))$ *for every subsets B_1, B_2 of Y;*
(4) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$ *for every open*

sets G_1, G_2 *of* Y;
(5) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(V_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(V_2)))) \subset F^-(\mathrm{Cl}(V_1)) \cup F^+(\mathrm{Cl}(V_2))$ *for every preopen sets* V_1, V_2 *of* Y;
(6) $\mathrm{pCl}(F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ *for every regular closed sets* K_1, K_2 *of* Y.

Proof. (1) ⇒ (2): Let B_1, B_2 be any subsets of Y. Then $\mathrm{Cl}_\theta(B_1)$ and $\mathrm{Cl}_\theta(B_2)$ are closed in Y. Therefore, by Lemma 1 and Theorem 2 we obtain

$$\mathrm{pCl}[F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B_2)))]$$
$$= [F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B_2)))] \cup \mathrm{Cl}(\mathrm{Int}[F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B_2)))]$$
$$\subset F^-(\mathrm{Cl}_\theta(B_1)) \cup F^+(\mathrm{Cl}_\theta(B_2)).$$

(2) ⇒ (3): This is obvious since $\mathrm{Cl}(B) \subset \mathrm{Cl}_\theta(B)$ for every subset B of Y.

(3) ⇒ (4): This is obvious since $\mathrm{Cl}(G) = \mathrm{Cl}_\theta(G)$ for every open set G of Y.

(4) ⇒ (5): Let V_1, V_2 be any preopen sets of Y. Then we have $V_i \subset \mathrm{Int}(\mathrm{Cl}(V_i))$ and $\mathrm{Cl}(V_i) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V_i)))$ for $i = 1, 2$. Now, set $G_i = \mathrm{Int}(\mathrm{Cl}(V_i))$, then G_i is open in Y and $\mathrm{Cl}(G_i) = \mathrm{Cl}(V_i)$. Therefore, by (4) we obtain $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(V_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(V_2)))) \subset F^-(\mathrm{Cl}(V_1)) \cup F^+(\mathrm{Cl}(V_2))$.

(5) ⇒ (6): Let K_1, K_2 be any regular closed sets of Y. Then we have $\mathrm{Int}(K_1) \in \mathrm{PO}(Y)$ and $\mathrm{Int}(K_2) \in \mathrm{PO}(Y)$ and hence by (5) $\mathrm{pCl}(F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2))) = \mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K_1)))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K_2))))) \subset F^-(\mathrm{Cl}(\mathrm{Int}(K_1))) \cup F^+(\mathrm{Cl}(\mathrm{Int}(K_2))) = F^-(K_1) \cup F^+(K_2)$.

(6) ⇒ (1): Let G_1, G_2 be any open sets of Y. Then $\mathrm{Cl}(G_1)$ and $\mathrm{Cl}(G_2)$ are regular closed sets of Y. Therefore, we obtain $\mathrm{pCl}(F^-(G_1) \cup F^+(G_2)) \subset \mathrm{pCl}[F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))] \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$. It follows from Theorem 2 that F is weakly precontinuous.

Corollary 2 *The following are equivalent for a multifunction* $f : X \to Y$:
(1) f *is weakly precontinuous;*
(2) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset f^{-1}(\mathrm{Cl}_\theta(B))$ *for every subsets* B *of* Y;
(3) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(B)))) \subset f^{-1}(\mathrm{Cl}_\theta(B))$ *for every subsets* B *of* Y;
(4) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))) \subset f^{-1}(\mathrm{Cl}(G))$ *for every open set* V *of* Y;
(5) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))) \subset f^{-1}(\mathrm{Cl}(V))$ *for every preopen set* V *of* Y;
(6) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(K))) \subset f^{-1}(K)$ *for every regular closed set* K *of* Y.

Theorem 4 *The following are equivalent for a multifunction* $F : X \to Y$:
(1) F *is weakly precontinuous;*
(2) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$ *for every* $G_1, G_2 \in \mathrm{SPO}(Y)$;
(3) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$ *for every* $G_1, G_2 \in \mathrm{SO}(Y)$;
(4) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$ *for every* $G_1, G_2 \in \mathrm{PO}(Y)$.

Proof. (1) ⇒ (2): Let $G_1, G_2 \in \mathrm{SPO}(Y)$. Then $G_i \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(G_i)))$ and $\mathrm{Cl}(G_i) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(G_i)))$ for $i = 1, 2$. Since $\mathrm{Cl}(G_1)$ and $\mathrm{Cl}(G_2)$ are regular closed sets, by Theorem 3 we have $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$.

(2) ⇒ (3): This is obvious since $\mathrm{SO}(Y) \subset \mathrm{SPO}(Y)$.

(3) ⇒ (4): For any $G \in \mathrm{PO}(Y)$, $\mathrm{Cl}(G)$ is regular closed and $\mathrm{Cl}(G) \in \mathrm{SO}(Y)$.

(4) ⇒ (1): Let G_1, G_2 be any open sets of Y, then $G_1, G_2 \in \mathrm{PO}(X)$ and we have

$$\mathrm{pCl}(F^-(G_1) \cup F^+(G_2)) \subset \mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2)).$$

It follows from Theorem 2 that F is weakly precontinuous.

Corollary 3 *The following properties are equivalent for a function $f : X \to Y$:*
(1) f is weakly precontinuous;
(2) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))) \subset f^{-1}(\mathrm{Cl}(G))$ *for every* $G \in \mathrm{SPO}(Y)$;
(3) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))) \subset f^{-1}(\mathrm{Cl}(G))$ *for every* $G \in \mathrm{SO}(Y)$;
(4) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))) \subset f^{-1}(\mathrm{Cl}(G))$ *for every* $G \in \mathrm{PO}(Y)$.

Theorem 5 *The following are equivalent for a multifunction $F : X \to Y$:*
(1) F is weakly precontinuous;
(2) $\mathrm{Cl}(\mathrm{Int}(F^-(G_1) \cup F^+(G_2))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$ *for every* $G_1, G_2 \in \mathrm{PO}(Y)$;
(3) $\mathrm{pCl}(F^-(G_1) \cup F^+(G_2)) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$ *for every* $G_1, G_2 \in \mathrm{PO}(Y)$;
(4) $F^+(G_1) \cap F^-(G_2) \subset \mathrm{pInt}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))$ *for every* $G_1, G_2 \in \mathrm{PO}(Y)$.

Proof. (1) ⇒ (2): Let G_1, G_2 be any preopen sets of Y. Since F is weakly precontinuous, by Theorem 2 we obtain

$$\mathrm{Cl}(\mathrm{Int}(F^-(G_1) \cup F^+(G_2))) \subset \mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2))))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2)).$$

(2) ⇒ (3): Let G_1, G_2 be any preopen sets of Y. By Lemma 1, we have

$$\mathrm{pCl}(F^-(G_1) \cup F^+(G_2)) = (F^-(G_1) \cup F^+(G_2)) \cup \mathrm{Cl}(\mathrm{Int}(F^-(G_1) \cup F^+(G_2))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2)).$$

(3) ⇒ (4): Let G_1, G_2 be any preopen sets of Y. Then we have

$$X - \mathrm{pInt}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2))) = \mathrm{pCl}(X - (F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))) =$$
$$\mathrm{pCl}((X - F^+(\mathrm{Cl}(G_1))) \cup (X - F^-(\mathrm{Cl}(G_2)))) = \mathrm{pCl}(F^-(Y - \mathrm{Cl}(G_1)) \cup F^+(Y - \mathrm{Cl}(G_2)))$$
$$\subset F^-(\mathrm{Cl}(Y - \mathrm{Cl}(G_1))) \cup F^+(\mathrm{Cl}(Y - \mathrm{Cl}(G_2))) =$$
$$X - F^+(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup (X - F^-(\mathrm{Int}(\mathrm{Cl}(G_2)))) =$$
$$X - (F^+(\mathrm{Int}(\mathrm{Cl}(G_1))) \cap F^-(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset X - (F^+(G_1) \cap F^-(G_2)).$$

This impies that $F^+(G_1) \cap F^-(G_2) \subset \mathrm{pInt}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))$.

(4) ⇒ (1): Since every open set is preopen, this follows from Theorem 2.

Corollary 4 *The following properties are equivalent for a function $f : X \to Y$:*
(1) f is weakly precontinuous;
(2) $\mathrm{Cl}(\mathrm{Int}(f^{-1}(G))) \subset f^{-1}(\mathrm{Cl}(G))$ *for every* $G \in \mathrm{PO}(Y)$;
(3) $\mathrm{pCl}(f^{-1}(G)) \subset f^{-1}(\mathrm{Cl}(G))$ *for every* $G \in \mathrm{PO}(Y)$;
(4) $f^{-1}(G) \subset \mathrm{pInt}(f^{-1}(\mathrm{Cl}(G)))$ *for every* $G \in \mathrm{PO}(Y)$.

For a multifunction $F : X \to Y$, the *graph* multifunction $G_F : X \to X \times Y$ is defined by $G_F(x) = \{x\} \times F(x)$ for each $x \in X$.

Lemma 2 (Noiri and Popa [14]) *The following hold for a multifunction $F : X \to Y$:*

$$(a)\ G_F^+(A \times B) = A \cap F^+(B) \text{ and } (b)\ G_F^-(A \times B) = A \cap F^-(B)$$

for every subsets $A \subset X$ and $B \subset Y$.

Theorem 6 *Let $F : X \to Y$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is weakly precontinuous if and only if $G_F : X \to X \times Y$ is weakly precontinuous.*

Proof. Necessity. Suppose that $F : X \to Y$ is weakly precontinuous. Let $x \in X$ and W_1, W_2 be any open sets of $X \times Y$ such that $G_F(x) \in W_1^+ \cap W_2^-$. Then $G_F(x) \subset W_1$ and $G_F(x) \cap W_2 \neq \emptyset$. Since $G_F(x) \subset W_1$, for each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W_1$. The family $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$ and there exists a finite number of points, say, $y_1, y_2, ..., y_n$ in $F(x)$ such that $F(x) \subset \cup_{i=1}^n V(y_i)$. Set $U_1 = \cap_{i=1}^n U(y_i)$ and $V_1 = \cup_{i=1}^n V(y_i)$. Then U_1 and V_1 are open in X and Y, respectively, and $G_F(x) = \{x\} \times F(x) \subset U_1 \times V_1 \subset W_1$. Since $G_F(x) \cap W_2 \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W_2$ and hence $(x, y) \in U_2 \times V_2 \subset W_2$ for some open sets $U_2 \subset X$ and $V_2 \subset Y$. Put $U = U_1 \cap U_2$. Then U is an open set containing x, $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$. Since F is weakly precontinuous, there exists $U_0 \in \mathrm{PO}(X, x)$ such that $U_0 \subset F^+(\mathrm{Cl}(V_1))$ and $U_0 \subset F^-(\mathrm{Cl}(V_2))$. It follows that $G = U \cap U_0 \in \mathrm{PO}(X, x)$. By Lemma 2 we obtain $G = U \cap U_0 \subset \mathrm{Cl}(U_1) \cap F^+(\mathrm{Cl}(V_1)) = G_F^+(\mathrm{Cl}(U_1) \times \mathrm{Cl}(V_1)) = G_F^+(\mathrm{Cl}(U_1 \times V_1)) \subset G_F^+(\mathrm{Cl}(W_1))$. Similarly, we obtain $G = U \cap U_0 \subset \mathrm{Cl}(U_2) \cap F^-(\mathrm{Cl}(V_2)) = G_F^-(\mathrm{Cl}(U_2) \times \mathrm{Cl}(V_2)) = G_F^-(\mathrm{Cl}(U_2 \times V_2)) \subset G_F^-(\mathrm{Cl}(W_2))$. Therefore, $G_F(g) \cap \mathrm{Cl}(W_2) \neq \emptyset$ for every $g \in G$. Then it follows that G_F is weakly precontinuous.

Sufficiency. Suppose that $G_F : X \to X \times Y$ is weakly precontinuous. Let $x \in X$ and G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. By $F(x) \subset G_1$, we have $G_F(x) \subset X \times G_1$ and $X \times G_1$ is open in $X \times Y$. Since $F(x) \cap G_2 \neq \emptyset$, we have $G_F(x) \cap (X \times G_2) = (\{x\} \times F(x)) \cap (X \times G_2) = \{x\} \times (F(x) \cap G_2)) \neq \emptyset$. Since $X \times G_2$ is open in $X \times Y$, there exists $U \in \mathrm{PO}(X, x)$ such that $G_F(U) \subset \mathrm{Cl}(X \times G_1) = X \times \mathrm{Cl}(G_1)$ and $G_F(u) \cap \mathrm{Cl}(X \times G_2) \neq \emptyset$ for every $u \in U$. By Lemma 2, we obtain $U \subset G_F^+(X \times \mathrm{Cl}(G_1)) = F^+(\mathrm{Cl}(G_1))$ and hence $F(U) \subset \mathrm{Cl}(G_1)$. Moreover, by Lemma 2 we obtain $U \subset G_F^-(X \times \mathrm{Cl}(G_2)) = F^-(\mathrm{Cl}(G_2))$ and hence $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. Therefore, it follows that F is weakly precontinuous.

Corollary 5 (Jafari-Noiri [5], Paul-Bhattacharyya [17]) *A function $f : X \to Y$ is weakly precontinuous if and only if the graph function $g_f : X \to X \times Y$, defined by $g_f(x) = (x, f(x))$ for each $x \in X$, is weakly precontinuous.*

Lemma 3 (Mashhour et at [11]) *Let U and X_0 be subsets of a space X. The following hold:*

(1) If $U \in \mathrm{PO}(X)$ and $X_0 \in \mathrm{SO}(X)$, then $U \cap X_0 \in \mathrm{PO}(X_0)$.

(2) If $U \in \mathrm{PO}(X_0)$ and $X_0 \in \mathrm{PO}(X)$, then $U \cap X_0 \in \mathrm{PO}(X_0)$.

Theorem 7 *Let $\{U_\alpha : \alpha \in \mathcal{A}\}$ be a cover of a space X by α-open sets of X. A multifunction $F : X \to Y$ is weakly precontinuous if and only if the restriction $F/U_\alpha : U_\alpha \to Y$ is weakly precontinuous for each $\alpha \in \mathcal{A}$.*

Proof. Necessity. Suppose that F is weakly precontinuous. Let $\alpha \in \mathcal{A}$ and x be any point in U_α. Let G_1, G_2 be any open sets of Y such that $(F/U_\alpha)(x) \in G_1^+ \cap G_2^-$. Since F is weakly precontinuous and $(F/U_\alpha)(x) = F(x)$, there exists $U_0 \in \mathrm{PO}(X, x)$ such that $F(U_0) \subset \mathrm{Cl}(G_1)$ and $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U_0$. Set $U = U_0 \cap U_\alpha$, then it follows from Lemma 3 that $U \in \mathrm{PO}(U_\alpha, x)$. Then $(F/U_\alpha)(U) = F(U) \subset \mathrm{Cl}(G_1)$ and $(F/U_\alpha)(u) \cap \mathrm{Cl}(G_2) = F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. Therefore, F/U_α is weakly precontinuous.

Sufficiency. Suppose that F/U_α is weakly precontinuous for each $\alpha \in \mathcal{A}$. Let $x \in X$ and G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. There exists $\alpha \in \mathcal{A}$ such that $x \in U_\alpha$. Then, we have $(F/U_\alpha)(x) = F(x)$ and hence $(F/U_\alpha)(x) \in G_1^+ \cap G_2^-$. Since F/U_α is weakly precontinuous, there exists $U \in \mathrm{PO}(U_\alpha, x)$ such that $(F/U_\alpha)(U) \subset \mathrm{Cl}(G_1)$ and $(F/U_\alpha)(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. Since U_α is α-open in X, it follows from Lemma 3 that $U \in \mathrm{PO}(X, x)$. Moreover, we have $F(U) \subset \mathrm{Cl}(G_1)$ and $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. Therefore, F is weakly precontinuous.

Corollary 6 (Popa and Noiri [21]) *Let $\{U_\alpha : \alpha \in \mathcal{A}\}$ be a cover of a space X by α-open sets of X. A function $f : X \to Y$ is weakly precontinuous if and only if the restriction $f/U_\alpha : U_\alpha \to Y$ is weakly precontinuous for each $\alpha \in \mathcal{A}$.*

For a multifunction $F : X \to Y$, by $\mathrm{Cl}(F) : X \to Y$ [2] (resp. $\mathrm{pCl}(F) : X \to Y$ [19]) we denote a multifunction defined as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ (resp. $\mathrm{pCl}(F)(x) = \mathrm{pCl}(F(x))$) for each $x \in X$.

Definition 3 A subset A of a space X is said to be

(1) *α-regular* [7] if for each $a \in A$ and any open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$,

(2) *α-almost regular* [8] if for each $a \in A$ and any regular open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$,

(3) *α-paracompact* [28] if every X-open cover of A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 4 (Popa and Noiri [24]) *If $F : X \to Y$ is a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$, then*

(1) $G^+(V) = F^+(V)$ for each open set V of Y,

(2) $G^-(K) = F^-(K)$ for each closed set K of Y, where G denotes $\mathrm{Cl}(F)$ or $\mathrm{pCl}(F)$.

Lemma 5 (Popa and Noiri [24]) *For a multifunction $F : X \to Y$, it follows that*

(1) $G^-(V) = F^-(V)$ for each open set V of Y,

(2) $G^+(K) = F^+(K)$ for each closed set K of Y,

where G denotes $\mathrm{Cl}(F)$ or $\mathrm{pCl}(F)$.

Theorem 8 *Let $F : X \to Y$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then the following are equivalent:*
(1) F is weakly precontinuous;
(2) $\mathrm{Cl}(F)$ is weakly precontinuous;
(3) $\mathrm{pCl}(F)$ is weakly precontinuous.

Proof We put $G = \mathrm{pCl}(F)$ or $\mathrm{Ci}(F)$ in the sequel.

Necessity. Suppose that F is weakly precontinuous. Then it follows from Theorem 2 and Lemmas 3 and 4 that for every open sets V_1 and V_2 of Y, $G^+(V_1) \cap G^-(V_2) = F^+(V_1) \cap F^-(V_2) \subset \mathrm{pInt}(F^+(\mathrm{Cl}(V_1)) \cap F^-(\mathrm{Cl}(V_2))) = \mathrm{pInt}(G^+(\mathrm{Cl}(V_1)) \cap G^-(\mathrm{Cl}(V_2)))$ and hence $G^+(V_1) \cap G^-(V_2) \subset \mathrm{pInt}(G^+(\mathrm{Cl}(V_1)) \cap G^-(\mathrm{Cl}(V_2)))$. By Theorem 2 G is weakly precontinuous.

Sufficiency. Suppose that G is weakly precontinuous. Then it follows from Theorem 2 and Lemmas 3 and 4 that for every open sets V_1 and V_2 of Y, $F^+(V_1) \cap F^-(V_2) = G^+(V_1) \cap G^-(V_2) \subset \mathrm{pInt}(G^+(\mathrm{Cl}(V_1)) \cap G^-(\mathrm{Cl}(V_2))) = \mathrm{pInt}(F^+(\mathrm{Cl}(V_1)) \cap F^-(\mathrm{Cl}(V_2)))$ and hence $F^+(V_1) \cap F^-(V_2) \subset \mathrm{pInt}(F^+(\mathrm{Cl}(V_1)) \cap F^-(\mathrm{Cl}(V_2)))$. It follows from Theorem 2 that F is weakly precontinuous.

4 Almost precontinuity and weak precontinuity

In this section, we obtain some sufficient conditions for a weakly precontinuous multifunction to be almost precontinuous.

Theorem 9 *If $F : X \to Y$ is weakly precontinuous and $F(x)$ is open in Y for each point $x \in X$, then F is almost precontinuous.*

Proof. Let $x \in X$ and G_1, G_2 be open sets in Y such that $F(x) \in G_1^+ \cap G_2^-$. Since F is weakly precontinuous, there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(G_1)$ and $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. Since $F(x)$ is open for each $x \in X$, $F(U)$ is open and $F(U) \subset \mathrm{Int}(\mathrm{Cl}(G_1)) = \mathrm{sCl}(G_1)$. Moreover, $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ and $F(u) \cap \mathrm{Int}(\mathrm{Cl}(G_2)) = F(u) \cap \mathrm{sCl}(G_2) \neq \emptyset$. Therefore, F is almost precontinuous.

Lemma 6 (Popa and Noiri [23]) *If A is an α-almost regular α-paracompact set of X and U is a regular open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

Lemma 7 (Popa [20]) *If A is an α-almost regular α-paracompact set of X and U is a regular open set such that $U \cap A \neq \emptyset$, then there exists an open set G of X such that $A \cap G \neq \emptyset$ and $\mathrm{Cl}(G) \subset U$.*

Theorem 10 *If $F : X \to Y$ is weakly precontinuous and $F(x)$ is an α-almost regular α-paracompact set of Y for each point $x \in X$, then F is almost precontinuous.*

Proof. Let V_1, V_2 be regular open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$. Since $F(x)$ is α-almost regular α-paracompact by Lemma 6 there exists an open set W_1 such that $F(x) \subset W_1 \subset \mathrm{Cl}(W_1) \subset V_1$. By Lemma 7, there exists an open set W_2 of Y such that $F(x) \cap W_2 \neq \emptyset$ and $\mathrm{Cl}(W_2) \subset V_2$. Since F is weakly precontinuous, there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(W_1) \subset V_1$ and $F(u) \cap \mathrm{Cl}(W_2) \neq \emptyset$ for every $u \in U$. Therefore, we have $x \in U \subset F^+(V_1) \cap F^-(V_2)$. This shows that $F^+(V_1) \cap F^-(V_2) \in \mathrm{PO}(X)$. It follows from [15, Theroem 2] that F is almost precontinuous.

5 Sufficient conditions for weak precontinuity

Definition 4 A multifunction $F : X \to Y$ is said to be
(1) *upper almost weakly continuous* [14] if for each $x \in X$ and each open set V containing $F(x)$, $x \in \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(V))))$;
(2) *lower almost weakly continuous* [14] if for each $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, $x \in \mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{Cl}(V))))$.

Definition 5 A multifunction $F : X \to Y$ is said to be
(1) *upper weakly continuous* [18] if for each $x \in X$ and each open set V containing $F(x)$, there exists an open neighborhood U of x such that $F(U) \subset \mathrm{Cl}(V)$;
(2) *lower weakly continuous* [18] if for each $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$.

Theorem 11 *If a multifunction $F : X \to Y$ is upper almost weakly continuous and lower weakly continuous, then it is weakly precontinuous.*

Proof. Let G_1, G_2 be any open set of Y. Then we have $F^+(G_1) \subset \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(G_1))))$ [14, Theorem 3.1] and $F^-(G_2) \subset \mathrm{Int}(F^-(\mathrm{Cl}(G_2)))$ [18, Theorem 4]. Therefore, we have

$$F^+(G_1) \cap F^-(G_2) \subset \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(G_1))) \cap \mathrm{Int}(F^-(\mathrm{Cl}(G_2))) \subset$$
$$\mathrm{Int}[\mathrm{Cl}(F^+(\mathrm{Cl}(G_1))) \cap \mathrm{Int}(F^-(\mathrm{Cl}(G_2)))] \subset \mathrm{Int}(\mathrm{Cl}[F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))]).$$

It follows from Theorem 2 that F is weakly precontinuous.

Theorem 12 *If a multifunction $F : X \to Y$ is lower almost weakly continuous and upper weakly continuous, then it is weakly precontinuous.*

Proof. This is shown in the same way as in Theorem 11 by utilizing [14, Theorem 3.2] and [18, Theorem 6].

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