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SOME NEW SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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Abstract

In this paper we introduce and examine some properties of new sequence spaces defined using a modulus function.

Introduction

By s denote the set of all complex sequences $x=(x_k)$. Let l_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ normed, a usual, by $\|x\| = \sup_k x_k < \infty$.

In the present note we introduce some new sequence spaces by using a modulus function f and examine some properties of these sequence spaces.

Main Results

Definition 1. A function $f: [0,\infty) \to [0,\infty)$ is called a modulus if

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x+y) \leq f(x) + f(y)$,

(iii) f is increasing and

(iv) f is continuous from the right at 0.

Let $p=(p_k)$ be a sequence of real numbers such that $p_k>0$ for all k and $\sup_k p_k = H < \infty$. This assumption is made throughout the rest of this paper.

Definition 2. Let f be a modulus. We define

$$l_\infty(p,f,s)=\{ x \in s : \sup_k k^{-s}\left[f(|x_k|)\right]^{p_k} < \infty \ , s \geq 0 \},$$

$$c_0(p,f,s)=\{ x \in s : \ k^{-s}\left[f(|x_k|)\right]^{p_k} \to 0 \ (k \to \infty) \ , s \geq 0 \},$$

$$c(p,f,s)=\{ x \in s : \ k^{-s}\left[f(|x_k - L|)\right]^{p_k} \to 0 \ (k \to \infty) \ , s \geq 0 \text{, for some } L\}.$$

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If s=0 and f(x)=x, we have the following sequence spaces which were defined by Maddox [4]

$l_\infty(p)=\{ x \in s : \sup_k |x_k|^{p_k} < \infty . \}$,

$c_o(p)=\{ x \in s : |x_k|^{p_k} \to 0 \ (k \to \infty)\}$,

$c(p)=\{ x \in s : |x_k - L|^{p_k} \to 0 \ (k \to \infty)$, for some L$\}$.

If f(x)=x, we have the following sequence spaces which were defined by Başarır [6].

$l_\infty(p,s)=\{ x \in s : \sup_k k^{-s}|x_k|^{p_k} < \infty \ , s \geq 0 \}$,

$c_o(p,s)=\{ x \in s : k^{-s}|x_k|^{p_k} \to 0 \ (k \to \infty) \ , s \geq 0 \}$,

$c(p,s)=\{ x \in s : k^{-s}|x_k - L|^{p_k} \to 0 \ (k \to \infty) \ , s \geq 0$, for some L$\}$.

If s=0 , f(x)=x, and $p_k=1$ for all k, we have l_∞, c_o, c.

Theorem 1. (i) $c_o(p,f,s)$, $c(p,f,s)$ and $l_\infty(p,f,s)$ are linear spaces over the complex field **C**.

(ii) Let f be any modulus. Then $c_o(p,f,s) \subset c(p,f,s) \subset l_\infty(p,f,s)$.

Proof:(i) We consider only $c_o(p,f,s)$.Others can be treated similarly. We have

$$|a_k + b_k|^{p_k} \leq C\left(|a_k|^{p_k} + |b_k|^{p_k}\right) \qquad (1)$$

where $C = \max(1,2^{H-1})$.

Let $x,y \in c_o(p,f,s)$. For $\lambda, \mu \in \mathbf{C}$, there exists M and N integer such that $|\lambda| \leq M$ and $|\mu| \leq N$. From (1), we have

$$k^{-s}[f(|\lambda x_k + \mu y_k|)]^{p_k} \leq C.M^H k^{-s}[f(|x_k|)]^{p_k} + C.N^H k^{-s}[f(|y_k|)]^{p_k}$$

This implies that $\lambda x + \mu y \in c_o(p,f,s)$. and completes the proof of (i).

(ii) Clearly $c_o(p,f,s). \subset c(p,f,s)$. Let $x \in c(p,f,s)$. Then there is some L such that

$$k^{-s}[f(|x_k - L|)]^{p_k} \to 0 \ (k \to \infty) \ , s \geq 0 .$$

Now by inequality (1), we have

$$k^{-s}[f(|x_k|)]^{p_k} = k^{-s}[f(|x_k - L + L|)]^{p_k}$$

$$\leq C.k^{-s}[f(|x_k - L|)]^{p_k} + C.k^{-s}[f(|L|)]^{p_k}$$

There exists an integer K such that $|L| \leq K$. Hence we have

$$k^{-s}[f(|x_k|)]^{p_k} \leq C.k^{-s}[f(|x_k - L|)]^{p_k} + C.k^{-s}[Kf(1)]^H$$

Since $x \in c(p.f.s)$, we get $x \in l_\infty(p.f,s)$.

If X is a linear space over the field **C**, then a paranorm on X is a function g: $g(\theta)=0, where\,\theta=(0,0,...)$, g(-x)=g(x), g(x+y) $\leq$ g(x)+g(y) and $|\lambda-\lambda_0|\to 0$ imply $g(\lambda x-\lambda_0 x_0)\to 0$, where $\lambda,\lambda_0\in$ **C**, x,x_0 $\in$X. A paranormed space is a linear space X with a paranorm g and is written (X,g).

Using the properties of modulus function, it is easy to verify that c_0 (p,f,s) is a linear topological space paranormed by g defined

$$g(x)=\sup_k k^{-s}\left[f(|x_k|)\right]^{p_k/M}$$

where M=max(1,H= $\sup_k p_k$). c (p,f,s) and l_∞ (p,f,s) are paranormed by g if inf p_k >0. Morever c_0 (p,f,s) ,c (p,f,s) and l_∞ (p,f,s) are complete in their paranorm topologies.

Theorem 2: Let inf p_k =h>0.

(i) $x_k\to L$ implies $x_k\to L$ [c (p,f,s)],

(ii) $x_k\to L$ [c (p,s)] implies $x_k\to L$ [c (p,f,s)],

(iii) $\beta=\lim_{t\to\infty}\dfrac{f(t)}{t}>0$ implies c (p,s)= [c (p,f,s)].

Proof:(i) Suppose that $x_k\to L$ $(k\to\infty)$. Since f modulus, then

$$\lim_{k\to\infty}\left[f(|x_k-L|)\right]=f\left[\lim_{k\to\infty}(|x_k-L|)\right]=0.$$

Since inf p_k =h>0 then,

$$\lim_{k\to\infty}\left[f(|x_k-L|)\right]^h=0,$$

so, for $0<\varepsilon<1$, $\exists\, k_0 \ni$ for all k > k_0,

$$\left[f(|x_k-L|)\right]^h<\varepsilon<1$$

and since $p_k\geq h$ for all k,

$$\left[f(|x_k-L|)\right]^{p_k}\leq\left[f(|x_k-L|)\right]^h<\varepsilon$$

then we get,

$$\lim_{k\to\infty}\left[f(|x_k-L|)\right]^{p_k}=0.$$

Since (k^{-s}) is bounded, we write

$$\lim_{k\to\infty}k^{-s}\left[f(|x_k-L|)\right]^{p_k}=0.$$

Therefore x $\in$ c (p,f s).

(ii) Let x $\in$ c (p,f,s), so that

$$S_k=k^{-s}\left[|x_k-L|\right]^{p_k}\to 0\ (k\to\infty).$$

Let ε >0 and choose δ with 0< δ < 1 such that f(t) < ε for $0\leq t\leq\delta$. Now we write

$$I_1=\{k\in N:\ |x_k-L|\leq\delta\ \},$$

$$I_2=\{k\in N:\ |x_k-L|\ >\delta\ \}.$$

For $|x_k-L|>\delta$,

$$|x_k-L|<|x_k-L|\delta^{-1}<1+\left|x_k-L|\delta^{-1}\right|.$$

where $k \in I_2$ and $[u]$ denotes the integer part of u. By definition 1(iii) and (ii) we have for $|x_k - L| > \delta$,

$$f(|x_k - L|) \le (1 + [|x_k - L|\delta^{-1}]).f(1) \le 2f(1)\,|x_k - L|\delta^{-1}.$$

For $|x_k - L| \le \delta$,

$$f(|x_k - L|) < \varepsilon$$

where $k \in I_1$. Hence

$$k^{-s}[f(|x_k - L|)]^{p_k} = k^{-s}[f(|x_k - L|)]^{p_k} + k^{-s}[f(|x_k - L|)]^{p_k}$$

where the first term over $k \in I_1$ and the second over $k \in I_2$. Then,

$$k^{-s}[f(|x_k - L|)]^{p_k} \le k^{-s}\varepsilon^H + [2f(1)\delta^{-1}]^H S_k \to 0 \quad (k \to \infty)$$

Since $x \in C(p,s)$, we get $x \in C(p,f,s)$.

(ii) In (ii), it was shown that $C(p,s) \subset C(p,f,s)$. We must show that $C(p,f,s) \subset C(p,s)$. For any modulus function, the existence of positive limit given with β in Maddox [5, Proposition 1]. Now $\beta > 0$ and let $x \in C(p,f,s)$. Since $\beta > 0$, for every $t > 0$, we write $f(t) \ge \beta t$. From this inequality, it is easy to see that $x \in C(p,s)$. This completes the proof.

Theorem 3: Let f and g be two modulus and $s, s_1, s_2 \ge 0$.
(i) $c(p,f,s) \cap c(p,g,s) \subset c(p,f+g,s)$,
(ii) $s_1 \le s_2$ implies $c(p,f,s_1) \subset c(p,f,s_2)$.

Proof:(i) Let $x=(x_k) \in c(p,f,s) \cap c(p,g,s)$. From (1), we have

$$[(f+g)(|x_k - L|)]^{p_k} = [f(|x_k - L|) + g(|x_k - L|)]^{p_k}$$
$$\le C\{[f(|x_k - L|)]^{p_k} + [g(|x_k - L|)]^{p_k}\}.$$

Since (k^{-s}) is bounded, we write

$$k^{-s}[(f+g)(|x_k - L|)]^{p_k} \le Ck^{-s}[f(|x_k - L|)]^{p_k} + Ck^{-s}[g(|x_k - L|)]^{p_k}.$$

Since $x=(x_k) \in c(p,f,s) \cap c(p,g,s)$, we get $x=(x_k) \in c(p,f+gs)$.

(ii) Let $s_1 \le s_2$. Then $k^{-s_2} \le k^{-s_1}$ for all $k \in N$. Since

$$k^{-s_2}[f(|x_k - L|)]^{p_k} \le k^{-s_1}[f(|x_k - L|)]^{p_k},$$

this inequality implies that $c(p,f,s_1) \subset c(p,f,s_2)$.

Theorem 4: Let f be a modulus, then
(i) $l_\infty \subset l_\infty(p,f,s)$,
(ii) If f is bounded then $l_\infty(p,f,s)$=s.

Proof: (i) $x=(x_k) \in l_\infty$. Since (x_k) is bounded, $(f(x_k))$ is also bounded, so that

$$k^{-s}[f(|x_k|)]^{p_k} \le k^{-s}[Kf(1)]^{p_k} \le k^{-s}[Kf(1)]^H < \infty.$$

Therefore $x=(x_k) \in l_\infty(p,f,s)$.

(ii) If f is bounded then, for any $x=(x_k) \in s$

$$k^{-s}\left[f\left(\left|x_k\right|\right)\right]^{p_k} \leq k^{-s} L^{p_k} \leq k^{-s} L^{H} < \infty ,$$

so that $l_\infty(p,f,s)=s$.

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