# NEW RESULTS ON SOFT SEMI-TOPOLOGICAL GROUPS

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Abstract. In this paper, the concept of soft semi-topological groups given in [**2**] by B. Tanay and N. Cakmak is improved by giving several theorems and properties related to soft semi-topological groups. Moreover relation between soft semi-topological groups and soft topological groups (soft groups) is studied. Finally soft image and soft inverse image of a soft semi-topological group (soft topological group) are examined.

## 1. Introduction

Decision making under conditions of uncertainty is a crucial step in many areas like economics, engineering, environmental science, medical science and social sciences. Such conditions cannot be overcome with classical methods, since the classical methods have certain restrictions. Moreover there are some theories, which have been used to solve these kinds of problems like probability theory, fuzzy set theory and rough set theory, but each of these theories has their own difficulties too. In order to deal with such kind of difficulties of uncertainties, Molodtsov [**5**] proposed a new approach called the soft set theory, which is different from probability theory, fuzzy set theory and rough set theory. After that, some mathematical aspects of soft sets are studied by Maji et al. in [**4**]. Soft algebraic structures such as soft groups are given by Aktas and Cagman in [**3**] and soft rings are defined by Acar et al. in [**9**]. As a continuation of these studies, definition of soft topological groups is given by Nazmul and Samanta [**7**]. The notion of a mapping on soft classes is introduced by Kharal and Ahmad [**1**]. They study several properties of images and inverse images of soft sets supported by examples and counter examples. They also applied these notions to the problem of medical diagnosis in medical expert systems.
In this paper our aim is to develop the notion of soft semi-topological group, establish several theorems and properties related to soft semi-topological group, and give a relation between soft semi-topological groups and soft topological groups (soft groups). Finally we define soft restricted image and kernel of a soft set on soft classes and then study several properties of soft image and soft inverse image of a soft semi-topological group (soft topological group). Some examples and counterexamples are also given.

Received May 28,2014.
2010 *Mathematics Subject Classification.* Primary 54A40, 22A05, 54E15, 03E72.
*Keywords and phrases.* Soft set, soft group, soft topological group, soft semi-topological group.

## 2. Preliminaries

Following [**3**]-[**5**] and [**7**], some definitions and preliminary results are presented in this section. Unless otherwise stated, $U$ is assumed to be an initial universal set, $A$ is the set of parameters, and $\mathcal{P}(U)$ denotes the power set of $U$.

### 2.1. Soft sets

**Definition 2.1.** [**5**] *A pair $(F, A)$ is called a soft set over $U$ if $F$ is a mapping given by $F : A \longrightarrow \mathcal{P}(U)$.*

**Definition 2.2.** [**7**] *Let $(F_1, A)$ and $(F_2, A)$ be two soft sets over a common universe $U$. Then $(F_1, A)$ is said to be a soft subset of $(F_2, A)$ if $F_1(x) \subseteq F_2(x)$ for all $x \in A$. This relation is denoted by $(F_1, A)\widetilde{\subseteq}(F_2, A)$ and $(F_1, A)$ is said to be soft equal to $(F_2, A)$ if $F_1(x) = F_2(x)$ for all $x \in A$.*

**Definition 2.3.** [**7**] *$\{(F_i, A); i \in I\}$ be a nonempty family of soft sets over a common universe $U$. The intersection denoted by $\widetilde{\bigcap}_{i\in I}$ is defined by $\widetilde{\bigcap}_{i\in I}(F_i, A) = (\widetilde{\bigcap}_{i\in I}F_i, A)$, where $(\widetilde{\bigcap}_{i\in I}F_i)(x) = \bigcap_{i\in I}(F_i(x))$ for all $x \in A$.*

**Definition 2.4.** [**7**] *$\{(F_i, A); i \in I\}$ be a nonempty family of soft sets over a common universe $U$. The union denoted by $\widetilde{\bigcup}_{i\in I}$ is defined by $\widetilde{\bigcup}_{i\in I}(F_i, A) = (\widetilde{\bigcup}_{i\in I}F_i, A)$, where $(\widetilde{\bigcup}_{i\in I}F_i)(x) = \bigcup_{i\in I}(F_i(x))$ for all $x \in A$.*

**Definition 2.5.** [**7**] *$\{(F_i, A); i \in I\}$ be a nonempty family of soft sets over a common universe $U$. The AND denoted by $\widetilde{\bigwedge}_{i\in I}$, is defined by $\widetilde{\bigwedge}_{i\in I}(F_i, A) = (\widetilde{\bigwedge}_{i\in I}F_i, A^I)$, where $(\widetilde{\bigwedge}_{i\in I}F_i)(x) = \bigcap_{i\in I}(F_i(x))$ for all $x = (x_i) \in A^I$.*

**Definition 2.6.** [**7**] *$\{(F_i, A); i \in I\}$ be a nonempty family of soft sets over a common universe $U$. The OR denoted by $\widetilde{\bigvee}_{i\in I}$ is defined by $\widetilde{\bigvee}_{i\in I}(F_i, A) = (\widetilde{\bigvee}_{i\in I}F_i, A^I)$, where $(\widetilde{\bigvee}_{i\in I}F_i)(x) = \bigcup_{i\in I}(F_i(x))$ for all $x = (x_i) \in A^I$.*

### 2.2. Soft groups

Throughout this section, unless otherwise is specified, $G$ denotes an arbitrary group and $A \neq \emptyset$ is the set of parameters. Moreover, all soft sets are considered over $G$.

**Definition 2.7.** [**3**] *Let $(F, A)$ be a soft set over $G$. Then $(F, A)$ is called a soft group over $G$ if $F(x)$ is a subgroup of $G$ for all $x \in A$, i.e. $F(x) \leq G$ for all $x \in A$.*

**Example 2.8.** [**9**] *Let $G = A = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be an additive group. Consider the set-valued function $F : A \longrightarrow \mathcal{P}(G)$ given by $F(x) = \{y \in R \mid x.y = 0\}$. Then $F(0) = G$, $F(1) = \{0\}$, $F(2) = \{0, 3\}$, $F(3) = \{0, 2, 4\}$, $F(4) = \{0, 3\}$ and $F(5) = \{0\}$. Clearly, all these sets are subgroups of $G$. Hence $(F, A)$ is a soft group over $G$.*

**Theorem 2.9.** [**3**] *Let $\{(F_i, A); i \in I\}$ be a nonempty family of soft groups over $G$, where $I$ is an index set. Then*
*i)* $\widetilde{\bigcap}_{i\in I}(F_i, A)$ *is a soft group over* $G$,
*ii)* $\widetilde{\bigwedge}_{i\in I}(F_i, A)$ *is a soft group over* $G$.

**Definition 2.10.** [**3**] *Let $(F, A)$ be a soft group over $G$. Then*
*i) $(F, A)$ is said to be an identity soft group over $G$ if $F(x) = \{e\}$ for all $x \in A$, where $e$ is the identity element of $G$.*
*ii) $(F, A)$ is said to be an absolute soft group over $G$ if $F(x) = G$ for all $x \in A$.*

### 2.3. Semi-topological group and topological group

**Definition 2.11.** [**8**] *i) A topological space $G$ that is also a group is called a semi-topological group if the mapping*

$$g_1 : (x, y) \longrightarrow xy$$

*of $G \times G$ into $G$ is continuous in each variable separately,*

*ii) A topological space $G$ that is also a group is called a topological group if the mapping $g_1$ is continuous in both variables and if the inversion mapping*

$$g_2 : x \longrightarrow x^{-1}$$

*of $G$ onto $G$ is also continuous.*

**Definition 2.12.** [**8**] *Let $(G, \tau)$ be a semi-topological group and $H$ be a subset of $G$. Then $H$, endowed with the topology induced from $G$ is called a semi-topological subgroup of $G$ if $H$ is a subgroup of $G$.*

## 3. Soft semi-topological group

In this section definition of soft semi-topological group is introduced and some of its properties are given. Also a relation between soft semi-topological groups and soft topological groups (soft groups) over a topological group is studied.

**Definition 3.1.** [**2**] *Let $(F, A)$ be a soft set over a semi-topological group $G$ with topology $\tau$. Then, $(F, A, \tau)$ is called soft semi-topological group over $G$ if for each $\alpha \in A$, $(F(\alpha), \tau_{F(\alpha)})$ is a semi-topological subgroup of $G$, where $\tau_{F(\alpha)}$ is the relativized topology on $F(\alpha)$ induced from $\tau$ for all $\alpha \in A$.*

**Example 3.2.** [**2**] *Let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$, $A = \{\alpha_1, \alpha_2, \alpha_3\}$, $(F, A)$ be the soft set defined by $F(\alpha_1) = \{0\}$, $F(\alpha_2) = \mathbb{Z}_4$, $F(\alpha_3) = \{0, 2\}$ and $\tau$ be the discrete topology on $\mathbb{Z}_4$. $(F(\alpha_1), \tau_{F(\alpha_1)})$ is a semi-topological subgroup of $G$ since $\tau_{F(\alpha_1)} = \{\emptyset, \{0\}\}$ and $F(\alpha_1)$ is a semi-topological subgroup of $G$. Similarly, $(F(\alpha_2), \tau_{F(\alpha_2)})$ and $(F(\alpha_3), \tau_{F(\alpha_3)})$ are semi-topological subgroups of $G$. So $(F, A, \tau)$ is a soft semi-topological group over $G$.*

**Definition 3.3.** [**7**] *Let $(F,A)$ be a soft set over a topological group $G$ with topology $\tau$ then $(F,A,\tau)$ is called soft topological group over $G$ if for each $\alpha \in A$, $(F(\alpha), \tau_{F(\alpha)})$ is a topological subgroup of $G$, where $\tau_{F(\alpha)}$ is the relatavized topology on $F(\alpha)$ induced from $\tau$ for all $\alpha \in A$.*

**Proposition 3.4.** [**2**] *Let $(G,\tau)$ be a topological group. Every soft topological group over $G$ is a soft semi-topological group over $G$.*

*Proof.* The proof is straightforward by the Definition 3.1 and Definition 2.11. □

**Remark.** *The converse of Proposition 3.4 is not true in general.*

**Example 3.5.** *Let $G = \mathbb{R}$, the real line as an additive abelian group and $G$ be endowed with a topology which has the system of left closed and right open intervals $\{[a,b) : -\infty \leq a \leq b \leq \infty\}$. Let $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the parameter set and $(F,A)$ be a soft set defined by $F(\alpha_1) = \mathbb{R}$, $F(\alpha_2) = \mathbb{Q}$, $F(\alpha_3) = \mathbb{Z}$, and $F(\alpha_4) = \{0\}$. It is clear that $(F,A)$ is a soft group over $G$. $F(\alpha_1)$ is a semi-topological group with the subspace topology, since the operation $g_1 : F(\alpha_1) \times F(\alpha_1) \longrightarrow F(\alpha_1)$ is separately continuous. Similarly $F(\alpha_2)$, $F(\alpha_3)$ and $F(\alpha_4)$ are semi-topological subgroups. Hence $(F,A,\tau)$ is a soft semi-topological group over $G$. But $(F,A)$ is not a soft topological group since the function $g_2 : x \longrightarrow x^{-1}$ is not continuous at $0$ which implies that $\mathbb{R} = G$ is not a topological group with respect to topology whose one of base is the set $\{[a,b) : -\infty \leq a \leq b \leq \infty\}$.*

**Definition 3.6.** [**2**] *Let $(F_1,A,\tau)$ and $(F_2,A,\tau)$ be two soft semi-topological groups over a semi-topological group $(G,\tau)$. Then their intersection (AND), is defined by $(F_1,A,\tau)\widetilde{\cap}(F_2,A,\tau) = (F_1\widetilde{\cap}F_2,A,\tau)$ $((F_1,A,\tau)\widetilde{\wedge}(F_2,A,\tau) = (F_1\widetilde{\wedge}F_2, A \times A, \tau))$, where $(F_1\widetilde{\cap}F_2)(x) = F_1(x) \cap F_2(x)$ $((F_1\widetilde{\wedge}F_2)(x,y) = F_1(x) \cap F_2(y))$ for all $x,y \in A$.*

**Theorem 3.7.** [**2**] *Let $(F_1,A,\tau)$ and $(F_2,A,\tau)$ be two soft semi-topological groups over a semi-topological group $(G,\tau)$. Then their intersection(AND) $(F_1\widetilde{\cap}F_2,A,\tau)((F_1\widetilde{\wedge}F_2, A \times A, \tau))$ is also a soft semi-topological group over $(G,\tau)$.*

*Proof.* Proof is direct consequence of Proposition 3.4 and Theorem 3.4 [**7**]. □

**Definition 3.8.** [**2**] *Let $(F,A,\tau)$ be a soft semi-topological group over a semi-topological group $(G,\tau)$. Then*
*i) $(F,A,\tau)$ is said to be an identity soft semi-topological group over $G$ if $F(\alpha) = \{e\}$ for all $\alpha \in A$, where $e$ is the identity element of $G$.*
*ii) $(F,A,\tau)$ is said to be an absolute soft semi-topological group over $G$ if $F(\alpha) = G$ for all $\alpha \in A$.*

**Definition 3.9.** [**2**] *Let $(F_1,A,\tau)$ and $(F_2,A,\tau)$ be two soft semi-topological groups over a semi-topological group $(G,\tau)$. Then $(F_1,A,\tau)$ is said to be a soft semi-topological subgroup(normal soft semi-topological subgroup) of $(F_2,A,\tau)$ if $F_1(\alpha)$ is*

*a subgroup (normal subgroup) of* $F_2(\alpha)$ *for all* $\alpha \in A$.
*This fact is denoted by* $(F_1, A, \tau)\widetilde{\leq}(F_2, A, \tau)$ $((F_1, A, \tau)\widetilde{\lhd}(F_2, A, \tau))$.

**Theorem 3.10.** [**2**] *Let* $(F_1, A, \tau)$ *and* $(F_2, A, \tau)$ *be two soft semi-topological groups over a semi-topological group* $(G, \tau)$. *If* $F_1(\alpha) \subseteq F_2(\alpha)$ *for all* $\alpha \in A$ *then* $(F_1, A, \tau)\widetilde{\leq}(F_2, A, \tau)$ *that is* $(F_1, A, \tau)$ *is soft semi-topological subgroup of* $(F_2, A, \tau)$.

**Theorem 3.11.** [**2**] *Let* $(F, A, \tau)$ *be a soft semi-topological groups over a semi-topological group* $(G, \tau)$ *and* $\{(H_i, A, \tau); i \in \Delta\}$ *be a nonempty family of soft semi-topological subgroups(normal soft semi-topological subgroup) of* $(F, A, \tau)$, *where* $\Delta$ *is an index set. Then*
*i)* $\widetilde{\bigcap}_{i\in\Delta}(H_i, A, \tau)$ *is a soft semi-topological subgroup(normal soft semi-topological subgroup) of* $(F, A, \tau)$.
*ii)* $\widetilde{\bigwedge}_{i\in I}(H_i, A, \tau)$ *is a soft semi-topological subgroup(normal soft semi-topological subgroup) of* $(F, A, \tau)$.

**Remark.** *Theorem 3.11 is not true in general for* $\widetilde{\bigcup}_{i\in\Delta}(H_i, A, \tau)$ *and* $\widetilde{\bigvee}_{i\in I}(H_i, A, \tau)$ *as one can see from the following example.*

**Example 3.12.** *Consider the set* $G = A = \{e, (12), (13), (23), (123), (132)\}$, *which is a group with function composition and set valued function* $F : A \longrightarrow \mathcal{P}(G)$ *defined by* $F(a) = G$ *for all* $a \in A$. *Let* $\tau$ *be the indiscreet topology on* $G$. *So* $(F, A, \tau)$ *is a soft semi-topological group over* $G$. *Let* $(H_1, A, \tau)$ *and* $(H_2, A, \tau)$ *be soft semi-topological subgroups whose set valued functions are defined by:*
$H_1(e) = \{e\}$, $H_1(12) = \{e, (12)\}$, $H_1(13) = \{e, (13)\}$, $H_1(23) = \{e, (23)\}$,
$H_1(123) = F(132) = \{e, (123), (132)\}$,
*and*
$H_2(e) = \{e\}$, $H_2(12) = \{e, (13)\}$, $H_2(13) = \{e, (13)\}$, $H_2(23) = \{e, (23)\}$,
$H_2(123) = F(132) = \{e, (123), (132)\}$.
*So the union of soft sets* $(H_1, A, \tau)$ *and* $(H_2, A, \tau)$ *is not a soft semi-topological group over* $G$ *since* $H_1 \cup H_2(12) = \{e, (12), (13)\}$ *is not a subgroup of* $G$.
*Similarly,* $\widetilde{\bigvee}_{i=1,2}(H_i, A, \tau)$ *is not a soft semi-topological group over* $G$.

**Definition 3.13.** [**2**] *Let* $(F_1, A, \tau_1)$ *and* $(F_2, A, \tau_2)$ *be two soft semi-topological groups over a semi-topological group* $(G_1, \tau_1)$ *and* $(G_2, \tau_2)$, *respectively. Then their product is defined by* $(F_1, A, \tau_1)\times(F_2, A, \tau_2) = (F_3, A\times A, \tau_1\times\tau_2)$, *where* $F_3(\alpha, \beta) = F_1(\alpha)\times F_2(\beta)$ *for all* $(\alpha, \beta) \in A \times A$.

**Theorem 3.14.** [**2**] *Let* $(F_1, A, \tau_1)$ *and* $(F_2, A, \tau_2)$ *be two soft semi-topological groups over semi-topological groups* $(G_1, \tau_1)$ *and* $(G_2, \tau_2)$, *respectively. Then their product* $(F_1, A, \tau_1) \times (F_2, A, \tau_2)$ *is a soft semi-topological group over* $(G_1 \times G_2)$.

*Proof.* The proof is straightforward. □

**Theorem 3.15.** [**2**] *Let $(G_1,\tau_1)$ and $(G_2,\tau_2)$ be two semi-topological groups. Let $(F_1,A,\tau_1)$ and $(F_2,A,\tau_1)$ be soft semi-topological groups over $(G_1,\tau_1)$. If $f$ is an algebraic homomorphism from $G_1$ to $G_2$ then $(f(F_1),A,\tau_2)$ and $(f(F_2),A,\tau_2)$ are both soft semi-topological groups over $(G_2,\tau_2)$, where $f(F_i)$ is defined by $f(F_i)(\alpha)=f(F(\alpha))$, $i=1,2$, for all $\alpha\in A$. Also if $(F_1,A,\tau_1)\widetilde{\leq}(F_2,A,\tau_1)$ then*

$$(f(F_1),A,\tau_2)\widetilde{\leq}(f(F_2),A,\tau_2).$$

*Proof.* The proof is straightforward. □

**Theorem 3.16.** [**2**] *Let $(F,A,\tau_1)$ be a soft semi-topological group over $(G_1,\tau_1)$ and $f:(G_1,\tau_1)\longrightarrow(G_2,\tau_2)$ be a homomorphism. Define the set $K_f(\alpha)$ by*

$$K_f(\alpha)=[Ker(f)]_{F(\alpha)}=(Kerf)\cap F(\alpha)=\{a\in F(\alpha);f(a)=\{e_{G_2}\}\}$$

*for all $\alpha\in A$. Then*
*i) $(K_f,A,\tau_1)$ is a soft semi-topological group over $(G_1,\tau_1)$,*
*ii) $(K_f,A,\tau_1)$ is a normal soft semi-topological subgroup of $(F,A,\tau_1)$.*

*Proof.* i) Obvious.
ii) By definition, $K_f(\alpha)=[Ker(f)]_{F(\alpha)}=\{a\in F(\alpha);f(a)=\{e_{G_2}\}\}$ for all $\alpha\in A$. Clearly $K_f(\alpha)$ is a normal subgroup of $F(\alpha)$ for all $\alpha\in A$. Therefore

$$(K_f,A,\tau_1)\widetilde{\triangleleft}(F,A,\tau_1).$$

□

Note that most of the results for soft topological groups given in [**7**] are also satisfied for soft semi-topological groups.

**Theorem 3.17.** [**2**] *$(F,A,\tau)$ is a soft semi-topological group(soft topological group) over a semi-topological group(topological group) $(G,\tau)$ if and only if $(F,A)$ is a soft group over semi-topological group(topological groups) $(G,\tau)$.*

*Proof.* Let $(F,A,\tau)$ be a soft semi-topological group over a topological group $(G,\tau)$. Then for all $\alpha\in A$, $(F(\alpha),\tau_{F(\alpha)})$ is a topological subgroup of $(G,\tau)$. So $F(\alpha)\leq G$ for all $\alpha\in A$. Hence $(F,A)$ is a soft group over $G$.
Now let $(F,A)$ be a soft group over the semi-topological group $(G,\tau)$. Since the restriction of a continuous function is continuous and $F(\alpha)\leq G$ for all $\alpha\in A$, then $(F(\alpha),\tau_{F(\alpha)})$ is a topological subgroup of $G$ for all $\alpha\in A$.
The proof for soft topological groups is similar. □

**Theorem 3.18.** [**2**] *Let $G$ be a semi-topological group(topological group) with two different topologies $\tau_1$ and $\tau_2$, $(F_1,A,\tau_1)$ and $(F_2,A,\tau_2)$ be two soft semi-topological groups(soft topological groups) over $(G,\tau_1)$ and $(G,\tau_2)$, respectively. Then*
*$(F_1\widetilde{\cap}F_2,A,\tau_1)$, $(F_1\widetilde{\cap}F_2,A,\tau_2)$, and $(F_1\widetilde{\cap}F_2,A,\tau_1\cap\tau_2)$ are soft semi-topological groups (soft topological groups) over $(G,\tau_1)$, $(G,\tau_2)$ and $(G,\tau_1\cap\tau_2)$, respectively, where $(F_1\widetilde{\cap}F_2)(\alpha)=F_1(\alpha)\cap F_2(\alpha)$ for all $\alpha\in A$.*

*Proof.* First let us prove that $(G, \tau_1 \cap \tau_2)$ is semi-topological group. We need to show that the mapping $g_1$ defined in Definition 2.11 is continuous with respect to topology $\tau_1 \cap \tau_2$. Let $U$ be an open set in $\tau_1 \cap \tau_2$. Then $U$ is in $\tau_1$ and $\tau_2$. Since $(G, \tau_1)$ and $(G, \tau_2)$ are semi-topological groups, $g_1^{-1}(U)$ is open in both topologies $\tau_1$ and $\tau_2$. Hence $g_1^{-1}(U)$ is open in $\tau_1 \cap \tau_2$. So $(G, \tau_1 \cap \tau_2)$ is semi-topological group.
Finally since $(F_1 \widetilde{\cap} F_2)(\alpha)$ is subgroup of $G$ for all $\alpha \in A$, by Theorem 3.17 $(F_1 \widetilde{\cap} F_2, A, \tau_1)$, $(F_1 \widetilde{\cap} F_2, A, \tau_2)$ and $(F_1 \widetilde{\cap} F_2, A, \tau_1 \cap \tau_2)$ are soft semi-topological groups (soft topological groups) over $(G, \tau_1)$, $(G, \tau_2)$, and $(G, \tau_1 \cap \tau_2)$, respectively. □

The following theorem gives us a characterization that if a soft set $(F, A)$ is a soft semi-topological group over a semi-topological group $(G, \tau)$ then $(F, A)$ is also a soft semi-topological group over $G$ with respect to all topologies which make $G$ a semi-topological group.

**Theorem 3.19.** *Let $(G, \tau_1)$ and $(G, \tau_2)$ be a semi-topological group (topological group) and $(F, A)$ be a soft set over $G$. Then $(F, A, \tau_1)$ is a soft semi-topological group (soft topological group) over $(G, \tau_1)$ if and only if $(F, A, \tau_2)$ is a soft semi-topological group (soft topological group) over $(G, \tau_2)$.*

*Proof.* The proof is straightforward by Theorem 3.17. □

**Definition 3.20.** *Let $G$ be a group and $(F, A)$ be a soft group over $G$. Then*
*i) Soft center for a fixed element $a \in G$ of a soft group $(F, A)$ is defined by*
*$SC_a(F, A) = (C_a F, A)$, where $C_a F(\alpha) = \{y : a.y = y.a,\ y \in F(\alpha)\}$ for all $\alpha \in A$.*
*ii) Soft center of a soft group $(F, A)$ is defined by*
*$SC_G(F, A) = (C_G F, A)$, where $C_G F(\alpha) = \{y : x.y = y.x,\ y \in F(\alpha),\ x \in G\}$ for all $\alpha \in A$.*

**Example 3.21.** *Suppose that $G = A = \{e, (12), (13), (23), (123), (132)\}$, which is the group with function composition. Consider the function $F : A \longrightarrow \mathcal{P}(G)$ defined by $F(e) = \{e\}$, $F((12)) = \{e, (12)\}$, $F(13) = \{e, (13)\}$, $F(23) = \{e, (23)\}$, $F(123) = F(132) = \{e, (123), (132)\}$. It is obvious that $(F, A)$ is a soft group over $G$ since $F(\alpha) \leq G$ for all $\alpha \in A$. Let take $(12) \in G$. Then soft center for $(12)$ of soft group $(F, A)$ is $SC_{(12)}(F, A) = (C_{(12)}F, A)$= $\{(e, \{e\})$,$((12)$,$\{e, (12)\})$,$((13)$,$\{e\})$,$((23), \{e\})$, $((123), \{e\})$,$((132), \{e\})\}$ and soft center of $(F, A)$ is $SC_G(F, A) = (C_G F, A)$ $= \{(e, \{e\}), ((12), \{e\}), ((13), \{e\}), ((23), \{e\}), ((123), \{e\})\}$.*

**Theorem 3.22.** *Let $G$ be a group and $(F, A)$ be a soft group over $G$. Then $SC_G(F, A) = \widetilde{\bigcap}_{x \in G} SC_x(F, A)$.*

*Proof.* For an arbitrary parameter $\alpha$ in $A$, take an element $y \in C_G F(\alpha)$. Then by definition, y commutes for all $x \in G$. So $y$ is in $\bigcap_{x \in G} C_x F(\alpha)$, which means that $SC_G(F, A)$ is soft subset of $\widetilde{\bigcap}_{x \in G} SC_x(F, A)$.

Similarly if $y \in \bigcap_{x\in G} C_xF(\alpha)$, y is commutative for all $x \in G$ then $y \in C_GF(\alpha)$. Hence $\widetilde{\bigcap}_{x\in G} SC_x(F,A)$ is soft subset of $SC_G(F,A)$. □

**Theorem 3.23.** *We have*
*i) If $(F,A)$ is a soft group over a group $G$, so is $SC_a(F,A)$ for all $a \in G$,*
*ii) If $(F,A)$ is a soft semi-topological group over a semi-topological group, so is $SC_a(F,A)$ for all $a \in G$,*
*iii) If $(F,A)$ is a soft topological group over a topological group, so is $SC_a(F,A)$ for all $a \in G$.*

*Proof.* The proof is straightforward. □

**Corollary 3.24.** *We have i) If $(F,A)$ is a soft group over a group $G$ so is $SC_G(F,A)$,*
*ii) If $(F,A)$ is a soft semi-topological group over a semi-topological group so is $SC_G(F,A)$,*
*iii) If $(F,A)$ is a soft topological group over a topological group so is $SC_G(F,A)$.*

*Proof.* The proof is obtained from Theorem 3.23 and Theorem 3.22. □

## 4. Mappings on soft (semi-)topological groups

In this section mappings on soft classes are studied following [**1**] but in our form. A new mapping soft restricted image on soft classes is defined. Soft image and soft inverse image of a soft semi-topological group are examined.

**Definition 4.1.** [**1**] *Let $X$ be a universe and $E$ be a set of parameters. Then the collection of all soft sets over $X$ with parameters from $E$ is called a soft class and is denoted by $(X,E)$.*

**Definition 4.2.** [**1**] *Let $(X,E)$ and $(X^{'},E^{'})$ be soft classes. Let $\phi : X \longrightarrow X^{'}$ and $\psi : E \longrightarrow E^{'}$ be two mappings. Then a mapping $(\phi,\psi) : (X,E) \longrightarrow (X^{'},E^{'})$ is defined as: for a soft set $(F,A)$ in $(X,E)$, $(\phi,\psi)(F,A) = (\phi F, B)$; $B = \psi(A) \subseteq E^{'}$ is a soft set in $(X^{'},E^{'})$ given by*

$$\phi F(\beta) = \phi\Big(\bigcup_{\alpha\in\psi^{-1}(\beta)\cap A} F(\alpha)\Big)$$

*for $\beta \in B \subseteq E^{'}$. $(\phi F, B)$ is called a soft image of a soft set $(F,A)$.*

**Definition 4.3.** [**1**] *Let $(\phi,\psi) : (X,E) \longrightarrow (X^{'},E^{'})$ be a mapping from a soft class $(X,E)$ to $(X^{'},E^{'})$ and $(H,C)$ be a soft set in $(X^{'},E^{'})$, where $C \subseteq E^{'}$. $\phi : X \longrightarrow X^{'}$ and $\psi : E \longrightarrow E^{'}$ be two mappings. Then $(\phi,\psi)^{-1}(H,C) = (\phi^{-1}H, D)$, where*

*$D = \psi^{-1}(C) \subseteq E$ is a soft set in $(X, E)$, defined as:*

$$\phi^{-1}H(\alpha) = \phi^{-1}(H(\psi(\alpha)))$$

*for $\alpha \in D \subseteq E$. $(\phi^{-1}H, D)$ is called a soft inverse image of $(H, C)$.*

**Theorem 4.4.** *Let $X$ and $X^{'}$ be two semi-topological groups (topological groups) with topologies $\tau_1$ and $\tau_2$, respectively, $(H, C)$ be a soft semi-topological group (soft topological group) over $X^{'}$, and $(\phi, \psi) : (X, E) \longrightarrow (X^{'}, E^{'})$ be a mapping from a soft class $(X, E)$ to $(X^{'}, E^{'})$ defined as in Definition 4.3. If the mapping $\phi : X \longrightarrow X^{'}$ is a group homomorphism then $(\phi, \psi)^{-1}(H, C)$ is a soft semi-topological group (soft topological group) over $X$.*

*Proof.* Let $\alpha$ be an arbitrary parameter in $D \subseteq E$. Then $H(\psi(\alpha))$ is a semi-topological subgroup of $X^{'}$, since $(H, C)$ is a soft semi-topological group. Hence $\phi^{-1}H(\psi(\alpha))$ is a semi-topological subgroup of $X$ by the facts that $\phi$ is homomorphism and $H(\psi(\alpha))$ is a semi-topological subgroup of $X^{'}$. This means that $\phi^{-1}H(\alpha)$ is semi-topological subgroup of $X$ for all $\alpha \in D$. So $(\phi, \psi)^{-1}(H, C)$ is a soft semi-topological group over $X$.
The proof is similar for the soft topological groups. □

**Example 4.5.** *Let $(X, +) = (\mathbb{Z}_4, +)$, $(X^{'}, \cdot) = (\{i, -i, 1, -1\}, \cdot)$ be topological groups endowed with discrete topologies. Let the parameter sets be $E_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $E_2 = \{\alpha_1^{'}, \alpha_2^{'}, \alpha_3^{'}\}$. Define $\phi : X \longrightarrow X^{'}$ and $\psi : E \longrightarrow E^{'}$ as*

$$\phi(0) = 1,\ \phi(1) = i,\ \phi(2) = -1,\ \phi(3) = -i,$$
$$\psi(\alpha_1) = \alpha_1^{'},\ \psi(\alpha_2) = \alpha_3^{'},\ \psi(\alpha_3) = \alpha_1^{'},\ \psi(\alpha_4) = \alpha_3^{'}.$$

*It is clear that $\phi$ is a group homomorphism and*

$$(H, C) = \{(\alpha_1^{'}, X^{'}), (\alpha_2^{'}, \{1\}), (\alpha_3^{'}, \{1, -1\})\}$$

*is soft topological groups over $X^{'}$.*
*The mapping $(\phi, \psi) : (X, E) \longrightarrow (X^{'}, E^{'})$ is given as: for a soft topological group $(H, C)$ in $(X^{'}, E^{'})$,*
*for the soft inverse image $(\phi^{-1}H, D)$, where $D = p^{-1} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ we have*
$(\phi, \psi)^{-1}(H, C)(\alpha_1) = \phi^{-1}(H(\psi(\alpha_1))) = \phi^{-1}(H(\alpha_1^{'})) = \phi^{-1}(X^{'}) = X,$
$(\phi, \psi)^{-1}(H, C)(\alpha_2) = \phi^{-1}(H(\psi(\alpha_2))) = \{0, 2\},$
$(\phi, \psi)^{-1}(H, C)(\alpha_3) = \phi^{-1}(H(\psi(\alpha_3))) = X,$
$(\phi, \psi)^{-1}(H, C)(\alpha_4) = \phi^{-1}(H(\psi(\alpha_4))) = \{0, 2\}.$
*Hence*

$$(\phi^{-1}H, D) = \{(\alpha_1, \mathbb{Z}_4), (\alpha_2, \{0, 2\}), (\alpha_3, \mathbb{Z}_4), (\alpha_4, \{0, 2\})\}$$

*is a soft topological group over $X$.*

**Remark.** *Theorem 4.4 is not true for the soft image of a soft semi-topological group (soft topological group).*

**Example 4.6.** *Let* $(X,+)=(\mathbb{Z},+)$, $(X^{'},+)=(\mathbb{Z}_6,+)$ *be semi-topological groups endowed with discrete topologies. Let the parameter sets be* $E_1=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ *and* $E_2=\{\alpha_1^{'},\alpha_2^{'},\alpha_3^{'}\}$. *Define* $\phi: X\longrightarrow X^{'}$ *and* $\psi: E\longrightarrow E^{'}$ *as*

$$\phi:\mathbb{Z}\longrightarrow\mathbb{Z}_6$$

$$a\longrightarrow\overline{a}$$

$$\psi(\alpha_1)=\alpha_1^{'},\ \psi(\alpha_2)=\alpha_3^{'},\ \psi(\alpha_3)=\alpha_1^{'},\ \psi(\alpha_4)=\alpha_3^{'}.$$

*Clearly,* $\phi$ *is a group homomorphism and* $(F,A)=\{(\alpha_1,2\mathbb{Z}),(\alpha_2,3\mathbb{Z}),(\alpha_3,5\mathbb{Z})\}$ *is a soft topological group over* $X$.
*Let choose* $\alpha_1^{'}\in B=\psi(A)=\{\alpha_1^{'},\alpha_3^{'}\}$. *We have* $\phi F(\alpha_1^{'})=\phi(\bigcup F(\{\alpha_1\})\cup F(\{\alpha_3\})=\{0,2,3,4\}$, *since* $\psi^{-1}(\alpha_1^{'})\cap A=\{\alpha_1,\alpha_3\}$. *Clearly,* $(\alpha_1^{'},\tau_{F(\alpha_1^{'})})$ *is not a topological subgroup of* $X^{'}$. *Hence* $(\phi F,B)$ *is not a soft topological group over* $X^{'}$.

**Definition 4.7.** *Let* $(X,E)$ *and* $(X^{'},E^{'})$ *be soft classes. Let* $\phi: X\longrightarrow X^{'}$ *and* $\psi: E\longrightarrow E^{'}$ *be two mappings. Then a mapping* $(\phi,\psi)_{\cap}:(X,E)\longrightarrow(X^{'},E^{'})$ *is defined as: for a soft set* $(F,A)$ *in* $(X,E)$, $((\phi F)_{\cap},B)$ *and* $B=\psi(A)\subseteq E^{'}$ *is a soft set in* $(X^{'},E^{'})$ *given by*

$$\phi F(\beta)=\phi\Big(\bigcap_{\alpha\in\psi^{-1}(\beta)\cap A}F(\alpha)\Big)$$

*for* $\beta\in B\subseteq E^{'}$. $(\phi,\psi)_{\cap}(F,A)=((\phi F)_{\cap},B)$ *is called a soft restricted image of a soft set* $(F,A)$.

**Theorem 4.8.** *Let* $X$ *and* $X^{'}$ *be two semi-topological groups (topological groups) with topologies* $\tau_1$ *and* $\tau_2$, *respectively,* $(F,A)$ *be a soft semi-topological group (soft topological group) over* $X$, *and* $(\phi,\psi)_{\cap}:(X,E)\longrightarrow(X^{'},E^{'})$ *be a mapping from a soft class* $(X,E)$ *to* $(X^{'},E^{'})$ *defined as in Definition 4.7. If the mapping* $\phi: X\longrightarrow X^{'}$ *is a group homomorphism then* $(\phi,\psi)_{\cap}(F,A)=((\phi F)_{\cap},B)$ *is a soft semi-topological group(soft topological group) over* $X^{'}$.

*Proof.* By the definition of restricted image of a soft set $((\phi F)_{\cap},B)$, for $\beta\in B$, $\phi F(\beta)=\phi\Big(\bigcap_{\alpha\in\psi^{-1}(\beta)\cap A}F(\alpha)\Big)$. Since $(F,A)$ is a soft semi-topological group over a semi-topological group $(X,\tau)$ for all $\alpha\in A$, $F(\alpha)$ is a soft group over $X$. By the fact that intersection of subgroups is a subgroup, $\Big(\bigcap_{\alpha\in p^{-1}(\beta)\cap A}F(\alpha)\Big)$ is a subgroup of $X$ and since $\phi$ is homomorphism, image of $\Big(\bigcap_{\alpha\in\psi^{-1}(\beta)\cap A}F(\alpha)\Big)$ is a subgroup of $X^{'}$. Hence for all $\beta\in B$, $(\phi F)_{\cap}(\beta)$ is a subgroup of $X^{'}$ so $((\phi F)_{\cap},B)$ is a soft group over $X^{'}$. By Theorem 3.17, $((\phi F)_{\cap},B)$ is soft semi-topological group over $X^{'}$.
The proof is similar for the soft topological groups. □

**Example 4.9.** *Let the groups* $(X,+)=(\mathbb{Z}_4,+)$, $(X^{'},\cdot)=(\{i,-i,1,-1\},\cdot)$ *be semi-topological group endowed with discrete topologies. Let the parameter sets be* $E_1=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ *and* $E_2=\{\alpha_1^{'},\alpha_2^{'},\alpha_3^{'}\}$. *Define* $\phi: X\longrightarrow X^{'}$ *and* $\psi: E\longrightarrow E^{'}$ *as*

$$\phi(0)=1,\ \phi(1)=i,\ \phi(2)=-1,\ \phi(3)=-i,$$
$$\psi(\alpha_1)=\alpha_1^{'},\ \psi(\alpha_2)=\alpha_3^{'},\ \psi(\alpha_3)=\alpha_1^{'},\ \psi(\alpha_4)=\alpha_3^{'}.$$

*One can easily see that* $\phi$ *is a group homomorphism and*

$$(F,A)=\{(\alpha_1,\{0\}),(\alpha_2,\{0,2\}),(\alpha_3,\mathbb{Z}_4)\}$$

*is a soft semi-topological group.*
*The mapping* $(\phi,\psi):(X,E)\longrightarrow(X^{'},E^{'})$ *is given as: for a soft topological group* $(F,A)$ *in* $(X,E)$, $((\phi F)_{\cap},B)$ *where* $B=\psi(A)=\{\alpha_1^{'},\alpha_3^{'}\}$ *is a soft topological group in* $(X^{'},E^{'})$ *obtained as follows:*
$(\phi F)_{\cap}(\alpha_1^{'})=\phi(F(\{\alpha_1\})\cap F(\{\alpha_3\}))=\{1\}$, *(since* $\psi^{-1}(\alpha_1^{'})\cap A=\{\alpha_1\}$*),*
$(\phi F)_{\cap}(\alpha_3^{'})=\phi(\bigcap F(\{\alpha_2\}))=\{1,-1\}$, *(since* $\psi^{-1}(\alpha_3^{'})\cap A=\{\alpha_2\}$*). Hence*

$$((\phi F)_{\cap},B)=\{(\alpha_1^{'},\{1\}),(\alpha_3^{'},\{1,-1\})\}.$$

**Theorem 4.10.** *The soft image given in Definition 4.2 and soft restricted image that is given in Definition 4.7 of a soft set* $(F,A)$ *are equal when one of the following conditions hold;*
*i) The map* $\psi: E\longrightarrow E^{'}$ *is injective,*
*ii) The map* $\phi$ *from* $X$ *to* $X^{'}$ *is a constant mapping,*
*iii) For each* $\beta\in B$ *and for each* $\alpha,\gamma\in\psi^{-1}(\beta)\cap A$, $F(\alpha)=F(\gamma)$.

*Proof.* i) Let $(\phi F,B)$ and $(\phi F_{\cap},B)$ be soft image and soft restricted image of the soft set $(F,A)$. Then for all $\beta\in B\subseteq E^{'}$, $\phi F(\beta)=\phi F_{\cap}(\beta)$, since the map $\psi: E\longrightarrow E^{'}$ is injective.
ii) and iii) can be proved similar. □

**Definition 4.11.** *Let* $(\phi,\psi)$ *be soft mapping from soft classes* $(X,E)$ *to* $(X^{'},E^{'})$, *where* $X^{'}$ *is a group. Then kernel of soft set* $(F,A)$ *over* $X$ *is defined by*

$$Ker(F,A)_{(\phi,\psi)}=\{(\alpha,F(\alpha))\mid \phi F(\beta)=\{e_{X^{'}}\},\ \beta\in\psi(A),\ \alpha\in\psi^{-1}(\beta)\cap A\},$$

*where* $\alpha\in A$ *and* $e_{X^{'}}$ *is the identity element of* $X^{'}$.
*It is clear that* $Ker(F,A)_{(\phi,\psi)}$ *is soft subset of* $(F,A)$.

**Example 4.12.** *The kernel of the soft set defined in Example 4.9 is* $Ker(F,A)_{(\phi,\psi)}=\emptyset$.

**Definition 4.13.** *Let* $(\phi,\psi)_{\cap}$ *be a soft restricted mapping from soft classes* $(X,E)$ *to* $(X^{'},E^{'})$, *where* $X^{'}$ *is a group. Then restricted kernel of soft set* $(F,A)$ *over* $X$ *is* *defined by*

$$Ker(F,A)_{(\phi,\psi)_\cap} = \{(\alpha, F(\alpha)) \mid (\phi F)_\cap(\beta) = \{e_{X'}\},\ \beta \in \psi(A),\ \alpha \in \psi^{-1}(\beta) \cap A\},$$

*where* $\alpha \in A$ *and* $e_{X'}$ *is the identity element of* $X^{'}$.
*It is clear that* $Ker(F,A)_{(\phi,\psi)_\cap}$ *is a soft subset of* $(F,A)$.

**Example 4.14.** *The restricted kernel of the soft set defined in Example 4.9 is* $Ker(F,A)_{(\phi,\psi)_\cap} = \{(\alpha_1, \{0\}\}$.

As one can obtain from above examples, kernel and restricted kernel of a soft set do not need to be equal in general. But if one of the conditions holds in Theorem 4.10, then the equality is always satisfied. Also note that if $\phi$ is a group homomorphism and $\psi$ is injective in Definition 4.11 and Definition 4.13, then the kernel and restricted kernel of $(F,A)$ is equal to the soft set $(K_\phi, A)$ given in Theorem 3.16 (we do not need to express a topology here).

**Theorem 4.15.** *Let* $X$ *be semi-topological group (topological group),* $X^{'}$ *be a group, and* $(\phi,\psi)$*(*$(\phi,\psi)_\cap$*) be the mapping given in Definition 4.2 and Definition in 4.7. Then the soft (restricted) kernel of a soft semi-topological group(soft topological group) over* $X$ *is a soft semi-topological group (soft topological group).*

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