Journal of Universal Mathematics
Vol.3 No.1 pp.77-82 (2020)
ISSN-2618-5660

ON THE SEQUENCE OF GELL NUMBERS

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Abstract. In this paper, we consider Pell numbers. We define the gell numbers which generalize the Pell numbers. Moreover, we derive Binet-like formula, generating function and exponential generating function for the gell sequence. Also, we obtain the gell series and some important identities for the gell sequence.

1. Introduction

Like the well-known Fibonacci, Pell numbers play important role in mathematics. Pell numbers too continue to amaze the mathematical community with their applications in analysis, trigonometry and various areas of discrete mathematics, such as number theory, graph theory, linear algebra and combinatorics. Also, the use of such special sequences has increased significantly in applied science [4]. Pell sequence $\{P_n\}_{n\geq 0}$ is defined by the initial values $P_0 = 0$ and $P_1 = 1$ and the recurrence relation

$$P_{n+2} = 2P_{n+1} + P_n, \qquad n \geq 0. \tag{1.1}$$

First few terms of this sequence are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378$. For convenience we take $P_{-1} = 1$ and $P_{-2} = -2$. The Pell number were called in honour of English mathematician John Pell [2, 3, 6]. The golden ratio has many applications in engineering, physics, architecture, arts and others [5]. In similar way, the ratio of two consecutive Pell numbers converges to

$$\lambda = 1 + \sqrt{2} \approx 2.4142135623,$$

that is called as "silver ratio". The silver number is the positive real root of the characteristic equation of Pell numbers

$$x^2 - 2x - 1 = 0.$$

The other root is

$$\mu = 1 - \sqrt{2}$$

so that,

$$\lambda + \mu = 2, \quad \lambda - \mu = 2\sqrt{2} \quad \text{and} \quad \lambda\mu = -1.$$

2000 *Mathematics Subject Classification.* 11B39; 05A15; 11R52.

Key words and phrases. Pell sequence, generalized Pell sequence, Binet-like formula, generating function, exponential generating function.

The Binet formula of Pell sequence is

$$P_n = \frac{\lambda^n - \mu^n}{\lambda - \mu},$$

Its the generating function is

$$G_P(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1 - 2x - x^2},$$

and the exponential generating function is

$$E_P(x) = \frac{e^{\lambda x} - e^{\mu x}}{\lambda - \mu} = \sum_{n=0}^{\infty} \frac{P_n}{n!} x^n.$$

Also, the Pell series is

$$S_P = \sum_{n=0}^{\infty} \frac{P_n}{t^{n+1}} = \frac{1}{t^2 - 2t - 1}.$$

The more properties and applications of Pell numbers can be found in [2, 3, 4, 5, 6]

2. The Gell Numbers

The gell sequence $\{\mathcal{G}I_n\}$ as a generalization of Pell sequence is defined by a two order recurrence;

$$\mathcal{G}I_{n+2} = 2\mathcal{G}I_{n+1} + \mathcal{G}I_n, \quad n \geq 1$$

with the initial conditions $\mathcal{G}I_1 = a, \quad \mathcal{G}I_2 = b$. The first few members of this sequence is given as follow ;

n	1	2	3	4	5	6	7	...
$\mathcal{G}I_n$	a	b	$a+2b$	$2a+5b$	$5a+12b$	$12a+29b$	$29a+70b$	...

Some investigations related the generalized Pell numbers were given in [1]. Here we obtain new results.

3. Main Results

In the present work we derive Binet-like formula, generating function and exponential generating function for the gell sequence. Also, we obtain the gell series and some important identities for the gell sequence.

Theorem 3.1. *Let $\mathcal{G}I_n$ be n th gell number. Then,*

$$\mathcal{G}I_n = aP_{n-2} + bP_{n-1}, \quad n \geq 3.$$

Proof. We will establish this using PMI. Since,

$$\mathcal{G}I_3 = aP_1 + bP_2 = a + 2b$$

and

$$\mathcal{G}I_4 = aP_2 + bP_3 = 5a + 12b,$$

the result is true when $n = 3, 4$. Suppose it is true for all positive integers $n \leq k$. Then,

$$\begin{aligned}
\mathcal{G}I_{k+2} &= 2\mathcal{G}I_{k+1} + \mathcal{G}I_k \\
&= 2aP_{k-1} + 2bP_k + aP_{k-2} + bP_{k-1} \\
&= a(2P_{k-1} + P_{k-2}) + b(2P_k + P_{k-1}) \\
&= aP_k + bP_{k+1}
\end{aligned}$$

Thus, by the strong version of PMI, the formula works for all positive integers $n \leq 3$. □

Theorem 3.2. *(Binet-like formula) Let $c = a\mu^2 - b\mu$ and $d = a\lambda^2 - b\lambda$. Then,*

$$\mathcal{G}I_n = \frac{c\lambda^n - d\mu^n}{\lambda - \mu}, \qquad n \geq 0. \tag{3.1}$$

Proof. Using 3.1 Theorem. We have,

$$\begin{aligned}
\mathcal{G}I_n &= aP_{n-2} + bP_{n-1} \\
(\lambda - \mu)\mathcal{G}I_n &= a\left(\lambda^{n-2} - \mu^{n-2}\right) + b\left(\lambda^{n-1} - \mu^{n-1}\right) \\
&= \lambda^n\left(\frac{a}{\lambda^2} + \frac{b}{\lambda}\right) - \mu^n\left(\frac{a}{\mu^2} + \frac{b}{\mu}\right) \\
&= \lambda^n\left(a\mu^2 - b\mu\right) - \mu^n\left(a\lambda^2 - b\lambda\right) \\
&= c\lambda^n - d\mu^n
\end{aligned}$$

This yields the desired formula. □

Denote that

$$\begin{aligned}
t = cd &= \left(a\mu^2 - b\mu\right)\left(a\lambda^2 - b\lambda\right) \\
&= a^2(\mu\lambda)^2 - ab\mu^2\lambda - ab\mu\lambda^2 + b^2\mu\lambda \\
&= a^2 + ab(\lambda + \mu) - b^2 \\
&= a^2 + 2ab - b^2.
\end{aligned}$$

Theorem 3.3. *(Catalan's Identity) Let $\mathcal{G}I_n$ denote the n th gell sequence. Then,*

$$\mathcal{G}I_{n+k}\mathcal{G}I_{n-k} - \mathcal{G}I_n^2 = t(-1)^{n-k+1}P_k^2, \qquad n \geq k.$$

Proof. Using (3.1) we write

$$\begin{aligned}
(\lambda - \mu)^2(\mathcal{G}I_{n+k}\mathcal{G}I_{n-k} - \mathcal{G}I_n^2) &= (c\lambda^{n+k} - d\mu^{n+k})(c\lambda^{n-k} - d\mu^{n-k}) - (c\lambda^n - d\mu^n)^2 \\
&= c^2\lambda^{2n} - cd\lambda^{n+k}\mu^{n-k} - cd\mu^{n+k}\lambda^{n-k} + d^2\mu^{2n} \\
&\quad - c^2\lambda^{2n} + cd\lambda^n\mu^n + cd\mu^n\lambda^n - d^2\mu^{2n} \\
&= -cd\lambda^{n+k}\mu^{n-k} - cd\lambda^{n-k}\mu^{n+k} + 2cd\lambda^n\mu^n \\
&= -cd(\lambda\mu)^n\left[\left(\frac{\lambda^k}{\mu^k} - 1\right) - \left(1 - \frac{\mu^k}{\lambda^k}\right)\right] \\
&= -cd(\lambda\mu)^{n-k}(\lambda^k - \mu^k)^2 \\
&= t(-1)^{n-k+1}(\lambda^k - \mu^k)^2
\end{aligned}$$

□

Theorem 3.4. *(Cassini Identity) Let $\mathcal{G}I_n$ denote the n th gell sequence. Then,*

$$\mathcal{G}I_{n+1}\mathcal{G}I_{n-1} - \mathcal{G}I_n^2 = t(-1)^n, \qquad n \geq 1$$

Proof. Taking $k = 1$ in the Catalan's Identity, the proof is completed. □

Theorem 3.5. *Let $\mathcal{G}I_n$ denote the n th gell sequence. Then,*

$$\mathcal{G}I_{n+m} = P_{m+1}\mathcal{G}I_n + P_m\mathcal{G}I_{n-1}, \qquad m \geq 0, n \geq 1.$$

Proof. Using (3.1) we get

$$\begin{aligned}
(\lambda-\mu)^2(P_{m+1}\mathcal{G}I_n + P_m\mathcal{G}I_{n-1}) &= (\lambda^{m+1}-\mu^{m+1})(c\lambda^n - d\mu^n) \\
&+ (\lambda^m - \mu^m)(c\lambda^{n-1} - d\mu^{n-1}) \\
&= c\lambda^{n+m+1} - d\lambda^{m+1}\mu^n - c\mu^{m+1}\lambda^n + d\mu^{m+n+1} \\
&+ c\lambda^{n+m-1} - d\lambda^m\mu^{n-1} - c\mu^m\lambda^{n-1} + d\mu^{m+n-1} \\
&= \left(c\lambda^{n+m} - d\mu^{n+m}\right)(\lambda-\mu)
\end{aligned}$$

□

Theorem 3.6. *(Gelin-Cesaro Identity) Let $\mathcal{G}I_n$ denote the n th gell sequence. Then,*

$$\mathcal{G}I_{n+2}\mathcal{G}I_{n+1}\mathcal{G}I_{n-1}\mathcal{G}I_{n-2} - \mathcal{G}I_n^4 = -t^2, \qquad n \geq 2$$

Proof. Using (3.1) we obtain

$$\begin{aligned}
\mathcal{G}I_{n+2}\mathcal{G}I_{n+1}\mathcal{G}I_{n-1}\mathcal{G}I_{n-2} - \mathcal{G}I_n^4 &= (\mathcal{G}I_{n+2}\mathcal{G}I_{n-2})(\mathcal{G}I_{n+1}\mathcal{G}I_{n-1}) - \mathcal{G}I_n^4 \\
&= \left(t(-1)^n + \mathcal{G}I_n^2\right)\left(-t(-1)^n + \mathcal{G}I_n^2\right) - \mathcal{G}I_n^4 \\
&= -t^2(-1)^{2n} + \mathcal{G}I_n^4 - \mathcal{G}I_n^4 = -t^2
\end{aligned}$$

□

Theorem 3.7. *(d'Ocagne's Identity) Let $\mathcal{G}I_n$ denote the n th gell sequence. Then,*

$$\mathcal{G}I_m\mathcal{G}I_{n+1} - \mathcal{G}I_{m+1}\mathcal{G}I_n = t(-1)^n P_{m-n}, \qquad m \geq n$$

Proof. Using (3.1) we have

$$\begin{aligned}
(\lambda-\mu)^2\mathcal{G}I_m\mathcal{G}I_{n+1} - \mathcal{G}I_{m+1}\mathcal{G}I_n &= (c\lambda^m - d\mu^m)\left(c\lambda^{n+1} - d\mu^{n+1}\right) \\
&- \left(c\lambda^{m+1} - d\mu^{m+1}\right)(c\lambda^n - d\mu^n) \\
&= c^2\lambda^{m+n+1} - cd\lambda^m\mu^{n+1} - cd\mu^m\lambda^{n+1} + d^2\mu^{m+n+1} \\
&- c^2\lambda^{m+n+1} + cd\lambda^{m+1}\mu^n + cd\mu^{m+1}\lambda^n - d^2\mu^{m+n+1} \\
&= (cd)\left[\lambda^m\mu^n(\lambda-\mu) - \lambda^n\mu^m(\lambda-\mu)\right] \\
&= cd(\lambda\mu)^n(\lambda-\mu)(\lambda^{m-n} - \mu^{m-n}) \\
&= t(-1)^n(\lambda-\mu)(\lambda^{m-n} - \mu^{m-n})
\end{aligned}$$

□

Theorem 3.8. *The generating function for the nth gell numbers is*

$$\mathcal{G}_{\mathcal{G}\mathcal{I}}(x) = \frac{ax - bx^2}{1 - 2x - x^2}.$$

Proof. Assume that the function

$$\mathcal{G}_{\mathcal{GI}}(x) = \sum_{n=1}^{\infty} \mathcal{G}I_n x^n = \mathcal{G}I_1 x + \mathcal{G}I_2 x^2 + \mathcal{G}I_3 x^3 + \ldots + \mathcal{G}I_n x^n + \ldots$$

be generating function of the gell numbers. Multiply both of side of the equality by the term $-2x$ such as

$$-2x\mathcal{G}_{\mathcal{GI}}(x) = -2\mathcal{G}I_1 x^2 - 2\mathcal{G}I_2 x^3 - 2\mathcal{G}I_3 x^4 - \ldots - 2\mathcal{G}I_n x^{n+1} + \ldots$$

and that is multipied every side with $-x^2$ such as

$$-x^2\mathcal{G}_{\mathcal{GI}}(x) = -\mathcal{G}I_1 x^3 - \mathcal{G}I_2 x^4 - \mathcal{G}I_3 x^5 - \ldots - \mathcal{G}I_n x^{n+2} + \ldots$$

Then, we write

$$\begin{aligned}(1-2x-x^2)\mathcal{G}_{\mathcal{GI}}(x) &= \mathcal{G}I_1 x - \mathcal{G}P_2 x^2 + (\mathcal{G}I_3 - 2\mathcal{G}I_2 - \mathcal{G}I_1)x^3 + \ldots \\ &\quad + (\mathcal{G}I_n - 2\mathcal{G}I_{n-1} - \mathcal{G}I_{n-2})x^n + \ldots\end{aligned}$$

Now, by using $\mathcal{G}I_1 = a$, $\mathcal{G}I_2 = b$, $\mathcal{G}I_3 = a + 2b$, $\mathcal{G}I_4 = 2a + 5b$, $\mathcal{G}I_5 = 5a + 12b \ldots$ we obtain that,

$$\mathcal{G}_{\mathcal{GI}}(x) = \frac{ax - bx^2}{1 - 2x - x^2}.$$

Thus, the proof is completed. □

Theorem 3.9. *The exponential generating function for the nth gell numbers is*

$$E_{\mathcal{GI}}(x) = \frac{ce^{\lambda x} - de^{\mu x}}{\lambda - \mu} = \sum_{n=0}^{\infty} \frac{\mathcal{G}I_n}{n!} x^n.$$

Proof. we know that,

$$e^{\lambda x} = \sum_{n=0}^{\infty} \frac{\lambda^n x^n}{n!} \quad \text{and} \quad e^{\mu x} = \sum_{n=0}^{\infty} \frac{\mu^n x^n}{n!}$$

Let's multiply each side of the first equation by c and the second equation by d, divide each side of both equations into $(\lambda - \mu)$ and subtract the second equation from the first equation we obtain

$$\frac{ce^{\lambda x} - de^{\mu x}}{\lambda - \mu} = \sum_{n=0}^{\infty} \frac{c\lambda^n - d\mu^n}{\lambda - \mu} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{\mathcal{G}I_n}{n!} x^n.$$

□

Theorem 3.10. *The Series for the nth gell numbers is*

$$S_{\mathcal{G}I} = \sum_{n=0}^{\infty} \frac{\mathcal{G}I_n}{t^{n+1}} = -\frac{2a}{t} + \left(\frac{a}{t^2} + \frac{b}{t}\right) \frac{t^2 - 2t}{t^2 - 2t - 1}.$$

Proof.

$$
\begin{aligned}
S_{\mathcal{G}I} = \sum_{n=0}^{\infty} \frac{\mathcal{G}I_n}{t^{n+1}} &= \sum_{n=0}^{\infty} \frac{aP_{n-2} + bP_{n-1}}{t^{n+1}} \\
&= a\sum_{n=0}^{\infty} \frac{P_{n-2}}{t^{n+1}} + b\sum_{n=0}^{\infty} \frac{P_{n-1}}{t^{n+1}} \\
&= a\sum_{n=-2}^{\infty} \frac{P_{n}}{t^{n+3}} + b\sum_{n=-1}^{\infty} \frac{P_{n}}{t^{n+2}} \\
&= -\frac{2a}{t} + \left(\frac{a}{t^2} + \frac{b}{t}\right)\sum_{n=-1}^{\infty} \frac{P_n}{t^{n+1}} \\
&= -\frac{2a}{t} + \left(\frac{a}{t^2} + \frac{b}{t}\right)\left(1 + \sum_{n=0}^{\infty} \frac{P_n}{t^{n+1}}\right)
\end{aligned}
$$

By using the Pell series we obtain

$$
S_{\mathcal{G}I} = -\frac{2a}{t} + \left(\frac{a}{t^2} + \frac{b}{t}\right)\frac{t^2 - 2t}{t^2 - 2t - 1}
$$

□

4. Conclusion

In the present work, we consider the Pell numbers sequence. We define the gell numbers as the generalization of Pell numbers. We give some algebraic identities for the gell numbers. Then, we derive the Binet formula, the generating and the exponential generating functions for the gell numbers sequence. Also, we obtain the gell series.

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