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WEAK FINITELY DUAL QUASI-CONJUGATIVE RELATIONS

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Abstract. The concept of 'weak finitely regular relations' was introduced by Shuzhen Luo and Xiaoquan Xu in 2019 and applied to partially ordered sets. In this article, this idea is extended to weak finality of dually quasi-conjugative relations was introduced in 2013 by this author.

1. Introduction and Preliminaries

For a set X, we call α a binary relation on X if $\alpha \subseteq X^2$. Let $\mathcal{B}(X)$ be denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X^2 : (\exists y \in X)((x, y) \in \alpha, (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $\mathcal{B}(X)$, with composition, is a monoid (semigroup with identity). Namely, $\triangle_X = \{(x, x) : x \in X\}$ is its identity element. For a binary relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X^2 : (y, x) \in \alpha\}$ and $\alpha^c = X^2 \setminus \alpha$. Thus $(\alpha^c)^{-1} = (\alpha^{-1})^c$ holds.

Let A be a subset of X. For $\alpha \in \mathcal{B}(X)$, set

$$A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\}, \ \ \alpha A = \{x \in X : (\exists b \in A)((x, b) \in \alpha)\}.$$

It is easy to see that $A\alpha = \alpha^{-1}A$ holds. Specially, we put $a\alpha$ instead of $\{a\}\alpha$ and αb instead of $\alpha\{b\}$.

Notions and notations used in this article that are not previously defined may be find by the reader in articles [3, 4, 5, 7, 10].

1.1. **Quasi-conjugative relations.** The fundamental works of K. A. Zareckii [9], B. M. Schein [7] and others on regular relations motivated several mathematicians to investigate similar classes of relations, obtained by putting α^{-1}, α^c or $(\alpha^c)^{-1}$ in place of one or both α's on the right side of the regularity equation

$$\alpha = \alpha \circ \beta \circ \alpha$$

(where β is some relation). The following class of elements in the semigroup $\mathcal{B}(X)$ have been investigated: normal relation in [1] by G. Jiang, L. Xu, J. Cai and G. Han;, dually normal relation in [2] by G. Jiang and L. Xu; quasi-conjugative and bi-normal relations [4, 5, 6, 8] by this author and M. Vinčić. For example:

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Definition 1.1. ([4], Definition 2.2) The relation $\alpha \in \mathcal{B}(X)$ is called a *quasi-conjugative* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha^c.$$

The class of duals of these relations are given in the following definition.

Definition 1.2. ([4], Remark 2.1) The relation $\alpha \in \mathcal{B}(X)$ is called a *dually quasi-conjugative* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^c \circ \beta \circ \alpha^{-1}.$$

In the following propositions we give an intrinsic characterization of any quasi-conjugative relation and its dual.

Proposition 1 ([4], Theorem 2.1)**.** For a binary relation α on a set X, the following conditions are equivalent:

(1) α is a quasi-conjugative relation; and
(2) For all $x, y \in X$, if $(x, y) \in \alpha$, there exist $u, v \in X$ such that:
(a) $(x, v) \in \alpha^c \wedge (y, u) \in \alpha$;
(b) $(\forall s, t \in X)((s, v) \in \alpha^c \wedge (t, u) \in \alpha) \implies (s, t) \in \alpha)$.

Proposition 2. For a binary relation α on a set X, the following conditions are equivalent:

(1) α is a dually quasi-conjugative relation; and
(2) For all $x, y \in X$, if $(x, y) \in \alpha$, there exist $u, v \in X$ such that:
(a) $(x, v) \in \alpha^{-1} \wedge (u, y) \in \alpha^c$;
(b) $(\forall s, t \in X)((s, v) \in \alpha^{-1} \wedge (u, t) \in \alpha^c) \implies (s, t) \in \alpha)$.

1.2. **Finitely quasi-conjugative relations.** For any set X, let $X^{(<\omega)} = \{F \subseteq X : F \text{ is finite}\}$. For any positive integer m, we write $\overline{m} = \{1, 2, ..., m\}$.

Definition 1.3. ([10]) For a binary relation $\rho \subseteq X \times Y$, define a relation $\rho^{(<\omega)} \subseteq X^{(<\omega)} \times Y^{(<\omega)}$ by

$$(\forall (F, G) \in X^{(<\omega)} \times Y^{(<\omega)})((F, G) \in \rho^{(<\omega)} \iff G \subseteq F\rho).$$

$\rho^{(<\omega)}$ is called finite extension of ρ.

For illustration purposes, we will show definition of finite extension of quasi-conjugative and dual quasi-conjugative relations:

Definition 1.4. ([5]) A binary relation α on a set X is called finitely quasi-conjugative if for all $(x, y) \in \alpha$, there are $u \in X$ and $\{v_1, v_2, ..., v_k\} \in X^{(<\omega)}$, such that

(i) $(u, y) \in \alpha^{-1} \wedge (\forall i \in \overline{k})((x, v_i) \in \alpha^c)$, and
(ii) for all $\{s_1, s_2, ..., s_k\} \in X^{(<\omega)}$ and $t \in X$,
if $(u, t) \in \alpha^{--1}$ and $(\forall i \in \overline{k})((s_i, v_i) \in \alpha^c)$ then there is $j \in \overline{k}$ such that $(s_j, t) \in \alpha$.

An important description of the finitely bi-conjugative relation is given in the following proposition

Proposition 3 ([5])**.** For a binary relation α on a set X, the following are equivalent:

(i) ρ is a finitely quasi-conjugative relation on X; and
(ii) there is a binary relation $\delta \subseteq X^{(<\omega)} \times X^{(<\omega)}$ such that $[(\alpha^{--1})^{(<\omega)} \circ \delta \circ (\alpha^c)^{(<\omega)} = \alpha^{(<\omega)}]$

Definition 1.5. ([5]) A binary relation α on a set X is called finitely dual quasi-conjugative if for all $(x,y) \in \alpha$, there are $u \in X$ and $\{v_1, v_2, ..., v_k\} \in X^{(<\omega)}$, such that

(i) $(u,y) \in \alpha^c \wedge (\forall i \in \overline{k})((x,v_i) \in \alpha^{-1})$, and

(ii) for all $\{s_1, s_2, ..., s_k\} \in X^{(<\omega)}$ and $t \in X$,

if $(u,t) \in \alpha^c$ and $(\forall i \in \overline{k})((s_i,v_i) \in \alpha^{-1})$ then there is $j \in \overline{k}$ such that $(s_j,t) \in \alpha$.

An important description of the finitely dual quasi-conjugative relation is given in the following proposition

Proposition 4 ([5])**.** For a binary relation α on a set X, the following are equivalent:

(i) ρ is a finitely dual quasi-conjugative relation on X; and

(ii) there is a binary relation $\delta \subseteq X^{(<\omega)} \times X^{(<\omega)}$ such that

$$(\alpha^c)^{(<\omega)} \circ \delta \circ (\alpha^{-1})^{(<\omega)} = \alpha^{(<\omega)}.$$

2. Weak finitely dual quasi-conjugative relations

In Definition 1.4 and 1.5, in statement (ii) the finite set $\{s_1, s_2, ..., s_k\}$, $(k \in \mathbb{N})$ appears as a very strong condition. In the just published article [3], in a situation where regular relations is discussed, this requirement is weakened by taking $\{s\}$ instead of $\{s_1, s_2, ..., s_k\}$. We impose an analogous requirement on the finality of dually quasi-conjugative relations.

Before that, we need the following definition

Definition 2.1. ([3], Definition 6) For a binary relation $\rho \subseteq X \times Y$, define a binary relation $\rho^{(<\omega)} \subseteq X \times Y^{(<\omega)}$, called the *right finite extension* of ρ, by

$$\rho_r^{(<\omega)} = \{(x,G) \in X \times Y^{(<\omega)} : G \subseteq x\rho\}.$$

2.1. **The concept.** We can now introduce the concept of weak finitely dual quasi-conjugative relations

Definition 2.2. A binary relation ρ on a set X is called *weak finitely dual quasi-conjugative* if for all $(x,y) \in \rho$, there are $u \in X$ and $\{v_1, v_2, ..., v_k\} \in X^{(<\omega)}$, such that

(i) $(u,y) \in \rho^c \wedge (\forall i \in \overline{k})((x,v_i) \in \rho^{-1})$, and

(ii) for all $s \in X$ and $t \in X$,

if $(u,t) \in \rho^c$ and $(\forall i \in \overline{k})((s,v_i) \in \rho^{-1})$ then $(s,t) \in \rho$.

2.2. **The main result.** In the following theorem, we give some characterizations of weak finality of dually quasi-conjugative relations.

Theorem 2.3. *Let $\rho \subseteq X \times X$ a binary relation. Then the following conditions are equivalent:*

(1) *ρ is weak finitely dual quasi-conjugative;*

(2) *there is a relation $\delta \subseteq X^{(<\omega)} \times X^{(<\omega)}$ such that*

$$(\rho^c)^{(<\omega)} \circ \delta \circ (\rho_r^{-1})^{(<\omega)} = \rho_r^{(<\omega)};$$

(3) *for all $(x,G) \in \rho_r^{(<\omega)}$ there is $(U,V) \in X^{(<\omega)} \times X^{(<\omega)}$ such that*

(i) *$G \subseteq U\rho^c \wedge V \subseteq \rho x$, and*

(ii) *for any $(s,T) \in X \times X^{(<\omega)}$,*

if $V \subseteq \rho s$ and $T \subseteq U\rho^c$, then $(s,T) \in \rho_r^{(<\omega)}$.

Proof. (1) $\Longrightarrow$ (2). Define a relation $\delta \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by $(G, F) \in \delta$ if and only if

$$(\forall (s,T) \in X \times X^{(<\omega)})((G \subseteq \rho s \wedge T \cap F\rho^c \neq \emptyset) \Longrightarrow T \cap s\rho \neq \emptyset).$$

For any $(h, W) \in X \times X^{(<\omega)}$, if $(h, W) \in (\rho^{-1})^{(<\omega)} \circ \delta \circ (\rho_r^{-1})^{(<\omega)}$, then there exists $(G, F) \in X^{(<\omega)} \times X^{(<\omega)}$ such that

$$(h, G) \in (\rho_r^{-1})^{(<\omega)} \wedge (G, F) \in \delta \wedge (F, W) \in (\rho^c)^{(<\omega)},$$

i.e., $G \subseteq h\rho^{-1} = \rho h$ and $W \subseteq F\rho^c$. Now we have to show that $W \subseteq h\rho$. For any $w \in W$, let $s = h$ and $T = \{w\}$. Then we have $G \subseteq \rho h = \rho s$ and $\emptyset \neq T \cap F\rho^c = \{w\} \cap F\rho^c = \{w\}$. From the definition of δ and $(G, F) \in \delta$, it follows $\emptyset \neq T \cap s\rho = \{w\} \cap h\rho$ what means that $w \in h\rho$. Therefore, $(h, W) \in \rho_r^{(<\omega)}$. We have proven inclusion $(\rho^c)^{(<\omega)} \circ \delta \circ (\rho_r^{-1})^{(<\omega)} \subseteq \rho_r^{(<\omega)}$.

For any $(h, W) \in X \times X^{(<\omega)}$, if $(h, W) \in \rho_r^{(<\omega)}$, then $W \subseteq h\rho$. This means for any $w \in W$ holds $(h, w) \in \rho$. Since ρ is weak finitely dual quasi-conjugative, there are $u_w \in X$ and $V_w \in X^{(<\omega)}$ such that

(i) $(u_w, w) \in \rho^c \wedge V_w \subseteq h\rho^{-1} = \rho h$, and

(ii) for all $s \in X$ and $t \in X$,

if $(u_w, t) \in \rho^c$ and $V_w \subseteq s\rho^{-1} = \rho s$ then $(s, t) \in \rho$.

Let $G = \bigcup_{w \in W} V_w$, $F = \{u_w : w \in W\}$. Then $G \subseteq h\rho^{-1}$ and $W \subseteq F\rho^c$ since $w \in u_w\rho^c \subseteq F\rho^c$. Let $s \in X$ and $T \in X^{(<\omega)}$ be arbitrary elements such that $G \subseteq s\rho^{-1}$ and $T \cap F\rho^c \neq \emptyset$. Then there exist $u_{w_0} \in F$ and $t_0 \in T$ such that $(u_{w_0}, t_0) \in \rho^c$. On the other hand, according to (ii), from $w \in W$ and $V_w \subseteq s\rho^{-1}$, follows $(s, t_0) \in \rho$. This means $T \cap s\rho \neq \emptyset$. By the definition of δ, we have $(G, F) \in \delta$. Hence $(h, W) \in (\rho^{-1})^{(<\omega)} \circ \delta \circ (\rho_r^{-1})^{(<\omega)}$. So, we have proven inclusion $\rho_r^{(<\omega)} \subseteq (\rho^{-1})^{(<\omega)} \circ \delta \circ (\rho_r^{-1})^{(<\omega)}$.

(2) $\Longrightarrow$ (3). This implication can be verified without difficulty.

(3) $\Longrightarrow$ (1). For any $(x, y) \in \rho$, there exist $V, U \in X^{(<\omega)}$ such that

(i) $y \in U\rho^c$ and $V \subseteq \rho x$; and

(ii) $(\forall (s,T) \in X \times X^{(<\omega)})((V \subseteq \rho s \wedge T \subseteq U\rho^c) \Longrightarrow (s,T) \in \rho_r^{(<\omega)})$.

Since $y \in U\rho^c$, there exists $u \in U$ such that $(u, y) \in \rho^c$. Let $V = \{v_1, v_2, ..., v_m\}$. Then $(u, y) \in \rho^c$ and $(\forall j \in \overline{m})((x, v_j) \in \rho^{-1})$, i.e., the condition (i) in Definition 2.2 is satisfied. Now we check the condition (ii) in Definition 2.2. For any $(s, t) \in X \times X$, if $(u, t) \in \rho^c$ and $(s, v_j) \in \rho^{-1}$ $(j = 1, 2, ..., m)$, i.e., $V \subseteq \rho s$ and $\{t\} \subseteq u\rho^c \subseteq U\rho^c$, then $t \in s\rho$ by the condition (ii). Thus ρ is a weak finitely dual quasi-conjugative relation. □

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