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A Poncelet Criterion for Special Pairs of Conics in $PG(2, p^m)$

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ABSTRACT

We study Poncelet's Theorem in finite projective planes over the field $GF(q)$, $q = p^m$ for p an odd prime and $m \geq 1$, for a particular pencil of conics. We investigate whether we can find polygons with n sides which are inscribed in one conic and circumscribed around the other, so-called Poncelet polygons. By using suitable elements of the dihedral group for these pairs, we prove that the length n of such Poncelet polygons is independent of the starting point. In this sense Poncelet's Theorem is valid. By using Euler's divisor sum formula for the totient function we determine the number of conic pairs which carry Poncelet polygons of length n. Moreover, we introduce polynomials whose zeros in $GF(q)$ yield information about the relation of a given pair of conics: In particular, we can decide for a given integer n, whether and how we can find Poncelet n-gons for pairs of conics in the projective plane $PG(2, q)$.

Keywords: *Poncelet's Theorem; finite projective planes; pencil of conics; quadratic residues.*
AMS Subject Classification (2020): *Primary: 51E15; Secondary: 05B25; 51A35; 51E20; 51M04.*

1. Introduction

In 1813 Jean-Victor Poncelet stated one of the most beautiful results in projective geometry, known as Poncelet's Theorem [14]. He proved that for two conics C and D in the real projective plane, the condition whether a polygon with n sides, which is inscribed in D and circumscribed around C is independent of the starting point of the polygon. Moreover, if for a pair of conics such polygons exist, they all share the same number of sides. A remarkable number of different proofs can be found in the literature, ranging from rather elementary proofs for special cases to proofs using measure theory or elliptic curves [6]. We refer the reader to the recent book [5] and [3] for an overview on Poncelet's Theorem. In addition to proving the statement itself, much work has been done to find criteria for the existence of such polygons for two given conics, the most advanced result given by Arthur Cayley in 1853 (see [4] and Section 5.2). In the context of finite geometries conics are replaced by ovals. In this case the situation becomes more delicate. E.g., it is known that only in one of the four finite projective planes of order 9 Poncelet's Theorem holds true (see [10]).

The aim of this paper is to look at Poncelet's Theorem for a specific pencil of conics in finite projective planes $PG(2, q)$, q a power of an odd prime. In particular, we look at pairs of conics O_α and O_β which lie in a nested position, i.e. they have the property that either all points of O_α are external points of O_β or all points of O_α are inner points of O_β, and vice versa. For such pairs, we show the following finite version of Poncelet's Theorem in $PG(2, q)$.

Theorem (cf. Theorem 3.6)**.** *Let (O_α, O_β) be a pair of conics in $PG(2, q)$ given by*

$$O_k : x^2 + ky^2 + ckz^2 = 0,\ k \in \{\alpha, \beta\},$$

for $\alpha, \beta \in GF(q)\backslash\{0\}$ and $-c$ a nonsquare in $GF(q)$. If an n-sided Poncelet polygon, i.e. a polygon with n sides such that the vertices are on O_β and the sides are tangents of O_α, can be constructed starting with a point $P \in O_\beta$, then an

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n-sided Poncelet polygon inscribed in O_β and circumscribed around O_α can be constructed starting with any other point $Q \in O_\beta$.

We also describe a criterion for the existence of Poncelet polygons in such planes, which turns out to be a number theoretic condition.

Theorem (cf. Theorem 3.8)**.** *If one point $P \in O_\beta$ is an external point of O_α, then all points of O_β are external points of O_α and a Poncelet polygon can be constructed. In particular, this is the case if and only if $\beta(\beta - \alpha)$ is a nonsquare in $GF(q)$.*

If a Poncelet polygon exists we are interested in the number of its sides. For example, if O_α and O_β carry a Poncelet triangle, we necessarily have $4\beta = \alpha$ (cf. Lemma 4.1). We are able to derive an algorithm to determine for each pair (O_α, O_β) in $PG(2, q)$ whether it carries an n-sided Poncelet polygon (the precise definitions of Poncelet polynomials and Poncelet coefficients can be found in Section 4):

Corollary (cf. Corollary 4.17)**.** *The following four steps give a complete description of n-sided Poncelet polygons for conic pairs (O_α, O_β) in $PG(2, q)$.*

1. *Determine all $n \geq 3$ with $n|(q + 1)$. For every such n, calculate $\frac{\phi(n)}{2}$, which gives the number of indices k, such that an n-sided Poncelet polygon can be constructed for (O_k, O_1).*
2. *For all values n obtained in Step 1, look up the Poncelet polynomial P_n.*
3. *For every Poncelet polynomial P_n from Step 2, solve $P_n(k) = 0$ in $GF(q)$. This gives the corresponding Poncelet coefficients k, such that an n-sided Poncelet polygon can be constructed for (O_k, O_1).*
4. *By using a coordinate transformation, the information obtained in Step 3 can be transferred to all pairs (O_α, O_β).*

In the final section, we take a brief look at the Euclidean plane and investigate some parallels to the formulas derived for finite planes, as for example the half-angle formula which can henceforth be interpreted in finite planes as well.

2. Preliminaries

In order to fix notations and to make the text self-contained, we briefly recollect the most important definitions and facts about finite projective planes (see, e.g., [9]) and finite fields (see, e.g., [8]).

Let $GF(q)$ denote the finite field with $q = p^m$ elements, where p is an odd prime and $m \geq 1$. Any finite field is cyclic, i.e. it can be written as

$$GF(q) = \{0, 1, a, a^2, \dots, a^{q-2}\}$$

for any primitive element a of $GF(q)$. An element $a^s \in GF(q)$ is a square in $GF(q)$ if and only if the exponent s is even.

The main number theoretic tool used in this paper is the quadratic residue theorem, in particular we need the following result.

Lemma 2.1. *If $q \equiv 1(4)$ then $-1 = a^{\frac{q-1}{2}}$ is a square in $GF(q)$ and hence*

$$-k \text{ is a square in } GF(q) \Leftrightarrow k \text{ is a square in } GF(q).$$

If $q \equiv 3(4)$ then $-1 = a^{\frac{q-1}{2}}$ is a nonsquare in $GF(q)$ and hence

$$-k \text{ is a square in } GF(q) \Leftrightarrow k \text{ is a nonsquare in } GF(q).$$

In this paper, we only deal with finite projective planes constructed over $GF(q)$. Those planes are denoted by $PG(2, q)$ and are also known as Desarguesian planes.

The set of points $\mathbb{P}$ of $PG(2, q)$ is defined by

$$\mathbb{P} = (GF(q) \setminus \{0\})^3 / \sim$$

where $\sim$ is the equivalence relation given by

$$x \sim y \iff x = \lambda y \text{ for } \lambda \in GF(q) \setminus \{0\}.$$

The set of lines of $PG(2,q)$ is formally the same set $\mathbb{B} = \mathbb{P}$. For $x = (x_1, x_2, x_3)^T \in (GF(q)\backslash\{0\})^3$ we will use the capital letter $X = [x] \in \mathbb{P}$ for the equivalence class, and similarly for lines $L = [l] \in \mathbb{B}$. Then the point $X \in \mathbb{P}$ is incident with the line $L \in \mathbb{B}$, if $l_1x_1 + l_2x_2 + l_3x_3 = 0$ in $GF(q)$.

All points, lines, pairs of lines and conics in $PG(2,q)$ are given by

$$\left\{[x] \in \mathbb{P} \middle| x^T A x = 0\right\}, \tag{1}$$

where $A \neq 0$ is a 3×3 matrix with coefficients in $GF(q)$. The set (1) corresponds to a conic if and only if the matrix A is regular. If O is a given conic and $L \in \mathbb{B}$ a line, we call L a *tangent* if it intersects O in one point, a *secant* if it intersects O in two points and an *external line* if it misses O. A point $X \in \mathbb{P}$ is called *inner point* of O if there is no tangent to O through X and *exterior point* if there are two tangents from X to O. The following Lemma describes a necessary tool used often in this paper.

Lemma 2.2. *Let O be a conic in $PG(2,q)$, A a matrix representing O, and $X = [x] \in \mathbb{P}$ a point in $PG(2,q)$. Then X is on O if and only if Ax is a tangent of O, X is an exterior point of O if and only if Ax is a secant of O, and X is an inner point of O if and only if Ax is an external line of O.*

3. A special pencil of conics in $PG(2,q)$

3.1. Construction and properties

In all of the following, we only consider conics of the form

$$O_k : x_1^2 + kx_2^2 + ckx_3^2 = 0, \tag{2}$$

for $k \in GF(q)\backslash\{0\}$ and $-c$ a nonsquare in $GF(q)$. Hence, in particular, $x_1 \neq 0$ for all points on O_k. The following results will explain our choice of conics. Compare also to Remark 3.7.

To understand the properties of a pair of such conics O_k above, we first have a closer look at a specific partition of the plane $PG(2,q)$. The idea is to start with the point $P = [(1,0,0)^T]$ and the line g through the points $[(0,1,0)^T]$ and $[(0,0,1)^T]$. We look at the *pencil* generated by P and g, i.e., the objects obtained by considering all nontrivial $GF(q)$-linear combinations of the equations corresponding to P and g. Clearly, such a pencil consists of $q+1$ objects, namely P and g as well as $q-1$ conics. These $q+1$ objects partition the plane $PG(2,q)$, which can be seen by the following results.

Lemma 3.1. *An equation of the point $P = [(1,0,0)^T]$ in the plane $PG(2,q)$ is given by*

$$P : x_2^2 + cx_3^2 = 0, \tag{3}$$

for $-c$ a nonsquare in $GF(q)$.

Proof. The components of $P = [(1,0,0)^T]$ clearly solve equation (3). As the associated matrix is singular, it describes a point, a line or a pair of lines. It is a point, if the polynomial $x_2^2 + cx_3^2$ is irreducible over $GF(q)$, which is the case if and only if $-c$ is a nonsquare in $GF(q)$. □

The mentioned partition is now as follows:

Lemma 3.2. *Let P be a point and g a line in $PG(2,q)$, such that $P \notin g$. Then the pencil generated by P and g forms a partition of the plane $PG(2,q)$. Moreover, P is the unique point in $PG(2,q)$ which is an inner point of all $q-1$ conics in the pencil.*

Proof. Since there exists a collineation of $PG(2,q)$ which maps three arbitrary noncollinear points to three given noncollinear points, we can restrict the proof without loss of generality to $P = [(1,0,0)^T]$ and g the line through $[(0,1,0)^T]$ and $[(0,0,1)^T]$. By Lemma 3.1, the corresponding equations are given by

$$g : x_1^2 = 0 \text{ and } P : x_2^2 + cx_3^2 = 0,$$

for $-c$ a nonsquare. Considering all nontrivial $GF(q)$-linear combinations of P and g leads to $q-1$ conics O_k, determined by $x_1^2 + kx_2^2 + kcx_3^2 = 0$ for $k \in GF(q)\backslash\{0\}$. The solutions of these equations are disjoint, since a common solution of any two equations would imply a common solution of P and g as well, which contradicts our assumption. Since there are $q+1$ points on every conic O_k and on g, the solutions of the $q+1$ equations form a partition of $PG(2,q)$. The second statement is a straight forward calculation. □

The next result is also well-known (compare to [9, Theorem 8.3.3]) and can easily be shown by using Lemma 2.1 and 2.2.

Lemma 3.3. *Let P be a point and g a line in $PG(2, q)$ with $P \notin g$. Let O_k, $k \in GF(q)\backslash\{0\}$, be the $q - 1$ pairwise disjoint conics in the pencil generated by P and g. Each line through P is a secant of O_i, if i is a square in $GF(q)$ and an external line of O_j, if j is a nonsquare in $GF(q)$, or vice versa.*

As an easy consequence for the $q - 1$ conics in the pencil generated by P and g we mention the following.

Corollary 3.4. *No two of the conics O_k, $k \in GF(q)\backslash\{0\}$, have tangents in common.*

Remark 3.5. The parameter $-c$ can be chosen among all nonsquares up to projective equivalence of the resulting pencil: Indeed, let $-c_1$ and $-c_2$ be nonsquares in $GF(q)$. Then c_1c_2 is a square in $GF(q)$ and the collineation given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{c_2{c_1}^{-1}} \end{pmatrix}$$

maps all conics O_k given by $x_1^2 + kx_2^2 + kc_1x_3^2 = 0$ to conics given by $x_1^2 + kx_2^2 + kc_2x_3^2 = 0$.

3.2. *Poncelet's Theorem for conics O_k*

The main goal in this section is to prove a finite version of Poncelet's Theorem, interpreted in $PG(2, q)$. Note that Poncelet's Theorem was proven by Marcel Berger in [2] for an arbitrary pair of conics in any projective plane constructed over a field of characteristic not equal to 2 with at least five elements. His proof uses a considerable part of the theory of projective geometry, e.g. the Desargues involution. As we restrict our attention to a special case of conic pairs, a much shorter proof is possible. Recall that we are only interested in pairs of conics of the form (2) described in the previous section.

Definition 3.1. Consider a pair of conics (O_α, O_β) given by (2). An *n-sided Poncelet polygon* is a polygon with n sides such that the vertices are on O_β and the sides are tangents of O_α.

Since the conics O_k are all disjoint and have no common tangents as mentioned in Corollary 3.4, we are in the particular situation that if we can find one line which is a tangent to O_α and a secant of O_β, then this leads necessarily to a Poncelet polygon. The finite version of Poncelet's Theorem we are going to prove here reads as follows.

Theorem 3.6. *Let (O_α, O_β) be a pair of conics in $PG(2, q)$ given by*

$$O_k : x^2 + ky^2 + ckz^2,\ k \in \{\alpha, \beta\}\,,$$

$k \in GF(q)\backslash\{0\}$ and $-c$ a nonsquare. If an n-sided Poncelet polygon can be constructed starting with a point $P \in O_\beta$, then an n-sided Poncelet polygon inscribed in O_β and circumscribed around O_α can be constructed starting with any other point $Q \in O_\beta$.

Proof. Consider the group G of all matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -cb \\ 0 & b & a \end{pmatrix}, \quad a, b \in GF(q),\ a^2 + cb^2 \neq 0.$$

The linear mapping associated to such a matrix maps the conic O_k to $O_{k'}$ with

$$k' = k(a^2 + cb^2). \tag{4}$$

This group acts transitively on the set of conics $\{O_k \mid k \in GF(q) \setminus \{0\}\}$. Indeed, let $P = [(1, y, z)^T]$ be a point on O_k and $Q = [(1, s, t)^T]$ be a point on $O_{k'}$. The group element which maps P to Q and hence O_k to $O_{k'}$ has the parameters

$$a = -k(sy + ctz), \quad b = -k(ty - sz).$$

This shows at the same time, that the stabilizer of a conic O_k acts regularly on its points.

Now, let $P_1, \ldots, P_n$ be points on O_β which form an n-sided Poncelet polygon with the conic O_α. If Q is an arbitrary point on O_β, the group element that maps P_1 to Q maps the given Poncelet polygon to another Poncelet polygon with points $Q = Q_1, Q_2, \ldots, Q_n$ on O_β and sides which are tangents of O_α. □

3.3. *Relations for pairs of conics*

In this section, we consider the mutual position of pairs of conics with regard to the existence of a Poncelet polygon.

Definition 3.2. Let O and O' be two conics in $PG(2,q)$. We say that O *lies inside* O' if O' consists of external points of O only. Notation: $O < O'$. Moreover, we say that O *lies outside* O' if O consists of external points of O' only. Notation: $O > O'$.

Note that $O < O'$ does not imply $O' > O$. In particular, in a finite projective plane we can have the unintuitive situation that $O < O'$ and $O' < O$ at the same time.

Remark 3.7. At a first glance, this choice of conic pairs O_α and O_β seems to be rather restrictive. In [12, Theorem 4] the structure of all possible pencils of conics is discussed. For two disjoint conics, there are only three different pencils up to collineations. It turns out that only the class of pencils studied in the present paper has the property that $O < O'$ or $O > O'$ for all pairs of conics. A similar result was also obtained by Abatangelo et al. in [1, Theorem 6.1] for $q \geq 17$.

Theorem 3.8. *If one point $P \in O_\beta$ is an external point of O_α, then $O_\alpha < O_\beta$. Moreover, we have $O_\alpha < O_\beta$ if and only if $\beta(\beta-\alpha)$ is a nonsquare in $GF(q)$.*

Proof. Recall that all points of $PG(2,q)$ with a zero x-coordinate lie on the line $g : x^2 = 0$ and hence, due to the partition, not on any conic O_k, $k \in GF(q)\backslash\{0\}$. A point P of O_β can therefore be considered as $P = [(1, p_2, p_3)^T]$. Using the conic equation, we have $p_2^2 = -\beta^{-1} - cp_3^2$. By Lemma 2.2, the conic O_α lies inside O_β if for all such points P, the line $O_\alpha P$ is a secant of O_α. This is the case if there exist two points $Q_1 = [(x_1, y_1, z_1)^T]$ and $Q_2 = [(x_2, y_2, z_2)^T]$ on O_α satisfying

$$\alpha^{-1}x_i + p_2y_i + cp_3z_i = 0 \tag{5}$$

for $i = 1, 2$. We can rewrite those points as $Q_i = [(1, \pm\sqrt{-\alpha^{-1} - cz_i^2}, z_i)^T]$ such that (5) becomes a quadratic equation in z:

$$z^2 - 2\alpha^{-1}\beta p_3 z + \alpha^{-1}c^{-1} + \alpha^{-1}\beta p_3^2 - \alpha^{-2}\beta c^{-1} = 0.$$

This equation has two solutions, if and only if its discriminant

$$\alpha^{-2}\beta^2 p_3^2 - \alpha^{-1}c^{-1} - \alpha^{-1}\beta p_3^2 + \alpha^{-2}\beta c^{-1}$$

is a nonzero square in $GF(q)$. Multiplying by α^2 and factorizing leads to the condition of

$$(\beta p_3^2 + c^{-1})(\beta - \alpha)$$

being a nonzero square in $GF(q)$. Using $P \in O_\beta$ gives the equivalent expression

$$(-\beta p_2^2 c^{-1})(\beta - \alpha).$$

Since $-c$ is a nonsquare by assumption, we need

$$\beta(\beta - \alpha)$$

to be a nonsquare in $GF(q)$. Since this expression is independent of the point P, we are done. □

As a direct consequence of Theorem 3.8, we can easily construct chains of nested conics.

Corollary 3.9. *Consider two conics O_α and O_β. Then*

$$O_\alpha < O_\beta \;\Leftrightarrow\; O_\beta < O_{\beta^2\alpha^{-1}}.$$

When calculating the relation $<$ for every pair (O_α, O_β) in a given plane, it is useful to apply the following result, which follows directly by the proof of Theorem 3.6:

Lemma 3.10. *Let (O_k, O_1) be a pair of conics in $PG(2,q)$. Then for each $\beta \in GF(q)\backslash\{0\}$ there exists a collinear transformation mapping (O_k, O_1) to $(O_{\beta k}, O_\beta)$. In particular, $O_k < O_1$ implies $O_{\beta k} < O_\beta$.*

	O_1	O_2	O_3	O_4	O_5	O_6
O_1			$<$	$<$	$<$	
O_2	$<$		$<$			$<$
O_3	$<$	$<$			$<$	
O_4		$<$			$<$	$<$
O_5	$<$			$<$		$<$
O_6		$<$	$<$	$<$		

Table 1. Mutual positions of conics in $PG(2,7)$.

Example 3.11. We want to investigate the relation $<$ in $PG(2,7)$. By looking at $\beta = 1$ and shifting the result by using Lemma 3.10, we obtain Table 1, showing all relations for the whole plane $PG(2,7)$. Using Corollary 3.9, we detect the following closed chains of conics $O_\alpha \to O_\beta \to O_{\beta^2\alpha^{-1}} \to \ldots$, namely:

$$O_1 \to O_3 \to O_2 \to O_6 \to O_4 \to O_5 \to O_1$$
$$O_1 \to O_4 \to O_2 \to O_1$$
$$O_3 \to O_5 \to O_6 \to O_3$$

Note that starting with two squares α and β results in a chain of conics with just squares as indices. Similarly, starting with two nonsquares as indices results in a chain of conics with only nonsquares as indices.

Since exactly half of all nonzero elements in $GF(q)$ are squares, the following is immediate.

Corollary 3.12. *For every conic O_β in $PG(2,q)$, there are $\frac{q-1}{2}$ conics O_α such that $O_\alpha < O_\beta$.*

Next, we have a closer look at the relations of the points on O_α and O_β.

Lemma 3.13. *Let $P = [(1, p_2, p_3)^T]$ be a point on O_β and $O_\alpha < O_\beta$. Then for the contact points $A_1 = [(1, y_1, z_1)^T]$ and $A_2 = [(1, y_2, z_2)^T]$ on O_α of the tangents through P we have*

$$z_{1,2} = \alpha^{-1}\beta p_3 \pm p_2\sqrt{\alpha^{-2}(-c^{-1}\beta)(\beta - \alpha)}$$

and

$$y_{1,2} = \begin{cases} p_2^{-1}(-\alpha^{-1} - cp_3z_{1,2}), & \text{if } p_2 \neq 0 \\ \pm\sqrt{-\alpha^{-1} - cz_{1,2}^2}, & \text{if } p_2 = 0. \end{cases}$$

Proof. To see this, we just have to solve the quadratic equation derived in Theorem 3.8. Since $O_\alpha < O_\beta$, we indeed get two solutions. □

Lemma 3.14. *Let $P = [(1, p_2, p_3)^T]$ and $Q = [(1, q_2, q_3)^T]$ be two points on O_β such that the line connecting P and Q is a tangent of O_α at the point $A = [(2, a_2, a_3)^T]$. Then*

$$(p_2, p_3) + (q_2, q_3) = (a_2, a_3).$$

Geometrically, this means that A is the midpoint of P and Q in the affine plane obtained by removing the line g through the points $[(0,1,0)^T]$ and $[(0,0,1)^T]$.

Proof. We have

$$1 + \beta p_2^2 + c\beta p_3^2 = 0 \text{ and } 1 + \beta q_2^2 + c\beta q_3^2 = 0. \tag{6}$$

The contact point $A \in Q_\alpha$ of the secant through P and Q is the only point

$$(1 + k, p_2 + kq_2, p_3 + kq_3), \quad k \in GF(q) \setminus \{0, -1\}, \tag{7}$$

which satisfies the equation for Q_α, i.e., by (6)

$$k^2 + \frac{2 + 2\alpha(p_2q_2 + cp_3q_3)}{1 - \alpha\beta^{-1}}k + 1 = 0.$$

Note that $1-\alpha\beta^{-1}\neq 0$, since otherwise $\alpha=\beta$. Solving for k yields

$$k=-\frac{1+\alpha(p_2q_2+cp_3q_3)}{1-\alpha\beta^{-1}}\pm\sqrt{\left(\frac{1+\alpha(p_2q_2+cp_3q_3)}{1-\alpha\beta^{-1}}\right)^2-1}.$$

So we have only one solution if the radicand is zero, i.e., if

$$\left(\frac{1+\alpha(p_2q_2+cp_3q_3)}{1-\alpha\beta^{-1}}\right)^2=1.$$

Hence $k=\pm 1$. Since we can exclud $k=-1$ by (7) we get $k=1$ which proves the claim. □

Corollary 3.15. *Let $P,Q\in O_\beta$ such that $[(1,0,0)^T]\notin\overline{PQ}$. Then there exists precisely one $\alpha\in GF(q)\backslash\{0\}$, $\alpha\neq\beta$, such that $\overline{PQ}$ is a tangent of O_α.*

Proof. For $P=[(1,p_2,p_3)^T]$ and $Q=[(1,q_2,q_3)^T]$ the contact point with O_α, if there is one, is $A=[(2,p_2+q_2,p_3+q_3)^T]$. As the characteristic of $GF(q)$ is odd, A is not on the line g through $[(0,1,0)^T]$ and $[(0,0,1)^T]$. Since we have a partition of the plane $PG(2,q)$, A must be a point on a conic O_α. We have to exclude the possibility of $\overline{PQ}$ being a secant of O_α. For this, note that there are $q+1$ points on $\overline{PQ}$, among them $P,Q\in O_\beta$ and a point on g. All the other $q-2$ points must lie on conics and there are at most two points on the same conic. Since $q-2$ is odd and by Lemma 3.4, there is exactly one conic with $\overline{PQ}$ as a tangent. By Lemma 3.14, we are done. □

In the following results, an n-sided Poncelet polygon for $O_\alpha<O_\beta$ with vertices B_i on O_β and contact points A_i on O_α is denoted by

$$B_1\xrightarrow{A_1}B_2\xrightarrow{A_2}B_3\xrightarrow{A_3}\ldots\xrightarrow{A_{n-1}}B_n\xrightarrow{A_n}B_1,$$

where $B_i\xrightarrow{A_i}B_{i+1}$ means that the line connecting B_i and B_{i+1} is the tangent of O_α in the point A_i. Note that by combining Lemma 3.13 and Lemma 3.14, we are able to calculate a Poncelet polygon by starting at a point on O_β. Before we analyze Poncelet polygons for different numbers of sides, we need some more properties of the points on O_k and their relations.

Lemma 3.16. *The conics O_α in $PG(2,q)$, $q\equiv 3(4)$, consist of the $q+1$ points*

$$\left\{\begin{bmatrix}1\\y_1\\z_1\end{bmatrix},\begin{bmatrix}1\\-y_1\\z_1\end{bmatrix},\begin{bmatrix}1\\y_1\\-z_1\end{bmatrix},\begin{bmatrix}1\\-y_1\\-z_1\end{bmatrix},\ldots,\begin{bmatrix}1\\y_{\frac{q+1}{4}}\\z_{\frac{q+1}{4}}\end{bmatrix},\begin{bmatrix}1\\-y_{\frac{q+1}{4}}\\z_{\frac{q+1}{4}}\end{bmatrix},\begin{bmatrix}1\\y_{\frac{q+1}{4}}\\-z_{\frac{q+1}{4}}\end{bmatrix},\begin{bmatrix}1\\-y_{\frac{q+1}{4}}\\-z_{\frac{q+1}{4}}\end{bmatrix}\right\}$$

if α is a square, and otherwise

$$\left\{\begin{bmatrix}1\\y_1\\z_1\end{bmatrix},\begin{bmatrix}1\\-y_1\\z_1\end{bmatrix},\begin{bmatrix}1\\y_1\\-z_1\end{bmatrix},\begin{bmatrix}1\\-y_1\\-z_1\end{bmatrix},\ldots,\begin{bmatrix}1\\y_{\frac{q-3}{4}}\\-z_{\frac{q-3}{4}}\end{bmatrix},\begin{bmatrix}1\\-y_{\frac{q-3}{4}}\\-z_{\frac{q-3}{4}}\end{bmatrix},\begin{bmatrix}1\\y\\0\end{bmatrix},\begin{bmatrix}1\\-y\\0\end{bmatrix},\begin{bmatrix}1\\0\\z\end{bmatrix},\begin{bmatrix}1\\0\\-z\end{bmatrix}\right\}.$$

Proof. For $y\neq 0$ and $z\neq 0$, we have that $[(1,y,z)^T]\in O_\alpha$ implies that $[(1,-y,z)^T]\in O_\alpha$, $[(1,y,-z)^T]\in O_\alpha$ and $[(1,-y,-z)^T]\in O_\alpha$. So we just have to check whether or not $[(1,0,z)^T]$ and $[(1,y,0)^T]$ are on O_α. We have

$$[(1,0,z)^T]\in O_\alpha\ \Leftrightarrow\ z^2=-\alpha^{-1}c^{-1}$$

and

$$[(1,y,0)^T]\in O_\alpha\ \Leftrightarrow\ y^2=-\alpha^{-1}.$$

As $q\equiv 3(4)$, c is a square in $GF(q)$ and -1 is not. Hence these points lie on O_α if and only if α is not a square. □

Lemma 3.17. *The conics O_α in $PG(2,q)$, $q\equiv 1(4)$, consist of the $q+1$ points*

$$\left\{\begin{bmatrix}1\\y_1\\z_1\end{bmatrix},\begin{bmatrix}1\\-y_1\\z_1\end{bmatrix},\begin{bmatrix}1\\y_1\\-z_1\end{bmatrix},\begin{bmatrix}1\\-y_1\\-z_1\end{bmatrix},\ldots,\begin{bmatrix}1\\y_{\frac{q-1}{4}}\\z_{\frac{q-1}{4}}\end{bmatrix},\begin{bmatrix}1\\-y_{\frac{q-1}{4}}\\z_{\frac{q-1}{4}}\end{bmatrix},\begin{bmatrix}1\\y_{\frac{q-1}{4}}\\-z_{\frac{q-1}{4}}\end{bmatrix},\begin{bmatrix}1\\-y_{\frac{q-1}{4}}\\-z_{\frac{q-1}{4}}\end{bmatrix},\begin{bmatrix}1\\y\\0\end{bmatrix},\begin{bmatrix}1\\-y\\0\end{bmatrix}\right\}$$

if α is a square, and otherwise

$$\left\{\begin{bmatrix}1\\y_1\\z_1\end{bmatrix},\begin{bmatrix}1\\-y_1\\z_1\end{bmatrix},\begin{bmatrix}1\\y_1\\-z_1\end{bmatrix},\begin{bmatrix}1\\-y_1\\-z_1\end{bmatrix},\dots,\begin{bmatrix}1\\y_{\frac{q-1}{4}}\\z_{\frac{q-1}{4}}\end{bmatrix},\begin{bmatrix}1\\-y_{\frac{q-1}{4}}\\z_{\frac{q-1}{4}}\end{bmatrix},\begin{bmatrix}1\\y_{\frac{q-1}{4}}\\-z_{\frac{q-1}{4}}\end{bmatrix},\begin{bmatrix}1\\-y_{\frac{q-1}{4}}\\-z_{\frac{q-1}{4}}\end{bmatrix},\begin{bmatrix}1\\0\\z\end{bmatrix},\begin{bmatrix}1\\0\\-z\end{bmatrix}\right\}$$

Proof. The proof is similar to the proof of Lemma 3.16. □

As a direct consequence of the previous Lemmas we obtain:

Corollary 3.18. *If $P_i = [(1, y_i, z_i)^T]$, $i = 1, \dots, q+1$, are the points on the conic O_α, then $\sum_{i=1}^{q+1}(y_i, z_i) = (0, 0)$ in $GF(q)$.*

Lemma 3.19. *Let $k, n \in \mathbb{N}$, $n > 2$, and $B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} B_3 \xrightarrow{A_3} \dots \xrightarrow{A_{kn-1}} B_{kn} \xrightarrow{A_{kn}} B_1$ be a kn-sided Poncelet polygon, where $B_i = [(1, y_i, z_i)^T]$ and $A_i = [(2, s_i, t_i)^T]$. Then we have for all $j \in \{1, 2, \dots k\}$*

$$\sum_{i=0}^{n-1}(y_{ki+j}, z_{ki+j}) = \sum_{i=0}^{n-1}(s_{ki+j}, t_{ki+j}) = (0, 0). \tag{8}$$

Proof. Consider the matrix $\tau \in G$ from the proof of Theorem 3.6 which maps $(1, y_i, z_i)^T$ to $(1, y_{i+k}, z_{i+k})^T$ and $(1, s_i, t_i)^T$ to $(1, s_{i+k}, t_{i+k})^T$, where we take indices cyclically. Then we have

$$\begin{pmatrix}1\\y\\z\end{pmatrix} := \sum_{i=0}^{n-1}\begin{pmatrix}1\\y_{ki+j}\\z_{ki+j}\end{pmatrix} = \sum_{i=0}^{n-1}\tau\begin{pmatrix}1\\y_{ki+j}\\z_{ki+j}\end{pmatrix} = \tau\begin{pmatrix}1\\y\\z\end{pmatrix}.$$

Since τ is not the identity matrix and $-c$ is a nonsquare it follows that $(y, z) = (0, 0)$. □

Note that the case $n = 2$ in Lemma 3.19 shows that in a Poncelet polygon with an even number of sides, carried by O_β and O_α, the line joining opposite vertices passes through $[(1, 0, 0)^T]$. This can bee seen as a generalization of Brianchon's Theorem [7].

4. A Poncelet Criterion

4.1. Poncelet coefficients

Here is a first result concerning the existence of n-sided Poncelet polygons, namely Poncelet triangles. It has already been observed by Luisi in [13] that there are restrictions to the existence of Poncelet triangles in $PG(2, q)$.

Lemma 4.1. *Let $O_\alpha < O_\beta$ be two conics in $PG(2, q)$ which carry a Poncelet triangle. Then $4\beta = \alpha$ in $GF(q)$.*

Proof. Let

$$B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} B_3 \xrightarrow{A_3} B_1$$

be a Poncelet triangle, $B_i = [(1, y_i, z_i)^T]$ and $A_i = [(2, s_i, t_i)^T]$. By Lemma 3.14, we therefore have $(y_1, z_1) + (y_2, z_2) = (s_1, t_1)$, $(y_2, z_2) + (y_3, z_3) = (s_2, t_2)$ and $(y_3, z_3) + (y_1, z_1) = (s_3, t_3)$. Moreover, by Lemma 3.19, we have $(y_1, z_1) + (y_2, z_2) + (y_3, z_3) = (0, 0)$, which gives the relations

$$(y_1, z_1) + (s_2, t_2) = (0, 0), \quad (y_2, z_2) + (s_3, t_3) = (0, 0), \quad (y_3, z_3) + (s_1, t_1) = (0, 0). \tag{9}$$

It follows that the lines $\overline{B_1A_2}$, $\overline{B_2A_3}$ and $\overline{B_3A_1}$ meet in $[(1, 0, 0)^T]$. Since there are no tangents through $[(1, 0, 0)^T]$, as seen in Lemma 3.2, these lines are secants of O_α and O_β. With Lemma 3.3 we know that α and β are either both squares or both nonsquares. To find the remaining intersection points of $\overline{B_1A_2}$, $\overline{B_2A_3}$ and $\overline{B_3A_1}$ with O_α and O_β, consider the points $\tilde{A}_i$ and $\tilde{B}_i$, where $\tilde{P} := [(x, -y, -z)^T]$ for a point $P = [(x, y, z)^T]$. Since $[(1, 0, 0)^T] \in \overline{B_i\tilde{B}_i}$ and $[(1, 0, 0)^T] \in \overline{A_i\tilde{A}_i}$, for $i = 1, 2, 3$, these are exactly the intersection points we are looking for. Note that this construction yields another Poncelet triangle. In particular, the second triangle is

$$\tilde{B}_1 \xrightarrow{\tilde{A}_1} \tilde{B}_2 \xrightarrow{\tilde{A}_2} \tilde{B}_3 \xrightarrow{\tilde{A}_3} \tilde{B}_1$$

as visualized in Figure 1.

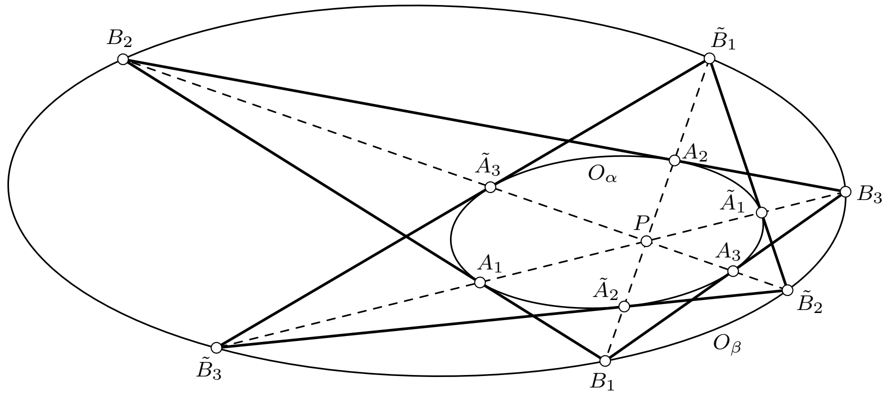

Figure 1. The triangle B_1, B_2, B_3 induces another triangle $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{B}_2$ via $P = [(1, 0, 0)^T]$.

The secant of O_β through B_1 and $\tilde{B}_1$ is given by

$$s_1 : z_1 y - y_1 z = 0.$$

In the case $z_1 \neq 0$ we get the relation $y = \frac{y_1}{z_1} z$. Intersecting this line with the conic O_α gives

$$z^2 = \frac{-z_1^2}{\alpha y_1^2 + c\alpha z_1^2}$$

and using $B_1 \in O_\beta$ gives

$$z^2 = \alpha^{-1}\beta z_1^2.$$

With this, we can calculate the intersection points for O_α, namely

$$A_2 = [(1, y_1\sqrt{\alpha^{-1}\beta}, z_1\sqrt{\alpha^{-1}\beta})^T] \text{ and } \tilde{A}_2 = [(1, -y_1\sqrt{\alpha^{-1}\beta}, -z_1\sqrt{\alpha^{-1}\beta})^T].$$

Using (9), we obtain the condition

$$(1 + 2\sqrt{\alpha^{-1}\beta})z_1 = 0.$$

Since we are in the case $z_1 \neq 0$ it follows $1 + 2\sqrt{\alpha^{-1}\beta} = 0$, which implies $\alpha = 4\beta$. In the case $z_1 = 0$, we directly deduce $z = 0$ for the secant through B_1 and $\tilde{B}_1$. Intersecting with O_α gives the two points

$$A_2 = [(1, \sqrt{-\alpha^{-1}}, 0)^T] \text{ and } \tilde{A}_2 = [(1, -\sqrt{-\alpha^{-1}}, 0)^T].$$

Applying (9) we get the condition $y_1 \pm 2\sqrt{-\alpha^{-1}} = 0$ and using $B_1 \in O_\beta$ yields again $4\beta = \alpha$. □

Remark 4.2. Recall that for $O_\alpha < O_\beta$ we have to check whether or not $\beta(\beta - \alpha)$ is a nonsquare. Hence, in the case $4\beta = \alpha$ we have to check whether or not $-3\beta^2$ is a nonsquare, which is the same as checking whether or not -3 is a nonsquare. For p an odd prime, we can compare this to well-known results from number theory (see [8]). For $p \equiv 1(4)$, we have that 3 is a nonsquare if and only if $3|(p+1)$ and for $p \equiv 3(4)$, we have that 3 is a square if and only if $3|(p+1)$. In both cases we therefore have

$$-3 \text{ nonsquare } \Leftrightarrow 3|(p+1).$$

This gives already a necessary condition for the existence of Poncelet triangles for pairs (O_α, O_β) in $PG(2, p)$, p an odd prime. By Poncelet's Theorem for such pairs, as seen in Theorem 3.6, the existence of a Poncelet triangle implies $3|(p+1)$, as there are $p+1$ points on the conic O_β. This is exactly the condition given by number theoretic results as well.

Using arguments as above, one easily checks the following result.

Lemma 4.3. *Let $O_\alpha < O_\beta$ be two conics in $PG(2, q)$, such that a 4-sided Poncelet polygon can be constructed. Then $2\beta = \alpha$ in $GF(q)$.*

The main goal is to find such a relation for all possible n-sided Poncelet polygons. For this, we first investigate which Poncelet n-gons occur in a given plane $PG(2,q)$. Note that this can be done just by applying Poncelet's Theorem and Euler's divisor sum formula, since we are dealing with a very special family of conics. First we need the following:

Lemma 4.4. *Let O_β be the conic given by $x+\beta y^2+c\beta z^2=0$ in $PG(2,q)$ where $-c$ is a nonsquare in $GF(q)$. Then for every $n|(q+1)$ there is a Poncelet n-gone with points $B_1,\dots,B_n$ on Q_β and sides which are tangents of a conic O_α.*

Proof. By Lemma 3.10 we may assume $\beta=1$ without loss of generality. The set of matrices

$$\begin{pmatrix} a & -cb \\ b & a \end{pmatrix}, \quad a,b\in GF(q),$$

equipped with matrix addition and multiplication in $GF(q)$ is a finite field. Observe, that the determinant is only zero for $a=b=0$ since $-c$ is a nonsquare. The multiplicative group of this field is cyclic. Since every subgroup of a cyclic group is also cyclic, we conclude that the group G of such matrices with determinant $a^2+cb^2=1$ is cyclic and has $q+1$ elements. Moreover, for every $n|(q+1)$ there is a cyclic subgroup H of G of order n. Let $\begin{pmatrix} a & -cb \\ b & a \end{pmatrix}$ be a generator of this subgroup H and

$$\tau=\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -cb \\ 0 & b & a \end{pmatrix}.$$

Let B_1 be an arbitrary point on O_1 and $B_2=\tau B_1$. According to Corollary 3.15, the line $\overline{B_1B_2}$ is tangent to some conic O_α. Iteration $B_{i+1}=\tau B_i$ yields a Poncelet n-gon $B_1,B_2,\dots,B_n$. □

Lemma 4.5. *For a given conic O_β in $PG(2,q)$ and every $n|(q+1)$ there are exactly $\frac{\phi(n)}{2}$ conics O_α, such that $O_\alpha<O_\beta$ carries a Poncelet n-gon.*

Proof. Consider the Poncelet polygon $B_1,B_2,\dots,B_n$ from Lemma 4.4, inscribed in O_β and circumscribed around some O_α. For each m, $1<m<\frac{n}{2}$, which is relatively prime to n we can construct another n-sided Poncelet polygon with the same points $B_1,\dots,B_n$, but a different inscribed conic O_{α_m}: For τ as above, the line $\overline{B_1\tau^mB_1}$ is tangent to some O_{α_m}, and so are the lines $\overline{\tau^{km}B_1\tau^{(k+1)m}B_1}$ for $k=1,2,\dots,n$. The lines $\overline{B_1\tau^mB_1}$ and $\overline{\tau^{-m}B_1}$ are pairs of tangents to O_{α_m}. Those pairs are different and therefore belong to different conics O_α. So there are at least $\frac{\phi(n)}{2}$ conics O_α which carry an n-sided Poncelet polygon.

By Lemma 3.12, there are exactly $\frac{q-1}{2}$ conics O_α, such that $O_\alpha<O_\beta$. Moreover, we know that once $O_\alpha<O_\beta$, starting with any point of O_β leads to a Poncelet polygon. Because of Theorem 3.6, the length of this Poncelet polygon has to divide $q+1$, i.e. the number of points on O_β. Recall now Euler's divisor sum formula for the totient function (see [8]), which states

$$\sum_{n|m}\phi(n)=m$$

for any integer m. Applied to the points of the conic, we have

$$\sum_{n|(q+1),\ n\geq 3}\phi(n)=q-1$$

which is the same as

$$\sum_{n|(q+1),\ n\geq 3}\frac{\phi(n)}{2}=\frac{q-1}{2}.$$

We conclude that there are exactly $\frac{\phi(n)}{2}$ conics O_α such that $O_\alpha<O_\beta$ carries a Poncelet n-gon for every divisor n of $q+1$. □

The next result reduces the problem of finding necessary relations for all n-sided Poncelet polygons, such as $4\beta=\alpha$ for $n=3$, to those with n odd.

Lemma 4.6. *Let $(O_{\beta k}, O_\beta)$ be a pair of conics in $PG(2,q)$ which carries an n-sided Poncelet polygon for k a square in $GF(q)$. Then $(O_{\beta\tilde{k}}, O_\beta)$ carries a $2n$-sided Poncelet polygon for*

$$\tilde{k} = \frac{2}{1 - \frac{1}{\sqrt{k}}}$$

where only those roots are taken such that $\tilde{k} \neq k$.

Proof. Let $O_\alpha < O_\beta$ be a pair of conics which carries a $2n$-sided Poncelet polygon with points $B_i = [(1, y_i, z_i)^T]$ on O_β and tangent points $A = [(2, s_i, t_i)^T]$ on O_α, as above. To calculate the relation between α and β we use that $(y_i, z_i) + (y_{i+1}, z_{i+1}) = (s_i, t_i)$ for two consecutive vertices of the polygon, as seen in Lemma 3.14. Hence, $[(2, y_1 + y_2, z_1 + z_2)^T] \in O_\alpha$ which gives immediately

$$\alpha = \frac{-4}{(y_1 + y_2)^2 + c(z_1 + z_2)^2}.$$

Since $B_1 \in O_\beta$ and $B_2 \in O_\beta$, we know that $y_i^2 + cz_i^2 = -\beta^{-1}$ for $i = 1, 2$ and we obtain

$$\alpha = \frac{2\beta}{1 - \beta(y_1 y_2 + c z_1 z_2)}.$$

The claim is $\alpha = \beta\tilde{k}$, hence we have to show the equality

$$\frac{2\beta}{1 - \beta(y_1 y_2 + c z_1 z_2)} = \frac{2\beta}{1 - \frac{1}{\sqrt{k}}}$$

which is equivalent to

$$\frac{1}{\sqrt{k}} + \beta(-y_1)y_2 + c\beta(-z_1)z_2 = 0.$$

The expression above can be interpreted as the incidence relation

$$(1, \beta y_2, c\beta z_2)\big(\frac{1}{\sqrt{k}}, -y_1, -z_1\big)^T = 0$$

which means that

$$[\big(\frac{1}{\sqrt{k}}, -y_1, -z_1\big)^T] \in T_\beta(B_2),$$

where $T_\beta(B_2)$ denotes the tangent of O_β in B_2. This can be done for all pairs of points $B_i, B_{i+1} \in O_\beta$. We get the conditions

$$[\big(\frac{1}{\sqrt{k}}, -y_{2\ell-1}, -z_{2\ell-1}\big)^T], [\big(\frac{1}{\sqrt{k}}, -y_{2\ell+1}, -z_{2\ell+1}\big)^T] \in T_\beta(B_{2\ell})$$

for $\ell = 1, \ldots, n$ where indices are taken cyclically. Exactly n tangents of the conic O_β are involved. The conditions above are equivalent to showing that the n intersection points are on some conic O_γ and form an n-sided Poncelet polygon with O_β. Observe that, by Lemma 3.19, $B_{i+n} = \tilde{B}_i$, and hence $[\big(\frac{1}{\sqrt{k}}, -y_{i+n}, -z_{i+n}\big)^T] = [\big(\frac{1}{\sqrt{k}}, y_i, z_i\big)^T]$. Therefore, we have to verify that

$$[\big(\frac{1}{\sqrt{k}}, \pm y_i, \pm z_i\big)^T] \in O_\gamma$$

for $i = 1, \ldots, n$ and $\beta = \gamma k$. For γ, we directly obtain

$$O_\gamma : \frac{x^2}{k} + \gamma y^2 + c\gamma z^2 = 0.$$

Since all the points $[1, y_i, z_i]^T$ lie on O_β, we indeed get $\beta = \gamma k$. By Lemma 3.10 , since $(O_{\beta k}, O_\beta)$ carries an n-sided Poncelet polygon, so does $(O_{\gamma k}, O_\gamma)$ which is what we wanted to show. □

Corollary 4.7. *Let O_α and O_β be conics in $PG(2,q)$ for which a $2n$-sided Poncelet polygon exists. Then there exists another conic O_γ such that the pair (O_γ, O_β) carries an n-sided Poncelet polygon.*

Proof. Let $O_\alpha < O_\beta$ such that a $2n$-sided Poncelet polygon can be constructed and $\alpha = h\beta$. By Lemma 3.8 we know that $\beta(\beta-\alpha)$ is a nonsquare, so $1-h$ is a nonsquare in our case. To show the statement above, we only have to show that for $\gamma = k\beta$, $1-h$ is a nonsquare if and only if $1-k$ is a nonsquare. This follows immediately by our formula for $2n$-sided Poncelet polygons seen in Lemma 4.6, namely

$$1-h = 1-\frac{2}{1-\frac{1}{\sqrt{k}}} = \frac{(\sqrt{k}+1)^2}{1-k}.$$

This gives us

$$(1-h)(1-k) = (\sqrt{k}+1)^2$$

and hence $(1-h)$ is a nonsquare if and only if $(1-k)$ is a nonsquare. □

Example 4.8. We have already seen in Lemma 4.3 that if (O_k, O_1) carries a 4-sided Poncelet polygon, we immediately have $k=2$. Hence by Lemma 4.6, we are able to compute the index h such that (O_h, O_1) carries an 8-sided Poncelet polygon, namely

$$h = \frac{2}{1-\frac{1}{\sqrt{k}}} = \frac{2}{1\pm\frac{1}{\sqrt{2}}} = 4 \pm 2\sqrt{2}.$$

This is only well defined if 2 is a square. For $GF(p)$, p an odd prime, we know from number theory (see [8]) that

$$2 \text{ is a square in } GF(p) \Leftrightarrow p \equiv \pm 1(8). \tag{10}$$

By Poncelet's Theorem, the existence of an 8-gon already implies $8|(p+1)$. Hence, the condition $p \equiv -1(8)$ is again equivalent to a purely number theoretic result.

The next goal is to deduce such relations for all n-sided Poncelet polygons, n odd. The main idea how to proceed lies already in the next result.

Lemma 4.9. *Let $O_k < O_1$ carry an n-sided Poncelet polygon for the points $B_1, \ldots, B_n \in O_1$, n odd. Then $O_{\frac{k^2}{(k-2)^2}} < O_1$ carries an n-sided Poncelet polygon as well, for the same points $B_1, \ldots, B_n \in O_1$.*

Proof. Let $O_k < O_1$ such that an n-sided Poncelet polygon can be constructed for n odd. By Lemma 3.14, we have that $(1, y_i, z_i) + (1, y_{i+1}, z_{i+1}) = (2, s_i, t_i)$ corresponds to a point on O_k for all $i = 1, \ldots, n$, where, as before, $B_i = [(1, y_i, z_i)^T]$. Hence we have

$$4 + k(y_i + y_{i+1})^2 + ck(z_i + z_{i+1})^2 = 0.$$

Using $1 + y_i^2 + cz_i^2 = 0$ for all $B_i \in O_1$ gives

$$k = \frac{2}{1-(y_i y_{i+1} + c z_i z_{i+1})},$$

which is equivalent to

$$\frac{k}{k-2} + \frac{k^2}{(k-2)^2} y_i(-y_{i+1}) + c\frac{k^2}{(k-2)^2} z_i(-z_{i+1}) = 0.$$

This can be read as the incidence relation

$$\Big(\frac{k}{k-2}, -\frac{k^2}{(k-2)^2} y_{i+1}, -c\frac{k^2}{(k-2)^2} z_{i+1}\Big)(1, y_i, z_i)^T = 0.$$

Hence we need

$$[(1, y_i, z_i)^T] \in T_{\frac{k^2}{(k-2)^2}}\left(\frac{k}{k-2}, -y_{i+1}, -z_{i+1}\right)$$

as well as

$$[(1, y_{i+1}, z_{i+1})^T] \in T_{\frac{k^2}{(k-2)^2}}\left(\frac{k}{k-2}, -y_i, -z_i\right).$$

In summary, this results in the condition

$$[(1, y_{i+1}, z_{i+1})^T], [(1, y_{i-1}, z_{i-1})^T] \in T_{\frac{k^2}{(k-2)^2}}\left(\frac{k}{k-2}, -y_i, -z_i\right).$$

This can be done for all $i = 1, \dots, n$ and since n is odd, for $O_{\frac{k^2}{(k-2)^2}} < O_1$, an n-sided Poncelet polygon is given via the same points $B_1, \dots, B_n$. □

Note that by Lemma 3.10 the conics $O_1 < O_{\beta^2}$ carry an n-sided Poncelet polygon if and only if $O_{\frac{1}{\beta^2}} < O_1$ carries an n-sided Poncelet polygon.

Remark 4.10. We have seen that for triangles there is only one conic O_k such that $O_k < O_1$ form a 3-sided Poncelet polygon, namely O_4. In this case, we should therefore have

$$k = \frac{k^2}{(k-2)^2},$$

which is equivalent to

$$k^2 - 5k + 4 = 0.$$

The only solutions are $k = 1$, which can be excluded, and $k = 4$, which we already computed in Lemma 4.1 by using other methods.

The procedure shown in the proof above can be iterated. To avoid long expressions, we use

$$t_{i+1} := \frac{t_i^2}{(t_i - 2)^2}, \tag{11}$$

for $t_0 := k$. Recall that for a given Poncelet n-gon using the points $B_1, \dots, B_n$ on O_1 and tangents of some O_α, there are $\frac{\phi(n)}{2} - 1$ other conics O_γ such that (O_γ, O_1) carries an n-sided Poncelet polygon.

Example 4.11. We know that for $O_\alpha < O_1$ a 5-sided Poncelet polygon for the same five points $B_1, \dots, B_5 \in O_\alpha$ can be constructed in two different ways, since $\frac{\phi(5)}{2} = 2$. Start with the polygon

$$B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} B_3 \xrightarrow{A_3} B_4 \xrightarrow{A_4} B_5 \xrightarrow{A_5} B_1.$$

The other 5-gon is then given by connecting B_i and B_{i+2}, namely

$$B_1 \xrightarrow{C_1} B_3 \xrightarrow{C_2} B_5 \xrightarrow{C_3} B_2 \xrightarrow{C_4} B_4 \xrightarrow{C_5} B_1.$$

Note that connecting B_i and B_{i+3} gives the same polygon, since we can read the above polygon by reversing the direction (see Figure 2).

For 5-sided Poncelet polygons, we therefore get the conditions $t_0 \neq t_1$ and $t_0 = t_2$. We have to solve

$$k = \frac{k^4}{(k^2 - 2(k-2)^2)^2},$$

which is equivalent to

$$(k-1)(k-4)(16 - 12k + k^2) = 0.$$

We obtain the four solutions

$$k \in \left\{1, 4, 6 + 2\sqrt{5}, 6 - 2\sqrt{5}\right\}.$$

Since $k = 1$ and $k = 4$ solve $t_0 = t_1$, we find that $k = 6 \pm 2\sqrt{5}$ implies that if $O_k < O_1$, then (O_k, O_1) carries a 5-sided Poncelet polygon. For $GF(p)$, p and odd prime, a result by Gauss about quadratic residues (see [8]) can be used, namely

$$5 \text{ is a square in } GF(p) \Leftrightarrow p \equiv \pm 1(5). \tag{12}$$

Hence in all planes $PG(2, p)$, in which 5 divides $p + 1$, the square root of 5 is well-defined and the indices of the Poncelet 5-gons given by $6 \pm 2\sqrt{5}$ can be computed.

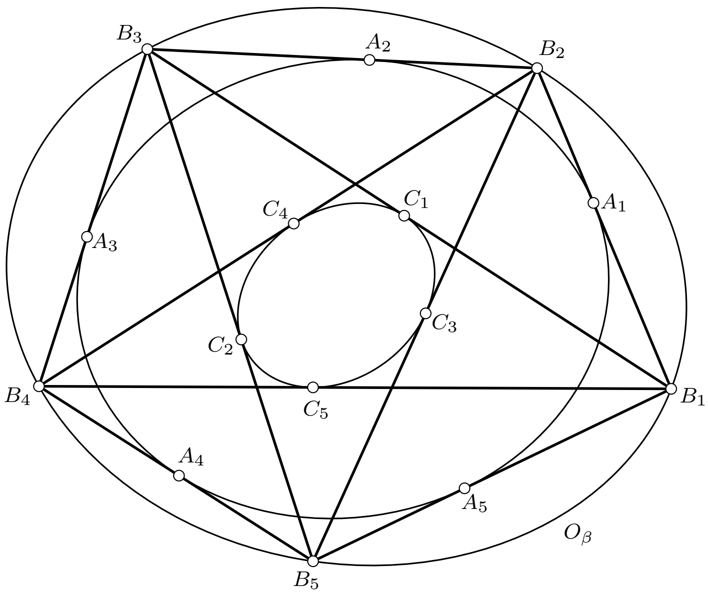

Figure 2. Two different 5-sided Poncelet polygons can be constructed using the same five points on the outer conic.

Finally, we can prove the theorem how to find the indices k, such that (O_k, O_1) carries an n-sided Poncelet polygon for n odd.

Theorem 4.12. *Let $n \geq 3$ be an odd number. Then the indices k such that (O_k, O_1) carries an n-sided Poncelet polygon in a plane $PG(2, q)$ are given by the solutions of*

$$t_0 = t_{\frac{\phi(n)}{2}}, \tag{13}$$

where we need $t_0 \neq t_i$ in $GF(q)$ for all $i < \frac{\phi(n)}{2}$. For a fixed plane $PG(2, q)$, these solutions are called Poncelet coefficients *for n-sided Poncelet polygons and denoted by k_n^i, $i = 1, \dots, \frac{\phi(n)}{2}$.*

Proof. Let $O_k < O_1$ carry an n-sided Poncelet polygon for the points $B_1, \dots, B_n$, n odd. Let the points be ordered such that for $O_{t_0} < O_1$ we have

$$B_1 \to B_2 \to B_3 \to \dots \to B_n \to B_1.$$

We have seen in the proof of Lemma 4.9 that the n-sided Poncelet polygon of $O_{t_1} < O_1$ is given by the order

$$B_1 \to B_3 \to B_5 \to \dots \to B_n \to B_2 \to \dots \to B_{n-1} \to B_1.$$

Iterating this, we see that the n-sided Poncelet polygon given by $O_{t_i} < O_1$ has the order

$$B_1 \to B_{1+2^i} \to B_{1+2\cdot 2^i} \to B_{1+3\cdot 2^i} \dots \to B_1,$$

where the indices are taken cyclically. We already know that there are exactly $\frac{\phi(n)}{2}$ different Poncelet n-gons. Since we are only working with n odd, we can apply Fermat's little Theorem (see [8]) and use

$$2^{\frac{\phi(n)}{2}} \equiv \pm 1 (n).$$

This shows directly that for $O_{t_{\frac{\phi(n)}{2}}}$ we start the polygon by $B_1 \to B_2$ or $B_1 \to B_n$ and hence the polygon is equivalent with the very first one. To determine the coefficients k such that (O_k, O_1) carries an n-sided Poncelet polygon, we therefore indeed have to solve (13). □

Remark 4.13. Note that for some values of n, the iteration needs fewer steps than $\frac{\phi(n)}{2}$, as the order of 2 modulo n can be smaller than $\frac{\phi(n)}{2}$. In these cases, not all indices can be constructed by starting with one Poncelet n-gon only. Nevertheless, the condition (13) stays the same but the same coefficients could be derived by computing less, i.e.

$$t_0 = t_{s_n}$$

and $t_0 \neq t_i$ for all $i < s_n$, where

$$s_n := \min\{s \mid 2^s \equiv \pm 1(n)\}.$$

The smallest example for $\frac{\phi(n)}{2} \neq s_n$ is $n = 17$, where we have $\frac{\phi(17)}{2} = 8$ but $2^4 \equiv -1(17)$, i.e. $s_{17} = 4$.

Example 4.14. We want to determine the indices k such that $O_k < O_1$ carries a 9-sided Poncelet polygon in $PG(2, 53)$. Since $\frac{\phi(9)}{2} = 3$, we have to solve

$$t_0 = t_3, \ t_1 \neq t_3, \ t_2 \neq t_3$$

in $GF(53)$. So, we need solutions of

$$t_0 - t_3 = k - \frac{k^8}{(128 + k(-256 + k(160 + (-32 + k)k)))^2} = 0.$$

Rewriting this equation, we have to solve

$$k^8 - k(128 + k(-256 + k(160 + (-32 + k)k)))^2 = 0.$$

We obtain the solutions

$$k \in \{1, 4, 13, 36, 40\}.$$

Since we can exclude the solutions 1 and 4, as they also solve $t_2 = t_3$, we deduce that

$$O_{13} < O_1, \ O_{36} < O_1, \ O_{40} < O_1$$

are the pairs of conics in $PG(2, 53)$ such that a 9-sided Poncelet polygon can be constructed.

4.2. *Poncelet polynomials*

We are now able to give an algorithm to determine for each pair (O_α, O_β) in $PG(2, q)$, whether it carries an n-sided Poncelet polygon for a given n. We use the iteration method described before to find polynomials $P_n(k)$ such that the zeros belong to the coefficients k of conics O_k, such that if $O_k < O_1$, then (O_k, O_1) carries an n-sided Poncelet polygon. By Lemma 3.10, this gives information about all pairs (O_α, O_β).

Definition 4.1. A polynomial P_n with integer coefficients is called Poncelet polynomial for n-sided Poncelet polygons, if the zeros in $GF(q)$ correspond to the coefficients k, such that $O_k < O_1$ carries an n-sided Poncelet polygon in $PG(2, q)$.

Example 4.15. We have already seen in Lemma 4.1 that $P_3(k) = k - 4$ and in Example 4.11 that $P_5(k) = 16 - 12k + k^2$.

By Lemma 4.5, we know that all these polynomials P_n are of degree $\frac{\phi(n)}{2}$, as the existence of one conic O_k, such that (O_k, O_1) carries an n-sided Poncelet polygon in $PG(2, q)$ leads to $\frac{\phi(n)}{2}$ such conics O_k. Until now, we only know how to produce Poncelet polynomials P_n for n odd, but similar to the Poncelet coefficients k, a doubling process can be applied for finding P_n with n even. Note that to find the coefficients for an odd n-sided Poncelet polygon, we look for indices k, such that

$$P_n(k) = 0 \text{ in } GF(q).$$

Applying Lemma 4.6 gives

$$P_{2n}(k) = \frac{(k-2)^{\phi(n)} P_n\left(\frac{k^2}{(k-2)^2}\right)}{P_n(k)}$$

for n odd and iterating once more, we get

$$P_{2n}(k) = (k-2)^{\phi(n)} P_n\left(\frac{k^2}{(k-2)^2}\right)$$

for n even.

Example 4.16. We have $P_3(k) = -4 + k$ and $\phi(3) = 2$. Hence we have to calculate

$$(k-2)^2 P_3\left(\frac{k^2}{(k-2)^2}\right) = (k-2)^2\left(-4 + \frac{k^2}{(k-2)^2}\right) = -(-4+k)(-4+3k).$$

Dividing by $P_3(k)$ gives $P_6(k) = 4 - 3k$.

For the general case, note that for numbers n and m which have the same value $\phi(n) = \phi(m)$, it has to be checked by hand, which polynomials of degree $\frac{\phi(n)}{2}$ given by the iteration belong to the n-gons and which to the m-gons. For example, the iteration for $\frac{\phi(n)}{2} = 3$ gives the polynomial

$$-(-4+k)(-1+k)k(-64+96k-36k^2+k^3)(-64+80k-24k^2+k^3).$$

Excluding the factors $(k-4)$ and $(k-1)$ which already occur at the first iteration, we find checking by hand

$$P_7(k) = -64 + 80k - 24k^2 + k^3 \text{ and } P_9(k) = -64 + 96k - 36k^2 + k^3.$$

With some computational effort, we are now able to create a list of all Poncelet polynomials P_n up to a chosen value of n. In Section 5 we will compare our findings to results in the real projective plane. This will finally lead to an explicit formula for the Poncelet polynomials (see Theorem 5.2).

Based on the list of Poncelet polynomials we are now able to formulate an algorithm to determine for each pair (O_α, O_β) in $PG(2,q)$ whether it carries an n-sided Poncelet polygon.

Corollary 4.17. *The following four steps give a complete description of n-sided Poncelet polygons for conic pairs (O_α, O_β) in $PG(2,q)$.*

1. *Determine all $n \geq 3$ with $n|(q+1)$. For every such n, calculate $\frac{\phi(n)}{2}$, which gives the number of indices k, such that an n-sided Poncelet polygon can be constructed for (O_k, O_1).*
2. *For all values n obtained in Step 1, look up the Poncelet polynomial P_n.*
3. *For every Poncelet polynomial P_n from Step 2, solve $P_n(k) = 0$ in $GF(q)$. This gives the corresponding Poncelet coefficients k, such that an n-sided Poncelet polygon can be constructed for (O_k, O_1).*
4. *By using the coordinate transformation described in Lemma 3.10, transform the information obtained in Step 3 to all pairs $O_\alpha < O_\beta$.*

Example 4.18. We want to deduce all relations of conic pairs $O_\alpha < O_\beta$ in the plane $PG(2,11)$ by using the algorithm above.

- Step 1: The values n, such that an n-sided Poncelet polygon can be constructed, are given by $n = 3, 4, 6, 12$. Moreover:

n	3	4	6	12
$\frac{\phi(n)}{2}$	1	1	1	2

- Step 2: We have the following Poncelet polynomials:

$$P_3(k) = -4 + k$$
$$P_4(k) = -2 + k$$
$$P_6(k) = -4 + 3k$$
$$P_{12}(k) = -16 + 16k - k^2$$

- Step 3: The zeros of the Poncelet polynomials in $GF(11)$ are given by:

$$P_3(k) = 0 \Leftrightarrow k = 4$$
$$P_4(k) = 0 \Leftrightarrow k = 2$$
$$P_6(k) = 0 \Leftrightarrow k = 5$$
$$P_{12}(k) = 0 \Leftrightarrow k = 6, 10$$

- Step 4: By suitable collinear transformations, we obtain all relations (see Table 2).

Remark 4.19. One can verify, that the Poncelet polygons sitting in the projective pencil $\{O_k \mid k \in GF(q) \setminus \{0\}\}$, can be considered as affinely regular polygons (choose the embedding of the affine plane in the projective plane given by $x = 1$). See [11] and the references therein for more information about this line of research.

	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8	O_9	O_{10}
O_1		12	3			4			6	12
O_2	4			12		3	6		12	
O_3					6	12	4	12	3	
O_4	3	4	6				12	12		
O_5	6			3		12		4		12
O_6	12		4		12		3			6
O_7			12	12				6	4	3
O_8		3	12	4	12	6				
O_9		12		6	3		12			4
O_{10}	12	6			4			3	12	

Table 2. Poncelet pairs in $PG(2,11)$. For example the pair $O_1 < O_3$ carries Poncelet triangles.

5. Comparison to other methods

5.1. Comparison to the Euclidean Plane

Recall that any point on $O_k : x^2 + ky^2 + ckz^2 = 0$ has a nonzero x-coordinate. Because of this, we can project these conics on the affine plane by setting $x = 1$. Moreover, we can look at real solutions of the equations. In the proof of Poncelet's Theorem for this family of conics, we have seen that there is an affine transformation which maps the whole family to a family of concentric circles. Let us therefore consider pairs of such circles in the Euclidean plane, i.e.

$$E_1 : x^2 + y^2 = 1$$
$$E_r : x^2 + y^2 = r^2, r > 1.$$

We try to find a suitable radius r for E_r, such that a regular n-sided polygon which is inscribed in E_r and circumscribed around E_1 can be constructed. It is elementary that one solution to this problem, namely the circumcircle radius r of a simple, regular n-sided polygon is given by

$$r = \frac{1}{\cos(\frac{\pi}{n})}.$$

In terms of Poncelet coefficients as defined for the finite case, this gives

$$k_n = \frac{1}{\cos^2(\frac{\pi}{n})}.$$

Example 5.1. The radius r for a simple, regular 5-gon is therefore given by $r = \frac{1}{\cos(\frac{\pi}{5})} = -1 + \sqrt{5}$. Note that $(-1 + \sqrt{5})^2 = 6 - 2\sqrt{5}$, which is exactly one of the zeros of the Poncelet polygon for 5-gons we obtained over finite fields (see Example 4.11). The second radius $\tilde{r}$, which corresponds to the complex 5-gon circumscribed about E_1, can be calculated as well, namely by $\tilde{r} = \frac{1}{\cos(\frac{2\pi}{5})}$, which leads to $\tilde{r} = 1 + \sqrt{5}$. Hence we obtain $\tilde{r}^2 = 6 + 2\sqrt{5}$, which belongs to the second coefficient for 5-gons obtained in the finite case.

Now we turn our attention to the formula deduced for the coefficients $\tilde{k}$ for $2n$-sided Poncelet polygons in Lemma 4.6. For this, note that

$$\cos^2\left(\frac{\phi}{2}\right) = \frac{1 + \cos(\phi)}{2}.$$

Hence we get

$$\tilde{k} = \frac{1}{\cos^2(\frac{\pi}{2n})} = \frac{2}{1 + \cos(\frac{\pi}{n})} = \frac{2}{1 - \frac{1}{\sqrt{1/\cos^2(\frac{\pi}{n})}}} = \frac{2}{1 - \frac{1}{\sqrt{k}}}, \tag{14}$$

which is exactly the formula derived for the finite case.

Since there does not exist a radical expression for $\cos(\frac{\pi}{n})$ for all integers n, it is convenient to look again at polynomials with roots $\frac{1}{\cos^2(\frac{k\pi}{n})}$. These are closely connected to the n-th cyclotomic polynomials $\Phi_n(x)$. Recall, that those polynomials can be written as

$$\Phi_n(x) = \prod_{1\le k\le n,(k,n)=1} (x - e^{\frac{2\pi ik}{n}}).$$

It is immediate that the degree of Φ_n is $\phi(n)$, the Euler totient function. The zeros of $\Phi_n(x)$ are given by $e^{\frac{2\pi ik}{n}}$ for $(k,n) = 1$. For a zero x of Φ_n, also $\overline{x} = \frac{1}{x}$ is a zero. Define

$$q_n\left(x + \frac{1}{x}\right) := \Phi_n(x)x^{-\frac{\phi(n)}{2}}.$$

The zeros of q_n are then given by $2\,\mathrm{Re}(e^{\frac{2\pi ik}{n}}) = 2\cos(\frac{2\pi k}{n})$. Next, define

$$r_n(x) := q_n(2x)$$

which has zeros $\cos(\frac{2\pi k}{n})$. In the next step, we consider

$$s_n(x) := r_n(2x - 1)$$

which has zeros $\frac{1+\cos(\frac{2\pi k}{n})}{2} = \cos^2(\frac{\pi k}{n})$ for $k = 1, \dots, n-1$. Finally, consider

$$\tilde{P}_n(x) = x^{\frac{\phi(n)}{2}} s_n\left(\frac{1}{x}\right)$$

with zeros $\frac{1}{\cos^2(\frac{\pi k}{n})}$, which is exactly the polynomial we wanted. Summarizing, we have the following explicit formula for the Poncelet polynomials.

Theorem 5.2. *The Poncelet polynomial $\tilde{P}_n$ for $n \ge 3$ is given by*

$$\tilde{P}_n(x) = x^{\frac{\phi(n)}{2}}\Phi_n(z)z^{-\frac{\phi(n)}{2}} \tag{15}$$

for $z = \frac{2-2\sqrt{1-x}}{x} - 1$. Moreover, the zeros of $\tilde{P}_n$ are $\frac{1}{\cos^2(\frac{\pi k}{n})}$ for $(k,n) = 1$.

Example 5.3. For $n = 5$, the cyclotomic polynomial is given by $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$, which leads to

$$\tilde{P}_5(x) = 16 - 12x + x^2$$

which indeed has the zeros $6 + 2\sqrt{5}$ and $6 - 2\sqrt{5}$ (see Example 4.11).

It remains to show that the polynomials $\tilde{P}_n$ given by (15) are the same as the Poncelet polynomials derived in the previous section. Remember that we started by iterating the result of Lemma 4.9, which states that if $O_k < O_1$ carries a Poncelet n-gon, then $O_{\frac{k^2}{(k-2)^2}} < O_1$ as well. Recall

$$t_0 := k,\ t_{i+1} := \frac{t_i^2}{(t_i - 2)^2}.$$

Note that

$$t_{i+1} = \frac{t_i^2}{(t_i - 2)^2} \Leftrightarrow t_i = \frac{2}{1 - \frac{1}{\sqrt{t_{i+1}}}}$$

and hence, by the result in (14), we actually double our angle at each step. We already know that the $\frac{\phi(n)}{2}$ zeros of $\tilde{P}_n$ are given by $\frac{1}{\cos^2(\frac{k\pi}{n})}$ for $(k,n) = 1$. Hence, we only have to show that $\frac{1}{\cos^2(\frac{k\pi}{n})}$ solves $t_0 = t_{\frac{\phi(n)}{2}}$. This is equivalent to showing that

$$\frac{1}{\cos^2(\frac{k\pi}{n})} = \frac{1}{\cos^2\left(\frac{2^{\frac{\phi(n)}{2}}k\pi}{n}\right)} \tag{16}$$

for all odd n and $k < n$, $(k,n) = 1$. For this, note first the following two immediate equations for any integer k, namely

$$\cos\left(\frac{\pi}{n}\right) = \cos\left(2k\pi \pm \frac{\pi}{n}\right)$$

and

$$\cos\left(\frac{\pi}{n}\right) = -\cos\left((2k+1)\pi \pm \frac{\pi}{n}\right).$$

Hence, for all integers k, we know

$$\left|\cos\left(\frac{\pi}{n}\right)\right| = \left|\cos\left(\frac{(kn\pm 1)\pi}{n}\right)\right|.$$

By Fermat's little Theorem, we know that for any odd integer n, we have

$$2^{\frac{\phi(n)}{2}} \equiv \pm 1 (n).$$

Hence, there exists a k, such that $2^{\frac{\phi(n)}{2}} = kn \pm 1$, which implies (16). For n odd, P_n and $\tilde{P}_n$ are both monic polynomials of degree $\frac{\phi(n)}{2}$ with the $\frac{\phi(n)}{2}$ zeros $\frac{1}{\cos^2(\frac{k\pi}{n})}$ for $(k,n) = 1$. Hence, they are indeed the same.

5.2. *Comparison to Cayley's Criterion*

The criterion deduced by Cayley in 1853 (see [5]) reads as follows.

Theorem 5.4. *Let C and D be the matrices corresponding to two conics generally situated in the projective plane. Consider the expansion*

$$\sqrt{\det(tC + D)} = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \ldots$$

Then an n-sided Poncelet polygon with vertices on C exists if and only if for $n = 2m+1$, we have

$$\det\begin{pmatrix} A_2 & \ldots & A_{m+1} \\ \vdots & \ldots & \vdots \\ A_{m+1} & \ldots & A_{2m} \end{pmatrix} = 0$$

and for $n = 2m$, we have

$$\det\begin{pmatrix} A_3 & \ldots & A_{m+1} \\ \vdots & \ldots & \vdots \\ A_{m+1} & \ldots & A_{2m} \end{pmatrix} = 0$$

In the discussion above, we were mainly interested in pairs of conics (O_k, O_1) with equations

$$O_k : x^2 + ky^2 + ckz^2 = 0$$

$$O_1 : x^2 + y^2 + cz^2 = 0$$

To apply Cayley's criterion, we therefore have to look at the expansion of the square root of

$$\det\begin{pmatrix} t+1 & 0 & 0 \\ 0 & t+k & 0 \\ 0 & 0 & c(t+k) \end{pmatrix}$$

which is given by

$$\sqrt{ck^2} + \frac{(k+2)\sqrt{ck^2}}{2k}t - \frac{(k-4)\sqrt{ck^2}}{8k}t^2 + \frac{(k-2)\sqrt{ck^2}}{16k}t^3 - \frac{(5k-8)\sqrt{ck^2}}{128k}t^4 + O(t)^5$$

Example 5.5. The condition for a 3-sided Poncelet polygon is given by vanishing of the coefficient of t^2 which is $A_2 = \frac{(k-4)\sqrt{ck^2}}{8k}$. This expression is zero if and only if $k - 4 = 0$, which is exactly the condition derived in Lemma 4.1 for the finite case.

Example 5.6. The condition for 5-sided Poncelet polygons is given by $A_2A_4 - A_3^2 = 0$, which is the same as $\frac{c((k-12)k+16)}{1024} = 0$. This is equivalent to $k^2 - 12k + 16 = 0$, so again, we obtain the same condition as for the finite case (compare to Example 4.11).

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