Mathematical Sciences and Applications E-Notes
8 (1) 1-10 (2020) ©MSAEN
https://doi.org/10.36753/mathenot.651519

Certain Semisymmetry Curvature Conditions on Paracontact Metric (k, μ)-Manifolds

Ahmet Yıldız* and Uday Chand De

Abstract
The object of the present paper is to characterize paracontact metric (k, μ)-manifolds satisfying some semisymmetry curvature conditions.

Keywords: *Paracontact metric (k, μ)-manifolds; Weyl semisymmetric manifolds; Projective semisymmetric manifolds; ϕ-Weyl semisymmetry; h-Weyl semisymmetry.*

AMS Subject Classification (2020): *Primary: 53B30; 53C15; 53C25; 53C50; Secondary: 53D10; 53D15*

***Corresponding author**

1. Introduction

In modern contact geometry, the study of nullity distribution on paracontact geometry is one among the most interesting topics. Paracontact metric structures have been introduced in [3], as a natural odd-dimensional counterpart to para-Hermitian structures, like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds have been studied by many authors in the recent years. A systematic study of paracontact metric manifolds was carried out by Zamkovoy [11].

An important class among paracontact metric manifolds is that of the (k, μ)-manifolds, which satisfy the nullity condition

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{1.1}$$

for all X, Y vector fields on M, where k and μ are constants and $h = \frac{1}{2}\mathcal{L}_\xi\phi$ [1]. This class includes the para-Sasakian manifolds [3, 11], the paracontact metric manifolds satisfying $R(X, Y)\xi = 0$ for all X, Y vector fields on M [12].

Among the geometric properties of manifolds symmetry is an important one. Local point of view it was introduced by Shirokov [5] as a Riemannian manifold with covariant constant curvature tensor R, that is, with $\nabla R = 0$, where ∇ is the Levi-Civita connection. An extensive theory of symmetric Riemannian manifolds was introduced by Cartan [2]. A manifold is called semisymmetric if the curvature tensor R satisfies $R(X, Y) \cdot R = 0$, where $R(X, Y)$ is considered to be a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y. Semisymmetric manifolds were locally classified by Szabó [7]. Also in [10] Yildiz and De studied h-Weyl semisymmetric, ϕ-Weyl semisymmetric, h-projectively semisymmetric and ϕ-projectively semisymmetric non-Sasakian (k, μ)-contact metric manifolds. Recently Mandal and De studied certain curvature conditions on paracontact (k, μ)-spaces [4].

The *projective curvature tensor* is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional semi-Riemannian manifold with metric g. The Ricci operator Q of (M, g) is defined by $g(QX, Y) = S(X, Y)$, where S denotes the Ricci tensor of type $(0, 2)$ on M. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the semi-Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes.

Received : *27–11–2019,* ***Accepted :*** *15–03–2020*

Here P is defined by [6]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.2}$$

for all X, Y, Z vector fields on M, where R is the curvature tensor and S is the Ricci tensor of M.

In fact, M is projectively flat if and only if it is of constant curvature [8]. Thus the projective curvature tensor is the measure of the failure of a semi-Riemannian manifold to be of constant curvature.

In semi-Riemannain geometry one of the important curvature properties is conformal flatness. The *Weyl conformal curvature tensor* is a measure of the curvature of spacetime and differs from the semi-Riemannian curvature tensor. It is the traceless component of the Riemannian tensor which has the same symmetries as the Riemannian tensor.The Weyl conformal curvature tensor is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z \\ &\quad -\frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad +\frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{1.3}$$

for all X, Y, Z vector fields on M, where $r = tr(S)$ is scalar curvature [9].

A paracontact metric (k,μ)-manifold is said to be an *Einstein* manifold if the Ricci tensor satisfies $S = ag$, where a a smooth function.

The outline of the article goes as follows: After introduction in section 2, we recall basic facts and some basic results of paracontact metric manifolds with characteristic vector field ξ belonging to the (k,μ)-nullity distribution. In section 3, we characterize paracontact metric (k,μ)-manifolds satisfying some semisymmetry curvature conditions. We prove that Weyl semisymmetric and projective semisymmetric paracontact metric (k,μ)-manifolds are Einstein manifolds and h-Weyl semisymmetric and ϕ-Weyl semisymmetric paracontact metric (k,μ)-manifolds are η-Einstein manifolds provided $k \neq -1$.

2. Preliminaries

An $(2n+1)$-dimensional manifold M is said to have an *almost paracontact structure* if it admits a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions ([3], [11]):

(i) $\eta(\xi) = 1,\ \phi^2 = I - \eta \otimes \xi$,

(ii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e., the ± 1-eigendistributions, $\mathcal{D}^{\pm} = \mathcal{D}_{\phi}(\pm 1)$ of ϕ have equal dimension n.

Thus from the definition it follows that $\phi\xi = 0$, $\eta \circ \phi = 0$ and the endomorphism ϕ has rank $2n$. The Nijenhius torsion tensor field $[\phi,\phi]$ is given by

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

When the tensor field $N_{\phi} = [\phi,\phi] - 2d\eta \otimes \xi = 0$, the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all X, Y vector fields on M, then we say that (M, ϕ, ξ, η, g) is an *almost paracontact metric manifold*. Notice that such a pseudo-Riemannian metric is necessarily of signature $(n+1, n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$, such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$, $g(X_i, Y_j) = 0$, $g(\xi, X_i) = g(\xi, Y_j) = 0$, and $Y_i = \phi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a ϕ-basis.

We define the *fundamental form* of the almost paracontact metric manifold by $\theta(X,Y) = g(X, \phi Y)$. If $d\eta(X,Y) = g(X, \phi Y)$, then M is said to be *paracontact metric manifold*. In a paracontact metric manifold one defines a symmetric, trace-free operator $h = \frac{1}{2}\mathcal{L}_{\xi}\phi$, where $\mathcal{L}_{\xi}$, denotes the Lie derivative. It is known [11] that h anti-commutes with ϕ and satisfies $h\xi = 0$, tr$h$ = tr$h\phi = 0$ and

$$\nabla\xi = -\phi + \phi h,$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold (M, g).

Moreover $h=0$ if and only if ξ is Killing vector field. In this case M is said to be a *K-paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. Also in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3. We also recall that any para-Sasakian manifold satisfies

$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X.$$

Now let M be a paracontact manifold. The (k,μ)-nullity distribution of a M for the pair (k,μ) is a distribution

$$N(k,\mu):p\rightarrow N_p(k,\mu)=\left\{\begin{array}{c} Z\in T_pM \mid R(X,Y)Z=k(g(Y,Z)X-g(X,Z)Y) \\ +\mu(g(Y,Z)hX-g(X,Z)hY), \end{array}\right\}, \tag{2.2}$$

for some real constants k and μ. If the characteristic vector field ξ belongs to the (k,μ)-nullity distribution we have (2.2).

Lemma 2.1. *[1] Let M be a paracontact metric (k,μ)-manifold of dimension $2n+1$. Then the following identities hold:*

$$h^2=(1+k)\phi^2, \tag{2.3}$$

$$(\nabla_X\phi)Y=-g(X,Y)\xi+g(hX,Y)\xi+\eta(Y)X-\eta(Y)hX, \quad \textit{for } k\neq -1, \tag{2.4}$$

$$(\nabla_X h)Y-(\nabla_Y h)X=-(1+k)(2g(X,\phi Y)\xi+\eta(X)\phi Y-\eta(Y)\phi X)+(1-\mu)(\eta(X)\phi hY-\eta(Y)\phi hX), \tag{2.5}$$

for any vector fields X, Y on M.

Lemma 2.2. *[1] In any $(2n+1)$-dimensional paracontact metric (k,μ)-manifold (M,ϕ,ξ,η,g) such that $k\neq -1$, the Ricci operator Q is given by*

$$Q=(2(1-n)+n\mu)I+(2(n-1)+\mu)h+(2(n-1)+n(2k-\mu))\eta\otimes\xi. \tag{2.6}$$

In particular, for $k>-1$, (M,g) is an η-Einstein manifold if and only if $\mu=2(1-n)$, or an Einstein manifold if and only if $k=0=\mu$ and $n=1$ (in this case the manifold is Ricci-flat).

For $k<-1$, (M,g) is an η-Einstein manifold if and only if $\mu=2(1-n)$, or an Einstein manifold if and only if $k=\frac{1-n^2}{n}$ and $\mu=2(1-n)$.

3. Main results

In this section we study some semisymmetry curvature conditions on paracontact metric (k,μ)-manifolds. Firstly we give the following:

Definition 3.1. A semi-Riemannian manifold $(M^{2n+1},g), n>1$, is said to be *Weyl semisymmetric if*

$$R(U,X)\cdot C=0,$$

holds on M for all U,X vector fields on M.

Let M be a Weyl semisymmetric paracontact metric (k,μ)-manifold with $k\neq -1$. Then above equation is equivalent to

$$(R(U,X)\cdot C)(W,Y)Z=0, \tag{3.1}$$

for any U,X,W,Y,Z vector fields on M. Thus we have

$$R(U,X)C(W,Y)Z-C(R(U,X)W,Y)Z-C(W,R(U,X)Y)Z-C(W,Y)R(U,X)Z=0. \tag{3.2}$$

Substituting $U=W=\xi$ in (3.2) yields

$$R(\xi,X)C(\xi,Y)Z-C(R(\xi,X)\xi,Y)Z-C(\xi,R(\xi,X)Y)Z-C(\xi,Y)R(\xi,X)Z=0, \tag{3.3}$$

where

$$C(\xi,Y)Z = (\frac{r-2nk}{2n(2n-1)})(g(Y,Z)\xi-\eta(Z)Y)-\frac{1}{2n-1}(S(Y,Z)\xi-\eta(Z)QY). \tag{3.4}$$

With the help of (3.3) and (3.4), we get

$$kS(X,Y)+\mu S(hX,Y)-2nk^2g(X,Y)-2nk\mu g(hX,Y)=0. \tag{3.5}$$

Putting $Y=hY$ in (3.5) and using (2.3), we obtain

$$\mu(k+1)S(X,Y)+kS(hX,Y)-2nk^2g(hX,Y)-2nk\mu(k+1)g(X,Y)=0. \tag{3.6}$$

Now suppose $k\neq -1$ and $\mu\neq 0$. Multiplying (3.5) by k and (3.6) by μ, we have

$$k^2S(X,Y)+\mu kS(hX,Y)-2nk^3g(X,Y)-2nk^2\mu g(hX,Y)=0, \tag{3.7}$$

and

$$\mu^2(k+1)S(X,Y)+\mu kS(hX,Y)-2nk^2\mu g(hX,Y)-2nk(k+1)\mu^2g(X,Y)=0, \tag{3.8}$$

respectively. Subtracting (3.8) from (3.7), we get

$$\{k^2-\mu^2(k+1)\}\{S(X,Y)-2nkg(X,Y)\}=0. \tag{3.9}$$

If $k\neq -1$, then $k^2-\mu^2(k+1)\neq 0$. Therefore from (3.9) it follows that $S(X,Y)=2nkg(X,Y)$, which implies that the manifold M is an Einstein manifold. Thus we have the following:

Theorem 3.1. *If M is a $(2n+1)$-dimensional Weyl semisymmetric paracontact metric (k,μ)-manifold with $k\neq -1$ then the manifold M is an Einstein manifold.*

Definition 3.2. A semi-Riemannian manifold $(M^{2n+1},g), n>1$, is said to be *projective semisymmetric if*

$$R(U,X)\cdot P=0,$$

holds on M for all U,X vector fields on M.

Let M be a projective semisymmetric paracontact metric (k,μ)-manifold with $k\neq -1$. Then above equation is equivalent to

$$(R(U,X)\cdot P)(W,Y)Z=0, \tag{3.10}$$

for any U,X,W,Y,Z vector fields on M. Thus we have

$$R(U,X)P(W,Y)Z-P(R(U,X)W,Y)Z-P(W,R(U,X)Y)Z-P(W,Y)R(U,X)Z=0. \tag{3.11}$$

Substituting $U=W=\xi$ in (3.11) yields

$$R(\xi,X)P(\xi,Y)Z-P(R(\xi,X)\xi,Y)Z-P(\xi,R(\xi,X)Y)Z-P(\xi,Y)R(\xi,X)Z=0, \tag{3.12}$$

where

$$P(\xi,Y)Z=kg(Y,Z)\xi+\mu(g(hY,Z)\xi-\eta(Z)hY)-\frac{1}{2n}S(Y,Z)\xi. \tag{3.13}$$

With help of (3.13) and (3.12), we get

$$\mu\{\eta(Z)g(R(\xi,X)hY,\xi)+g(R(\xi,X)Y,hZ)+g(R(\xi,X)Z,hY)\}+\frac{1}{2n}\{S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z)\}=0,$$

which implies that

$$\begin{aligned}&\mu\{kg(hX,Z)\eta(Y)+\mu g(hX,hZ)\eta(Y)\}\\&+\frac{k}{2n}\{S(Z,\xi)g(X,Y)+S(Y,\xi)g(X,Z)\\&-S(X,Z)\eta(Y)-S(X,Y)\eta(Z)\}\\&+\frac{\mu}{2n}\{S(Z,\xi)g(hX,Y)+S(Y,\xi)g(hX,Z)\\&-S(hX,Z)\eta(Y)-S(hX,Y)\eta(Z)\}=0.\end{aligned} \tag{3.14}$$

Putting $Z=\xi$ in (3.14), we have

$$kS(X,Y)+\mu S(hX,Y)-2nk^2g(X,Y)-2nk\mu g(hX,Y)=0. \tag{3.15}$$

Putting $X=hX$ in (3.15) and using $h^2=(k+1)\phi^2$, we obtain

$$\mu(k+1)S(X,Y)+kS(hX,Y)-2nk^2g(hX,Y)-2nk(k+1)\mu g(X,Y)=0. \tag{3.16}$$

Multiplying (3.15) by k and (3.16) by μ, we have

$$k^2S(X,Y)+k\mu S(hX,Y)-2nk^3g(X,Y)-2nk^2\mu g(hX,Y)=0, \tag{3.17}$$

and

$$\mu^2(k+1)S(X,Y)+\mu kS(hX,Y)-2nk^2\mu g(hX,Y)-2nk(k+1)\mu^2g(X,Y)=0. \tag{3.18}$$

respectively. Subtracting (3.18) from (3.17), we get

$$\{k^2-\mu^2(k+1)\}\{S(X,Y)-2nkg(X,Y)\}=0. \tag{3.19}$$

If $k\neq-1$ then $k^2-\mu^2(k+1)\neq 0$. Therefore from (3.19) it follows that $S(X,Y)=2nkg(X,Y)$. Thus the manifold M is an Einstein manifold. Hence we have the following:

Theorem 3.2. *If M is a $(2n+1)$-dimensional projective semisymmetric paracontact metric (k,μ)-manifold with $k\neq-1$ then the manifold M is an Einstein manifold.*

Definition 3.3. A semi-Riemannian manifold $(M^{2n+1},g), n>1$, is said to be h-Weyl *semisymmetric if*

$$C(X,Y)\cdot h=0, \tag{3.20}$$

holds on M.

Now let M be a h-Weyl semisymmetric paracontact metric (k,μ)-manifold with $k\neq-1$. Then equation (3.20) is equivalent to

$$C(X,Y)hZ-hC(X,Y)Z=0,$$

for any X,Y,Z vector fields on M. Firstly, we get

$$\begin{aligned} R(X,Y)hZ-hR(X,Y)Z &= \mu(k+1)\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ &\quad +\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\} \\ &\quad +k\{g(hY,Z)\eta(X)\xi-g(hX,Z)\eta(Y)\xi \\ &\quad +\eta(X)\eta(Z)hY-\eta(Y)\eta(Z)hX \\ &\quad +g(\phi Y,Z)\phi hX-g(\phi X,Z)\phi hY\} \\ &\quad +(\mu+k)\{g(\phi hX,Z)\phi Y-g(\phi hY,Z)\phi X\} \\ &\quad +2\mu g(\phi X,Y)\phi hZ. \end{aligned} \tag{3.21}$$

Then we can write

$$\begin{aligned} C(X,Y)hZ-hC(X,Y)Z &= R(X,Y)hZ-hR(X,Y)Z \\ &\quad -\frac{1}{2n-1}\{S(Y,hZ)X-S(X,hZ)Y+g(Y,hZ)QX \\ &\quad -g(X,hZ)QY-S(Y,hZ)hX+S(X,hZ)hY \\ &\quad -g(Y,hZ)hQX+g(X,hZ)hQY\} \\ &\quad +\frac{r}{2n(2n-1)}\{g(Y,hZ)X-g(X,hZ)Y \\ &\quad -g(Y,hZ)hX+g(X,hZ)hY\}=0. \end{aligned} \tag{3.22}$$

Using (3.21) in (3.22), we get

$$
\begin{aligned}
&\mu(k+1)\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\
&+k\{g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi + \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \\
&+g(\phi Y,Z)\phi hX - g(\phi X,Z)\phi hY\} \\
&+(\mu+k)\{g(\phi hX,Z)\phi Y - g(\phi hY,Z)\phi X\} + 2\mu g(\phi X,Y)\phi hZ \\
&-\frac{1}{2n-1}\{S(Y,hZ)X - S(X,hZ)Y + g(Y,hZ)QX \\
&-g(X,hZ)QY - S(Y,hZ)hX + S(X,hZ)hY \\
&-g(Y,hZ)hQX + g(X,hZ)hQY\} \\
&+\frac{r}{2n(2n-1)}\{g(Y,hZ)X - g(X,hZ)Y - g(Y,hZ)hX + g(X,hZ)hY\} = 0.
\end{aligned} \tag{3.23}
$$

Putting $Y = hY$ in (3.23), we have

$$
\begin{aligned}
&\mu(k+1)\{g(hY,Z)\eta(X)\xi + \eta(X)\eta(Z)hY\} \\
&+k\{g(h^2Y,Z)\eta(X)\xi + \eta(X)\eta(Z)h^2Y \\
&+g(\phi hY,Z)\phi hX - g(\phi X,Z)\phi h^2Y\} \\
&+(\mu+k)\{g(\phi hX,Z)\phi hY - g(\phi h^2Y,Z)\phi X\} \\
&+2\mu g(\phi X,Y)\phi h^2Z \\
&-\frac{1}{2n-1}\{S(hY,hZ)X - S(X,hZ)hY + \\
&g(hY,hZ)QX - g(X,hZ)QhY - S(hY,hZ)hX \\
&+S(X,hZ)h^2Y - g(hY,hZ)hQX + g(X,hZ)hQhY\} \\
&+\frac{r}{2n(2n-1)}\{g(hY,hZ)X - g(X,hZ)hY \\
&-g(hY,hZ)hX + g(X,hZ)h^2Y\} = 0.
\end{aligned} \tag{3.24}
$$

Multyping with ξ in (3.24), we obtain

$$
\begin{aligned}
&(k+1)\eta(X)[\mu g(hY,Z) + k\{g(Y,Z) - \eta(Y)\eta(Z)\} \\
&-\frac{1}{2n-1}\{S(Y,Z) - 2nk\eta(Y)\eta(Z) + 2nkg(Y,Z) - 2nk\eta(Y)\eta(Z)\} \\
&+\frac{r}{2n(2n-1)}\{g(Y,Z) - \eta(Y)\eta(Z)\}] = 0,
\end{aligned}
$$

i.e.,

$$
\mu g(hY,Z) + (k + \frac{r}{2n(2n-1)} + 2nk)\{g(Y,Z) - \eta(Y)\eta(Z)\} - \frac{1}{2n-1}\{S(Y,Z) - 2nk\eta(Y)\eta(Z)\} = 0. \tag{3.25}
$$

Now from (2.6), we have

$$
g(hY,Z) \quad = \quad \frac{1}{2(n-1)+\mu}S(Y,Z) - \frac{2(1-n)+n\mu}{2(n-1)+\mu}g(Y,Z) - \frac{(2(n-1)+n(2k-\mu)}{2(n-1)+\mu}\eta(Y)\eta(Z). \tag{3.26}
$$

Thus from (3.25) and (3.26), we get

$$
\begin{aligned}
&(\frac{\mu}{2(n-1)+\mu} - \frac{1}{2n-1})S(Y,Z) \\
&-(\frac{\mu(2(1-n)+n\mu)}{2(n-1)+\mu} - k - \frac{r}{2n(2n-1)} - 2nk)g(Y,Z) \\
&-(\frac{\mu(2(n-1)+n(2k-\mu))}{2(n-1)+\mu} + k + \frac{r}{2n(2n-1)} + 2nk - \frac{2nk}{2n-1})\eta(Y)\eta(Z) = 0,
\end{aligned}
$$

which turns to

$$S(Y,Z) = \frac{\lambda_2}{\lambda_1} g(Y,Z) + \frac{\lambda_3}{\lambda_1}\eta(Y)\eta(Z),$$

where

$$\begin{aligned}
\lambda_1 &= \frac{\mu}{2(n-1)+\mu} - \frac{1}{2n-1},\\
\lambda_2 &= \frac{\mu(2(1-n)+n\mu)}{2(n-1)+\mu} - k - \frac{r}{2n(2n-1)} - 2nk,\\
\lambda_3 &= \frac{\mu(2(n-1)+n(2k-\mu))}{2(n-1)+\mu} + k + \frac{r}{2n(2n-1)} + 2nk - \frac{2nk}{2n-1}.
\end{aligned}$$

Thus the manifold M is an η-Einstein manifold. Hence we state the following:

Theorem 3.3. *If M is a $(2n+1)$-dimensional h-Weyl semisymmetric paracontact metric (k, μ)-manifold with $k \neq -1$ then M is an η-Einstein manifold.*

Definition 3.4. A semi-Riemannian manifold $(M^{2n+1}, g), n > 1$, is said to be ϕ-Weyl *semisymmetric if*

$$C(X,Y)\cdot\phi = 0,$$

holds on M.

Let M be a ϕ-Weyl semisymmetric paracontact metric (k, μ)-manifold with $k \neq -1$. Then above equation is equivalent to

$$C(X,Y)\phi Z - \phi C(X,Y)Z = 0.$$

for any X, Y, Z vector fields on M. Firstly we get

$$\begin{aligned}
R(X,Y)\phi Z - \phi R(X,Y)Z &= g(X,\phi Z)Y - g(Y,\phi Z)X + g(Y,Z)\phi X\\
&-g(X,Z)\phi Y - g(X,\phi Z)hY + g(Y,\phi Z)hX\\
&+g(hY,\phi Z)X - g(hX,\phi Z)Y - g(Y,Z)\phi hX\\
&+g(X,Z)\phi hY - g(hY,Z)\phi X + g(hX,Z)\phi Y\\
&+\frac{-1-\frac{\mu}{2}}{k+1}\{g(hY,\phi Z)hX - g(hX,\phi Z)hY - g(hY,Z)\phi hX\\
&+g(hX,Z)\phi hY\} - \frac{-k+\frac{\mu}{2}}{k+1}\{g(hX,\phi Z)\phi hY - g(hY,\phi Z)\phi hX\\
&-g(\phi hY,Z)hX + g(\phi hX,Z)hY\}\\
&+(k+1)\{g(\phi X,Z)\eta(Y)\xi - g(\phi Y,Z)\eta(X)\xi\\
&+\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\}\\
&+(\mu-1)\{g(\phi hX,Z)\eta(Y)\xi - g(\phi hY,Z)\eta(X)\xi\\
&+\eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX\}.
\end{aligned} \tag{3.27}$$

Then we have

$$\begin{aligned}
C(X,Y)\phi Z-\phi C(X,Y)Z &= g(X,\phi Z)Y-g(Y,\phi Z)X+g(Y,Z)\phi X\\
&-g(X,Z)\phi Y-g(X,\phi Z)hY+g(Y,\phi Z)hX\\
&+g(hY,\phi Z)X-g(hX,\phi Z)Y-g(Y,Z)\phi hX\\
&+g(X,Z)\phi hY-g(hY,Z)\phi X+g(hX,Z)\phi Y\\
&+\frac{-1-\frac{\mu}{2}}{k+1}\{g(hY,\phi Z)hX-g(hX,\phi Z)hY-g(hY,Z)\phi hX\\
&+g(hX,Z)\phi hY\}-\frac{-k+\frac{\mu}{2}}{k+1}\{g(hX,\phi Z)\phi hY-g(hY,\phi Z)\phi hX\\
&-g(\phi hY,Z)hX+g(\phi hX,Z)hY\}\\
&+(k+1)\{g(\phi X,Z)\eta(Y)\xi-g(\phi Y,Z)\eta(X)\xi\\
&+\eta(X)\eta(Z)\phi Y-\eta(Y)\eta(Z)\phi X\}\\
&+(\mu-1)\{g(\phi hX,Z)\eta(Y)\xi-g(\phi hY,Z)\eta(X)\xi\\
&+\eta(X)\eta(Z)\phi hY-\eta(Y)\eta(Z)\phi hX\}\\
&-\frac{1}{2n-1}\{S(Y,\phi Z)X-S(X,\phi Z)Y+g(Y,\phi Z)QX\\
&-g(X,\phi Z)QY-S(Y,\phi Z)\phi X+S(X,\phi Z)\phi Y\\
&-g(Y,\phi Z)\phi QX+g(X,\phi Z)\phi QY\}\\
&+\frac{r}{2n(2n-1)}\{g(Y,\phi Z)X-g(X,\phi Z)Y\\
&-g(Y,\phi Z)\phi X+g(X,\phi Z)\phi Y\}=0.
\end{aligned} \tag{3.28}$$

Putting $X=\phi X$ and multyplying with W in (3.28), we obtain

$$\begin{aligned}
&g(\phi X,\phi Z)g(Y,W)-g(Y,\phi Z)g(\phi X,W)-g(Y,Z)g(\phi X,\phi W)\\
&-g(\phi X,Z)g(\phi Y,W)-g(\phi X,\phi Z)g(hY,W)+g(Y,\phi Z)g(h\phi X,W)\\
&+g(hY,\phi Z)g(\phi X,W)-g(h\phi X,\phi Z)g(Y,W)-g(Y,Z)g(\phi h\phi X,W)\\
&+g(\phi X,Z)g(\phi hY,W)+g(hY,Z)g(\phi X,\phi W)+g(h\phi X,Z)g(\phi Y,W)\\
&+\frac{-1-\frac{\mu}{2}}{k+1}\{g(hY,\phi Z)g(h\phi X,W)-g(h\phi X,\phi Z)g(hY,W)\\
&-g(hY,Z)g(\phi h\phi X,W)+g(h\phi X,Z)g(\phi hY,W)\}\\
&-\frac{-k+\frac{\mu}{2}}{k+1}\{g(h\phi X,\phi Z)g(\phi hY,W)-g(hY,\phi Z)g(\phi h\phi X,W)\\
&-g(\phi hY,Z)g(h\phi X,W)+g(\phi h\phi X,Z)g(hY,W)\}\\
&-(k+1)\{g(\phi X,\phi Z)\eta(Y)\eta(W)-\eta(Y)\eta(Z)g(\phi X,\phi W)\}\\
&-(\mu-1)\{g(h\phi X,\phi Z)\eta(Y)\eta(W)-\eta(Y)\eta(Z)g(\phi h\phi X,W)\}\\
&-\frac{1}{2n-1}\{S(Y,\phi Z)g(\phi X,W)-S(\phi X,\phi Z)g(Y,W)+g(Y,\phi Z)S(\phi X,W)\\
&-g(\phi X,\phi Z)S(Y,W)-S(Y,\phi Z)g(\phi^2X,W)+S(\phi X,\phi Z)g(\phi Y,W)\\
&+g(Y,\phi Z)S(\phi X,\phi W)-g(\phi X,\phi Z)S(Y,\phi W)\}\\
&+\frac{r}{2n(2n-1)}\{g(Y,\phi Z)g(\phi X,W)-g(\phi X,\phi Z)g(Y,W)\\
&+g(Y,\phi Z)g(\phi X,\phi W)+g(\phi X,\phi Z)g(\phi Y,W)\}=0.
\end{aligned} \tag{3.29}$$

Putting $Y=W=\xi$ in (3.29), we get

$$(-k+\frac{2nk}{2n-1}-\frac{r}{2n(2n-1)})g(\phi X,\phi Z)+\mu g(\phi hX,\phi Z)+\frac{1}{2n-1}S(\phi X,\phi Z)=0. \tag{3.30}$$

Using (2.1) and (2.6) in (3.30), we have

$$\begin{aligned}
&(k - \frac{2nk}{2n-1} + \frac{r}{2n(2n-1)})\{g(X,Z) - \eta(X)\eta(Z)\} \\
&-S(X,Z) + 2nk\eta(X)\eta(Z) \\
&-(\frac{\mu(2n+1) + 4(n-1)}{2n-1})\{\frac{1}{2(n-1)+\mu}S(X,Z) - \frac{(2(1-n)+n\mu)}{2(n-1)+\mu}g(X,Z) \\
&-\frac{(2(n-1)+n(2k-\mu)}{2(n-1)+\mu}\eta(X)\eta(Z)\} = 0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&[4(n-1) + \mu - \frac{2(n-1)(2-\mu) + (2-2n+n\mu)}{2(n-1)+\mu}]S(X,Z) \\
= &[\frac{2nk(2n-1) - 4n^2k + r}{2n} - \frac{2(n-1)(2-\mu) + (2-2n+n\mu)}{2(n-1)+\mu}]g(X,Z) \\
&-[\frac{2nk(2n-1) - 4n^2k + r - 4n^2k}{2n} + \frac{2(n-1)(2-\mu) + (2-2n+2nk-n\mu)}{2(n-1)+\mu}]\eta(X)\eta(Z).
\end{aligned}$$

Hence we have

$$S(X,Z) = \frac{\lambda_2^{'}}{\lambda_1^{'}}g(X,Z) + \frac{\lambda_3^{'}}{\lambda_1^{'}}\eta(X)\eta(Z),$$

where

$$\begin{aligned}
\lambda_1^{'} &= 4(n-1) + \mu - \frac{2(n-1)(2-\mu) + (2-2n+n\mu)}{2(n-1)+\mu}, \\
\lambda_2^{'} &= \frac{2nk(2n-1) - 4n^2k + r}{2n} - \frac{2(n-1)(2-\mu) + (2-2n+n\mu)}{2(n-1)+\mu}, \\
\lambda_3^{'} &= \frac{2nk(2n-1) - 4n^2k + r - 4n^2k}{2n} + \frac{2(n-1)(2-\mu) + (2-2n+2nk-n\mu)}{2(n-1)+\mu}.
\end{aligned}$$

Thus the manifold M is an η-Einstein manifold. Hence we can state the following:

Theorem 3.4. *If M is a $(2n+1)$-dimensional ϕ-Weyl semisymmetric paracontact metric (k, μ)-manifold with $k \neq -1$ then M is an η-Einstein manifold.*

Acknowledgements

The authors are grateful to the referees for their valuable comments and suggestions.

References

[1] Montano, B.C., Erken, I.K., Murathan, C.: *Nullity conditions in paracontact geometry.* Diff. Geom. Appl. **30**, 665–693 (2012).

[2] Cartan, E.: *Sur une classes remarquable d'espaces de Riemann.* Bull. Soc. Math. France. **54**, 214–264 (1926), .

[3] Kaneyuki, S., Williams, F.L.: *Almost paracontact and parahodge structures on manifolds.* Nagoya Math. J. **99**, 173-187 (1985).

[4] Mandal, K., De, U.C.: *Paracontact metric (k, μ)-spaces satisfying certain curvature conditions.* Kyungpook Math. J. **59**, 163–174 (2019).

[5] Shirokov, P. A.: Collected works of geometry. Kazan Univ. Press, Kazan (1966).

[6] Soos, G.: *Über die geodätischen Abbildungen von Riemannaschen Räumen auf projektiv symmetrische Riemannsche Räume.* Acta. Math. Acad. Sci. Hungar. Tom. **9**, 359–361 (1958).

[7] Szabó, Z.I.: *Structure theorems on Riemannian spaces satisfying R(X,Y).R=0, the local version.* J. Diff. Geometry. **17**, 531-582 (1982).

[8] Yano, K., Bochner, S.: Curvature and Betti numbers. Annals of Mathematics Studies, 32, Princeton University Press, (1953).

[9] Yano, K., Kon, M.: Structures on Manifolds. Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, (1984).

[10] Yıldız, A., De, U. C.: *A classification of* (k,μ)*-contact metric manifolds.* Commun Korean Math. Soc. **27**, 327–339 (2012).

[11] Zamkovoy, S.: *Canonical connections on paracontact manifolds.* Ann. Glob. Anal. Geom. **36**, 37-60 (2009).

[12] Zamkovoy, S., Tzanov, V.: *Non-existence of flat paracontact metric structures in dimension greater than or equal to five.* Annuaire Univ. Sofia Fac. Math. Inform. **100**, 27-34 (2011).

Affiliations

Ahmet Yildiz
Address: Inonu University, Education Faculty, Dept. of Mathematics, 44280, Malatya-Turkey.
e-mail: a.yildiz@inonu.edu.tr
ORCID ID: 0000-0002-9799-1781

Uday Chand De
Address: University of Calcutta, Dept. of Pure Mathematics, 700019, West Bengal-India.
e-mail: uc_de@yahoo.com
ORCID ID: 0000-0002-8990-4609