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Simple, Double and Isoclinic Rotations with a Viable Algorithm

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Abstract
The main topic of this study is to investigate rotation matrices in four dimensional Euclidean space in two different ways. The first of these ways is Rodrigues formula and the second is Cayley formula.The most important common point of both formulas is the use of skew symmetric matrices. However, depending on the skew symmetric matrix used, it is possible to classify the rotation matrices by both formulas. Therefore, it is also revealed how the rotation matrices obtained by both formulas are classified as simple, double or isoclinic rotation. Eigenvalues of skew symmetric matrices play the major role in this classification. With the use of all results, it is also seen which skew symmetric matrix is obtained from a given rotation matrix by Rodrigues and Cayley formula, respectively. Finally, an algorithm for classification of rotations is given with the help of the obtained data and explained with an example.

Keywords: *Euclidean 4-Space; Rotation Matrix; Rodrigues Rotation Formula; Cayley Rotation Formula.*

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1. Introduction

Linear transformations appear in many areas of mathematical sciences. Among the linear transformations, orthogonal transformations are the well known ones. These transformations contribute to the solution of many problems in kinematics, physics and computer graphics [1-8]. There are different types of orthogonal transformations: reflections, rotations and their various combinations. In this study, We aim to investigate the second kind of orthogonal transformations i.e. rotations in four dimensional Euclidean space. In the literature, there are many studies dealing with rotation matrices in three dimensional space [10-16] while there are limited studies examining rotation matrices in higher dimensions [6-9]. Mostly, three dimensional rotation matrices have been analyzed with the help of skew symmetric matrices [14-16]. For any given skew-symmetric matrix

$$A = \begin{bmatrix} 0 & -a_{12} & a_{13} \\ a_{12} & 0 & -a_{23} \\ -a_{13} & a_{23} & 0 \end{bmatrix},$$

we obtain exponential of θA as follows:

$$R = Rod(\theta A) = I + \sin(\theta)A + (1 - \cos(\theta))\, A^2$$

with the use of the property $A^3 = -A$. The matrix R corresponds to a rotation matrix where the unit vector $\mathbf{u} = (a_{23}, -a_{13}, a_{12})$ is axis of rotation and θ is the angle of rotation. This is called Rodrigues rotation formula which requires to evaluate exponential of a skew-symmetric matrix θA. An uncomplicated procedure of computing e^A for a given skew-symmetric matrix A is stated in [5]. Additionally, the derivative of a rotation with Rodrigues

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rotation formula is investigated in [22] by exponential coordinates. Cayley rotation formula is an admitted formula to compose a rotation matrix by using any skew-symmetric matrix A. According to this formula,

$$R = Cay(A) = (I + A)(I - A)^{-1}$$

corresponds to a rotation matrix [17-19].

In this study, we investigate 4×4 rotation matrices by two different methods: Rodrigues formula and Cayley formula. Both formulas use the skew symmetric matrices to determine the rotation matrices. However, we classify the rotation matrices as simple, double or isoclinic rotations by depending on the skew symmetric matrices used. It is essential to note that eigenvaluesof the skew symmetric matrices play the major role in this classification. Then, it is also given how to find the skew symmetric matrix which generates a given rotation matrix by Rodrigues and Cayley formula, respectively. Furthermore, an algorithm for classification of rotation matrices is given with the help of the obtained data and explained with an example. This is an example of a numbered first-level heading.

2. Preliminaries

We need the axis of rotation to characterize three-dimensional rotation matrices. In four dimensional case, this situation is very different. Instead of axis of rotations, the concept of plane of rotation is revealed. Plane (or planes) of a given rotation help us describe the rotations in $\mathbb{E}^4$. Simply, the plane of rotation is the plane that transforms itself under the rotation which is not fixed. But all vectors in the plane of rotation are transformed to other vectors in the same plane by the rotation. We can classify the rotations depending on the number of planes of the rotation.

i. Simple Rotation: A rotation, which has one plane of rotation, is called a simple rotation. There should be a another plane which is orthogonal to the plane of rotation. All vectors in this plane are transformed to themselves. In this case, the rotation takes place in the plane of rotation. Consider the rotation matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix},$$

xy-plane is the orthogonal plane to the plane of rotation and zw-plane is the plane of rotation. The points in zw-plane are rotated through an angle α.

ii. Double Rotation: A rotation, which has two planes of rotation, is called a double rotation. The planes of rotation are orthogonal to each other. The rotation is said to take place in both planes of rotation. There are two nonzero angles of rotation, one for each plane of rotation. Points in the first plane rotate through θ, while points in the second plane rotate through β. All other points rotate through an angle between θ and β. For example;

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{bmatrix}$$

is a double rotations where the plane of rotations are xy and zw planes with the angles of rotations β and θ, respectively.

iii. Isoclinic Rotation: Isoclinic rotations are considered as a specific case of the double rotation. In this case, angles of rotations should be equal. But it differs from the double rotation, since the planes of rotation are not unique but identifiable. There are infinitely many number of orthogonal pairs of planes which are considered as planes of rotation [21].

In the following two sections, we will need a special kind of matrix decomposition for skew symmetric matrices. This is a well known matrix decomposition and deeply discussed in the study [3] for n dimensional case. But, we will give the proof of this property by a different approach in four dimensional case. Firstly, it is necessary to express the eigenvalues of 4×4 skew symmetric matrices to understand the decomposition. In following parts of the paper, we denote 4×4 identity and zero matrix by I_4 and 0_4, respectively.

Let $A=(a_{jk})\in M_{4\times 4}(\mathbb{R})$ be any skew-symmetric matrix. Since eigenvalue of skew symmetric matrix A is either 0 or purely imaginary, then the eigenvalues of A take the form

$$\theta_1 i, -\theta_1 i, \theta_2 i, -\theta_2 i \;\text{ where } \theta_1 \geqslant 0,\; \theta_2 \geqslant 0 \tag{1}$$

By direct computations, the eigenvalues satisfy the characteristic equation

$$P_A(\theta)=\theta^4+2a\theta^2+b^2=0$$

where

$$a=\frac{1}{2}\sum_{j<k} a_{jk}^2 \text{ and } b=a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}.$$

Hence

$$\theta_1=\sqrt{a-\sqrt{a^2-b^2}},$$
$$\theta_2=\sqrt{a+\sqrt{a^2-b^2}}.$$

A skew symmetric matrix $A=0$ if and only if $\theta_1=\theta_2=0$.

Lemma 2.1. *Assume that $A\in M_{4\times 4}(\mathbb{R})$ is a nonzero skew-symmetric matrix with the eigenvalues (1).*

i) *If $\theta_1>0, \theta_2>0$ and $\theta_1\neq\theta_2$, then we have*

$$A=\theta_1 A_1+\theta_2 A_2$$

where A_1 and A_2 are skew-symmetric matrices and satisfy the following properties

$$A_1A_2=A_2A_1=0_4,\;\; A_1^3=-A_1 \;\; \textit{and} \;\; A_2^3=-A_2.$$

Moreover, we have the unique expressions of A_1 and A_2 as follows:

$$A_1=\frac{1}{\theta_1(\theta_2^2-\theta_1^2)}\left(\theta_2^2A+A^3\right) \;\textit{and}\; A_2=\frac{1}{\theta_2(\theta_1^2-\theta_2^2)}\left(\theta_1^2A+A^3\right).$$

ii) *If $\theta_1=\theta_2=\theta$, then we obtain*

$$A^2=-\theta^2 I_4.$$

iii) *If $\theta_1=0$ and $\theta_2=\theta$, then we get*

$$A^3=-\theta^2 A.$$

Proof. It is clear that the skew-symmetric matrix $A$is unitary diagonalizable because it a normal matrix. Therefore, we write

$$A=UDU^*$$

where D is a diagonal and U is a unitary matrix.
i) Suppose that $\theta_1>0, \theta_2>0$ and $\theta_1\neq\theta_2$. We have

$$D=diag\{\theta_1 i,\;\; -\theta_1 i,\; \theta_2 i,\;\; -\theta_2 i\}.$$

If we define

$$D_1=diag\{i\;, -i\;, 0, 0\} \text{ and } D_2=diag\{0,0,i,-i\},$$

then we get

$$D=\theta_1 D_1+\theta_2 D_2.$$

Let us define the skew-symmetric matrices $A_1=UD_1U^*$and $A_2=UD_2U^*$. Thus we get

$$A=\theta_1 A_1+\theta_2 A_2. \tag{2}$$

Since $D_1^3 = -D_1$ and $D_2^3 = -D_2$, it follows that $A_1^3 = -A_1$ and $A_2^3 = -A_2$. By using these properties, we have

$$\begin{aligned} A^2 &= (\theta_1 A_1 + \theta_2 A_2)(\theta_1 A_1 + \theta_2 A_2) = \theta_1^2 A_1^2 + \theta_2^2 A_2^2 \\ A^3 &= (\theta_1 A_1 + \theta_2 A_2)(\theta_1^2 A_1^2 + \theta_2^2 A_2^2) = -\theta_1^3 A_1 - \theta_2^3 A_2 \end{aligned} \tag{3}$$

Solving the equations (2) and (3), we find

$$A_1 = \frac{1}{\theta_1(\theta_2^2 - \theta_1^2)}\left(\theta_2^2 A + A^3\right) \text{ and } A_2 = \frac{1}{\theta_2(\theta_1^2 - \theta_2^2)}\left(\theta_1^2 A + A^3\right).$$

ii) Assume that $\theta_1 = \theta_2 = \theta$. Then we get

$$D = diag\{\theta i, -\theta i, \theta i, -\theta i\}$$

is the diagonal matrix. Therefore we have

$$A^T = A^* = U^* \overline{D} U.$$

Then we find

$$AA^T = AA^* = -A^2 = U^* D \overline{D} U.$$

Since $D\overline{D} = \theta^2 I_4$, then we see that

$$A^2 = -\theta^2 I_4.$$

iii) Let $\theta_1 = 0$ and $\theta_2 = \theta$.Then we have $D = diag\{0, 0, \theta i, -\theta i\}$. Therefore we obtain

$$A^T = A^* = U\overline{D}U^*.$$

Then we get

$$-A^3 = AA^T A = UD\overline{D}DU^*.$$

Since $D\overline{D}D = \theta^2 D$, then we see that

$$-A^3 = U(\theta^2 D)U^* = \theta^2 A.$$

Thus, we get

$$A^3 = -\theta^2 A.$$

□

3. Rotations by Rodrigues Formula

Rodrigues rotation formula, which is named after Olinde Rodrigues, gives an efficient method for computing rotation matrices in three dimensional Euclidean space by exponential of skew symmetric matrices. In this part, we will give Rodrigues rotation formula for four dimensional Euclidean space by using Lemma 2.1. We will also use the relation between eigenvalues of a skew symmetric matrix and eigenvalues of its exponential matrix. If $\{\theta_1 i, -\theta_1 i, \theta_2 i, -\theta_2 i\}$ is the set of eigenvalues of a given skew symmetric matrix, then the eigenvalues of matrix exponential $R = e^A$ are $e^{\theta_1 i}, e^{-\theta_1 i}, e^{\theta_2 i}$ and $e^{-\theta_2 i}$. This relation helps us to classify the rotation matrices. For details of matrix exponential, the readers are referred to [5] and [3].

Theorem 3.1. *Suppose that $A = \theta_1 A_1 + \theta_2 A_2 \in M_{4\times 4}(\mathbb{R})$ is a nonzero skew-symmetric matrix with the eigenvalues (1) where $\theta_1 > 0, \theta_2 > 0$ and $\theta_1 \neq \theta_2$.Then*

$$R = Rod(A) = e^A = I_4 + \sin\theta_1 A_1 + (1 - \cos\theta_1)A_1^2 + \sin\theta_2 A_2 + (1 - \cos\theta_2)A_2^2$$

is a rotation matrix.

Proof. We know that $A_1 A_2 = A_2 A_1 = 0_4$ by i) of Lemma 2.1. Therefore, we write

$$e^A = e^{\theta_1 A_1 + \theta_2 A_2} = e^{\theta_1 A_1} e^{\theta_2 A_2}.$$

The property $A_1^3 = -A_1$ implies that

$$\begin{aligned} e^{\theta_1 A_1} &= I_4 + \sum_{k\geq 1} \frac{\theta_1^k A_1^k}{k!} \\ &= I_4 + \left(\frac{\theta_1}{1!} - \frac{\theta_1^3}{3!} + \frac{\theta_1^5}{5!} - \cdots\right) A_1 - \left(-\frac{\theta_1^2}{2!} + \frac{\theta_1^4}{4!} - \frac{\theta_1^6}{6!} - \cdots\right) A_1^2 \\ &= I_4 + \sin\theta_1 A_1 + (1-\cos\theta_1)\, A_1^2. \end{aligned}$$

Similarly, the property $A_2^3 = -A_2$ yields that

$$\begin{aligned} e^{\theta_2 A_2} &= I_4 + \sum_{k\geq 1} \frac{\theta_2^k A_2^k}{k!} \\ &= I_4 + \left(\frac{\theta_2}{1!} - \frac{\theta_2^3}{3!} + \frac{\theta_2^5}{5!} - \cdots\right) A_2 - \left(-\frac{\theta_2^2}{2!} + \frac{\theta_2^4}{4!} - \frac{\theta_2^6}{6!} - \cdots\right) A_2^2 \\ &= I_4 + \sin\theta_2 A_2 + (1-\cos\theta_2)\, A_2^2. \end{aligned}$$

Using the property $A_1A_2 = A_2A_1 = 0_4$, we obtain

$$\begin{aligned} R &= [I_4 + \sin\theta_1 A_1 + (1-\cos\theta_1)\, A_1^2][I_4 + \sin\theta_2 A_2 + (1-\cos\theta_2)\, A_2^2] \\ &= I_4 + \sin\theta_1 A_1 + (1-\cos\theta_1)A_1^2 + \sin\theta_2 A_2 + (1-\cos\theta_2)A_2^2. \end{aligned}$$

Since $A^T = -A$, then we have $e^{A^T}e^A = I_4$. This yields $R^TR = I_4$. We know that $trace(A) = 0$. Therefore, we have

$$\det R = \det e^A = e^{trace(A)} = e^0 = 1.$$

□

Theorem 3.2. *Assume that $A \in M_{4\times 4}(\mathbb{R})$ is a nonzero skew-symmetric matrix with the eigenvalues (1) where $\theta_1 = \theta_2 = \theta$. Then*

$$R = Rod(A) = e^A = (\frac{1}{\theta}\sin\theta)A + (\cos\theta)I_4$$

is a rotation.

Proof. By ii) of Lemma 2.1, we have $A^2 = -\theta^2 I_4$. This implies that

$$\begin{aligned} R = e^A &= I_4 + A - \frac{\theta^2}{2!}I_4 - \frac{\theta^2}{3!}A + \frac{\theta^4}{4!}I_4 + \frac{\theta^4}{5!}A - \frac{\theta^6}{6!}I_4 - \frac{\theta^6}{7!}A + \frac{\theta^8}{8!}I_4 + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} - \cdots\right) I_4 + \frac{1}{\theta}\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) A \\ &= (\cos\theta)I_4 + (\frac{1}{\theta}\sin\theta)A. \end{aligned}$$

Similar to the proof of above theorem, one can easily see that R is a rotation matrix. □

Theorem 3.3. *Let $A \in M_{4\times 4}(\mathbb{R})$ be a skew-symmetric matrix with the eigenvalues (1) where $\theta_1 = 0$, $\theta_2 = \theta > 0$. Then*

$$R = Rod(A) = e^A = I_4 + \frac{\sin\theta}{\theta}A + \frac{1-\cos\theta}{\theta^2}A^2$$

is a rotation matrix.

Proof. By iii) of Lemma 2.1, we have $A^3 = -\theta^2 A$. This implies that

$$\begin{aligned} R = e^A &= I_4 + A + \frac{1}{2!}A^2 - \frac{\theta^2}{3!}A - \frac{\theta^2}{4!}A^2 + \frac{\theta^4}{5!}A + \frac{\theta^4}{6!}A^2 - \frac{\theta^6}{7!}A - \frac{\theta^6}{8!}A^2 + \cdots \\ &= I_4 + \frac{1}{\theta}(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots)A + \frac{1}{\theta^2}\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \frac{\theta^8}{8!} + \cdots\right) A^2 \\ &= I_4 + \frac{\sin\theta}{\theta}A + \frac{1-\cos\theta}{\theta^2}A^2. \end{aligned}$$

Similar to proof of Theorem 3.1 and Theorem 3.2, it is easily seen that R is a rotation matrix. □

Note that, construction of rotation matrices with Rodrigues rotation formula was discussed for four dimensional Euclidean space under the condition $a = 1$ in the study [20].

Remark. Note that $A = \theta(A_1 + A_2)$ in Theorem 3.2 and $A = \theta A_2$ in Theorem 3.3, where A_1, A_2 were defined in Lemma 2.1. In other words, the exponential e^A is continues at 0 with respect to θ.

Theorem 3.4. *Let $R \in SO(4)$ be given where $\{e^{\theta_1 i}, e^{-\theta_1 i}, e^{i\theta_2}, e^{-i\theta_2} : \theta_1, \theta_2 \neq \pi\}$ is the set of eigenvalues of R. Then we obtain the skew-symmetric matrix A such that $R = e^A$ as follows:*

i. *If $\theta_1 \neq 0$, $\theta_2 \neq 0$ and $\theta_1 \neq \theta_2$, then*

$$A = \theta_1 \frac{R^2 - \left(R^2\right)^T - 2\cos\theta_2(R - R^T)}{4\sin\theta_1(\cos\theta_1 - \cos\theta_2)} + \theta_2 \frac{R^2 - \left(R^2\right)^T - 2\cos\theta_1(R - R^T)}{4\sin\theta_2(\cos\theta_2 - \cos\theta_1)}.$$

ii. *If $\theta_1 = \theta_2 = \theta \neq 0$, then*

$$A = \frac{\theta}{2\sin\theta}(R - R^T).$$

iii. *$\theta_1 = 0$, $\theta_2 = \theta \neq 0$ then*

$$A = \frac{\theta}{2\sin\theta}(R - R^T).$$

Proof. **i.** Suppose that $\theta_1 \neq \theta_2$, then we have

$$R = I_4 + \sin\theta_1 A_1 + (1 - \cos\theta_1)A_1^2 + \sin\theta_2 A_2 + (1 - \cos\theta_2)A_2^2.$$

Since we have $A_1^T = -A_1$ and $A_2^T = -A_2$, then we find

$$R^T = I_4 - \sin\theta_1 A_1 + (1 - \cos\theta_1)A_1^2 - \sin\theta_2 A_2 + (1 - \cos\theta_2)A_2^2$$

So, we obtain

$$R - R^T = 2\sin\theta_1 \; A_1 + 2\sin\theta_2 \; A_2. \tag{4}$$

On the other hand, we get

$$R^2 = I_4 + 2\sin\theta_1\cos\theta_1 A_1 + 2\sin^2\theta_1 A_1^2 + 2\sin\theta_2\cos\theta_2 A_2 + 2\sin^2\theta_2 A_2^2$$

and

$$\left(R^2\right)^T = I_4 - 2\sin\theta_1\cos\theta_1 A_1 + 2\sin^2\theta_1 A_1^2 - 2\sin\theta_2\cos\theta_2 A_2 + 2\sin^2\theta_2 A_2^2.$$

Therefore, we find

$$R^2 - \left(R^2\right)^T = 4\sin\theta_1\cos\theta_1 \; A_1 + 4\sin\theta_2\cos\theta_2 \; A_2. \tag{5}$$

If we solve equations (4) and (5), then we obtain

$$A_1 = \frac{R^2 - \left(R^2\right)^T - 2\cos\theta_2(R - R^T)}{4\sin\theta_1(\cos\theta_1 - \cos\theta_2)}.$$

and

$$A_2 = \frac{R^2 - \left(R^2\right)^T - 2\cos\theta_1(R - R^T)}{4\sin\theta_2(\cos\theta_2 - \cos\theta_1)}.$$

Thus, we get

$$A = \theta_1 \frac{R^2 - \left(R^2\right)^T - 2\cos\theta_2(R - R^T)}{4\sin\theta_1(\cos\theta_1 - \cos\theta_2)} + \theta_2 \frac{R^2 - \left(R^2\right)^T - 2\cos\theta_1(R - R^T)}{4\sin\theta_2(\cos\theta_2 - \cos\theta_1)}.$$

ii. Suppose that $\theta_1 = \theta_2 = \theta$, then we get

$$R = (\cos\theta)I_4 + \frac{1}{\theta}(\sin\theta)A.$$

Using the property $A^T = -A$, we get

$$R^T = (\cos\theta)I_4 - \frac{1}{\theta}(\sin\theta)A.$$

We obtain

$$A = \frac{\theta}{2\sin\theta}(R - R^T).$$

iii. Assume that $\theta_1 = 0$, $\theta_2 = \theta$, then we have then

$$R = I_4 + \frac{1}{\theta}(\sin\theta)A + \frac{1}{\theta^2}(1 - \cos\theta)A^2$$

By the property $A^T = -A$, we obtain

$$A = \frac{\theta}{2\sin\theta}(R - R^T).$$

□

Remark 3.1. Let $R \in SO(4)$ be given where $\{e^{\theta_1 i}, e^{-\theta_1 i}, e^{i\pi}, e^{-i\pi}\}$ is the set of eigenvalues of R.
Case:1 If $\theta_1 = 0$, then we have

$$R = I_4 + \frac{2}{\pi^2}A^2.$$

Thus, we have

$$A^2 = \frac{\pi^2}{2}(R - I_4).$$

If we denote the matrix $\frac{\pi^2}{2}(R - I_4)$ by B, then it is needed to find skew-symmetric matrix A^2 such that $A^2 = B$ where B is known and $A^3 = -\pi^2 A$. Since A has the set of eigenvalues $\{0, 0, \pi i, -\pi i\}$, then we obtain that the eigenvalues of B are $0, -\pi^2$. This means that there exists an orthogonal matrix P such that

$$B = PKP^T$$

where $K = diag\{0, 0, -\pi^2, -\pi^2\}$. If we choose

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & -\pi & 0 \end{bmatrix}.$$

then, we have

$$K = E^2$$

So, we get

$$A^2 = PE^2P^T = PEP^TPEP^T.$$

That is

$$A = PEP^T.$$

Case: 2 If $\theta_1 \neq 0, \pi$, then we have

$$R = I_4 + \sin\theta_1 A_1 + (1 - \cos\theta_1)A_1^2 + 2A_2^2. \tag{6}$$

By using the properties $A_1^T = -A_1$ and $A_2^T = -A_2$ we get

$$R - R^T = 2\sin\theta_1 A_1.$$

So, we have

$$A_1 = \frac{1}{2\sin\theta_1}(R - R^T).$$

If we substitute A_1 in equation (6), we get

$$A_2^2 = \frac{1}{2}\left[\frac{1}{2}(R^T + R - 2I_4) + \frac{\cos\theta_1 - 1}{4\sin^2\theta_1}(R^2 - (R^T)^2)\right].$$

Let us denote

$$\frac{1}{2}\left[\frac{1}{2}(R^T+R-2I_4)+\frac{\cos\theta_1-1}{4\sin^2\theta_1}(R^2-(R^T)^2)\right]$$

by C. Similar to case1, it is necessary to obtain skew-symmetric matrix A_2 satisfies $A_2^3=-A_2$ such that $A_2^2=C$ where C is known. We can find A_2 by same method given in case 1.

Case:3 If $\theta_1=\pi$, then we have $R=-I_4$. This case corresponds to the identity transformation. Simply, we can choose

$$A=UDU^*$$

for any unitary matrix U where $D=diag\{\pi i,-\pi i,\pi i,-\pi i\}$. Since $e^D=-I_4$.

4. Rotations by Cayley Formula

Cayley rotation formula was discovered by Arthur Cayley in 1846. This formula also express special orthogonal matrices with skew-symmetric matrices. In this section, we will discuss Cayley rotation formula for four dimensional Euclidean space. In this manner, we will again use Lemma 2.1.

Theorem 4.1. *Suppose that $A=\theta_1A_1+\theta_2A_2\in M_{4\times 4}(\mathbb{R})$ is a nonzero skew-symmetric matrix with the eigenvalues (1) where $\theta_1>0$, $\theta_2>0$ and $\theta_1\neq\theta_2$. Then*

$$R=Cay(A)=(I_4+A)(I_4-A)^{-1}=I_4+\frac{2\theta_1}{1+\theta_1^2}A_1+\frac{2\theta_1^2}{1+\theta_1^2}A_1^2+\frac{2\theta_2}{1+\theta_2^2}A_2+\frac{2\theta_2^2}{1+\theta_2^2}A_2^2$$

is a rotation matrix.

Proof. We need to compute $(I_4-A)^{-1}$ by using the properties in Lemma 2.1. It is easily seen that

$$\begin{aligned}
A^2&=\theta_1^2A_1^2+\theta_2^2A_2^2,\\
A^3&=-\theta_1^3A_1-\theta_2^3A_2,\\
A^4&=-\theta_1^4A_1^2-\theta_2^4A_2^2,\\
A^5&=\theta_1^5A_1+\theta_2^5A_2,\\
&\vdots
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
(I_4-A)^{-1}&=I_4+\sum_{k\geq 1}A^k\\
&=I_4+(\theta_1-\theta_1^3+\theta_1^5-\cdots)A_2+\left(\theta_1^2-\theta_1^4+\theta_1^6-\cdots\right)A_1^2\\
&+(\theta_2-\theta_2^3+\theta_2^5-\cdots)A_2+\left(\theta_2^2-\theta_2^4+\theta_2^6-\cdots\right)A_2^2\\
&=I_4+\frac{\theta_1}{1+\theta_1^2}A_1+\frac{\theta_1^2}{1+\theta_1^2}A_1^2+\frac{\theta_2}{1+\theta_2^2}A_2+\frac{\theta_2^2}{1+\theta_2^2}A_2^2.
\end{aligned}$$

By using the properties of A_1 and A_2 in Lemma 2.1, we obtain

$$\begin{aligned}
R&=(I_4+\theta_1A_1+\theta_2A_2)\left(I_4+\frac{\theta_1}{1+\theta_1^2}A_1+\frac{\theta_1^2}{1+\theta_1^2}A_1^2+\frac{\theta_2}{1+\theta_2^2}A_2+\frac{\theta_2^2}{1+\theta_2^2}A_2^2\right)\\
&=I_4+\frac{2\theta_1}{1+\theta_1^2}A_1+\frac{2\theta_1^2}{1+\theta_1^2}A_1^2+\frac{2\theta_2}{1+\theta_2^2}A_2+\frac{2\theta_2^2}{1+\theta_2^2}A_2^2.
\end{aligned}$$

By using the properties in Lemma 2.1, we get

$$R^T=I_4-\frac{2\theta_1}{1+\theta_1^2}A_1+\frac{2\theta_1^2}{1+\theta_1^2}A_1^2-\frac{2\theta_2}{1+\theta_2^2}A_2+\frac{2\theta_2^2}{1+\theta_2^2}A_2^2.$$

Again with the use of the properties in i) of Lemma 2.1, then we can easily see that $R^TR=I_4$. On the other hand, it is easily seen that $\det(I_4-A)=\det(I_4+A)$. Thus, it is seen that R is a rotation matrix. $\square$

Theorem 4.2. *Assume that $A \in M_{4\times 4}(\mathbb{R})$ is a nonzero skew-symmetric matrix with the eigenvalues (1) where $\theta_1 = \theta_2 = \theta$. Then*

$$R = Cay(A) = (I_4 + A)(I_4 - A)^{-1} = \frac{1-\theta^2}{1+\theta^2} I_4 + \frac{2}{1+\theta^2} A.$$

is a rotation matrix.

Proof. By Lemma 2.1, we have $A^2 = -\theta^2 I_4$. This implies that

$$\begin{aligned}
(I_4 - A)^{-1} &= I_4 + \sum_{k\geq 1} A^k \\
&= I_4 + A - \theta^2 I_4 - \theta^2 A + \theta^4 I_4 + \theta^4 A - \theta^6 I_4 - \theta^6 A + \theta^8 I_4 + \theta^8 A + \cdots \\
&= (1 - \theta^2 + \theta^4 - \theta^6 + \theta^8 \cdots) I_4 + (1 - \theta^2 + \theta^4 - \theta^6 + \theta^8 \cdots) A \\
&= \frac{1}{1+\theta^2}(I_4 + A).
\end{aligned}$$

Then we have

$$\begin{aligned}
R = Cay(A) = (I_4 + A)(I_4 - A)^{-1} &= \frac{1}{1+\theta^2}(I_4 + A)(I_4 + A) \\
&= \frac{1}{1+\theta^2}(I_4 + A^2 + 2A) \\
&= \frac{1}{1+\theta^2}(I_4 - \theta^2 I_4 + 2A) \\
&= \frac{1-\theta^2}{1+\theta^2} I_4 + \frac{2}{1+\theta^2} A.
\end{aligned}$$

Similarly, we can easily see that R is a rotation matrix as in the proof of above theorem. □

Theorem 4.3. *Let $A \in M_{4\times 4}(\mathbb{R})$ be a nonzero skew-symmetric matrix with the eigenvalues (1) where $\theta_1 = 0, \theta_2 = \theta >$. Then*

$$R = Cay(A) = (I_4 + A)(I_4 - A)^{-1} = I_4 + \frac{2}{1+\theta^2} A + \frac{2}{1+\theta^2} A^2$$

is a rotation matrix.

Proof. By Lemma 2.1, we have $A^3 = -\theta^2 A$. This implies that

$$\begin{aligned}
(I_4 - A)^{-1} &= I_4 + \sum_{k\geq 1} A^k \\
&= I_4 + A + A^2 - \theta^2 A - \theta^2 A^2 + \theta^4 A + \theta^4 A^2 - \theta^6 A - \theta^6 A^2 + \theta^8 A + \theta^8 A^2 + \cdots \\
&= I^4 + (1 - \theta^2 + \theta^4 - \theta^6 + \theta^8 \cdots) A + (1 - \theta^2 + \theta^4 - \theta^6 + \theta^8 \cdots) A^2 \\
&= I_4 + \frac{1}{1+\theta^2}(A + A^2).
\end{aligned}$$

Then we have

$$\begin{aligned}
R = Cay(A) = (I_4 + A)(I_4 - A)^{-1} &= (I_4 + A)(I_4 + \frac{1}{1+\theta^2}(A + A^2).) \\
&= I_4 + (\frac{1}{1+\theta^2} + 1 - \frac{\theta^2}{1+\theta^2}) A + (\frac{1}{1+\theta^2} + \frac{1}{1+\theta^2}) A^2 \\
&= I_4 + \frac{2}{1+\theta^2} A + \frac{2}{1+\theta^2} A^2.
\end{aligned}$$

Similarly, we can easily see that R is a rotation matrix as in the proof of above theorem. □

Lemma 4.1. *Assume that $A = \theta_1 A_1 + \theta_2 A_2 \in M_{4\times 4}(\mathbb{R})$ is a nonzero skew-symmetric matrix with the eigenvalues (1). Then the set of eigenvalues $R = Cay(A)$ is obtained as follows*

$$\{\frac{(1+\theta_1 i)^2}{1+\theta_1^2}, \frac{(1-\theta_1 i)^2}{1+\theta_1^2}, \frac{(1+\theta_2 i)^2}{1+\theta_2^2}, \frac{(1-\theta_2 i)^2}{1+\theta_2^2}\}.$$

Proof. It is clear that the skew-symmetric matrix $A$is unitary diagonalizable because it is a normal matrix. We write

$$A = UDU^*$$

where $D = diag\{\theta_1 i, -\theta_1 i, \theta_2 i, -\theta_2 i\}$ and U is a unitary matrix. Therefore, we obtain

$$R = (I_4 + UDU^*)(I_4 - UDU^*)^{-1} = U(I + D)(I - D)^{-1}U^*.$$

Since we have

$$(I + D)(I - D)^{-1} = diag(\frac{(1+\theta_1 i)^2}{1+\theta_1^2}, \frac{(1-\theta_1 i)^2}{1+\theta_1^2}, \frac{(1+\theta_2 i)^2}{1+\theta_2^2}, \frac{(1-\theta_2 i)^2}{1+\theta_2^2}),$$

then R is also diagonalizable with the set of eigenvalues

$$\left\{\frac{(1+\theta_1 i)^2}{1+\theta_1^2}, \frac{(1-\theta_1 i)^2}{1+\theta_1^2}, \frac{(1+\theta_2 i)^2}{1+\theta_2^2}, \frac{(1-\theta_2 i)^2}{1+\theta_2^2}\right\}.$$

□

Theorem 4.4. *Let $R \in SO(4)$ with the set of eigenvalues*

$$\{\frac{(1+\theta_1 i)^2}{1+\theta_1^2}, \frac{(1-\theta_1 i)^2}{1+\theta_1^2}, \frac{(1+\theta_2 i)^2}{1+\theta_2^2}, \frac{(1-\theta_2 i)^2}{1+\theta_2^2}\}.$$

Then we can find the skew-symmetric matrix A such that $R = Cay(A)$ as follows:

i. *if $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_1 \neq \theta_2$, then*

$$\begin{aligned} A = {} & \frac{(1+\theta_1^2)^2(1+\theta_2^2)}{16(\theta_2^2-\theta_1^2)}\left(R^2 - (R^2)^T - \frac{2(1-\theta_2^2)}{1+\theta_2^2}(R - R^T)\right) \\ & + \frac{(1+\theta_2^2)^2(1+\theta_1^2)}{16(\theta_1^2-\theta_2^2)}\left(R^2 - (R^2)^T - \frac{2(1-\theta_1^2)}{1+\theta_1^2}(R - R^T)\right), \end{aligned}$$

ii. *if $\theta_1 = \theta_2 = \theta$, then*

$$A = \frac{1+\theta^2}{4}(R - R^T)$$

iii. *if $\theta_1 = 0$, $\theta_2 = \theta > 0$, then*

$$A = \frac{1+\theta^2}{4}(R - R^T).$$

Proof. **i.** If $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_1 \neq \theta_2$, then we have

$$R = I + \frac{2\theta_1}{1+\theta_1^2}A_1 + \frac{2\theta_1^2}{1+\theta_1^2}A_1^2 + \frac{2\theta_2}{1+\theta_2^2}A_2 + \frac{2\theta_2^2}{1+\theta_2^2}A_2^2.$$

Since $A_1^T = -A_1$ and $A_2^T = -A_2$, then we find

$$R - R^T = \frac{4\theta_1}{1+\theta_1^2}A_1 + \frac{4\theta_2}{1+\theta_2^2}A_2, \tag{7}$$

$$R^2 - (R^2)^T = \frac{8\theta_1(1-\theta_1^2)}{(1+\theta_1^2)^2}A_1 + \frac{8\theta_2(1-\theta_2^2)}{(1+\theta_2^2)^2}A_2. \tag{8}$$

If we solve the equations (7) and (8), then we obtain

$$\begin{aligned} A_1 &= \frac{(1+\theta_1^2)^2(1+\theta_2^2)}{16\theta_1(\theta_2^2-\theta_1^2)}\left(R^2 - (R^2)^T - \frac{2(1-\theta_2^2)}{1+\theta_2^2}(R - R^T)\right) \text{ and} \\ A_2 &= \frac{(1+\theta_2^2)^2(1+\theta_1^2)}{16\theta_2(\theta_1^2-\theta_2^2)}\left(R^2 - (R^2)^T - \frac{2(1-\theta_1^2)}{1+\theta_1^2}(R - R^T)\right). \end{aligned}$$

That is

$$\begin{aligned} A &= \theta_1 A_1 + \theta_2 A_2 \\ &= \frac{(1+\theta_1^2)^2(1+\theta_2^2)}{16(\theta_2^2-\theta_1^2)}\left(R^2-(R^2)^T-\frac{2(1-\theta_2^2)}{1+\theta_2^2}(R-R^T)\right) \\ &+ \frac{(1+\theta_2^2)^2(1+\theta_1^2)}{16(\theta_1^2-\theta_2^2)}\left(R^2-(R^2)^T-\frac{2(1-\theta_1^2)}{1+\theta_1^2}(R-R^T)\right). \end{aligned}$$

The proof of ii. and iii. can be done by using Theorem 4.2 and Theorem 4.3. □

5. Classifications of Rotations

By using matrix decomposition of skew-symmetric matrices, we have given two different methods to generating rotation matrices with skew-symmetric matrices in $\mathbb{E}^4$ in previous sections. One of them is called Rodrigues rotation formula and the other one is called Cayley rotation formula. The explicit form of the rotation matrices, which are generated by these formulas, are obtained as follows;

$$R = e^A = I_4 + \sum_{k=1}^{2} \sin\theta_k \, A_k + (1-\cos\theta_k)A_k^2,$$

$$R = (I_4+A)(I_4-A)^{-1} = I_4 + \sum_{k,j=1}^{2} \frac{2\theta_k^j}{1+\theta_k^2} A_k^j,$$

with the set of eigenvalues

$$\left\{e^{\theta_1 i}, e^{-\theta_1 i}, e^{\theta_2 i}, e^{-\theta_2 i}\right\},$$

$$\left\{\frac{(1+\theta_1 i)^2}{1+\theta_1^2}, \frac{(1-\theta_1 i)^2}{1+\theta_1^2}, \frac{(1+\theta_2 i)^2}{1+\theta_2^2}, \frac{(1-\theta_2 i)^2}{1+\theta_2^2}\right\},$$

respectively. As a conclusion of this result, we classify rotations according to the values of θ_1 and θ_2 as follows:

i. If $\theta_j = 0$ and $\theta_k \neq 0$ $(j \neq k)$, then formulas generates simple rotations;

ii. If θ_1, θ_2 are nonzero and $\theta_1 \neq \theta_2$, then formulas generates double rotations;

iii. If θ_1, θ_2 are nonzero and $\theta_1 = \theta_2$, then formulas generates isoclinic rotations.

6. Algorithm and application

In this part, we will give an algorithm to generate rotations by Rodrigues and Cayley formulas with mathematica. The algorithm starts to work by giving entries of the upper triangular part of the 4×4 skew-symmetric matrix. Firstly, it finds the values of θ_1 and θ_2 and gives the type of rotations. Then, it obtains the skew-symmetric matrices A_1 and A_2 and generates two different rotation matrices, namely R_{rod} and R_{cay}. This simple algorithm is given as follows:

```
>f[i_, j_] = Which[i == j, 0, i < j, a[i, j], i > j, -a[j, i]];

>A = Table[f[i, j], {i, 4}, {j, 4}];

>A // MatrixForm

>θ1 = Min[SingularValueList[A, 4]]

>θ2 = Max[SingularValueList[A, 4]]

>If[θ1 == 0, Print[This is a simple rotation],
```

If[$\theta_1 == \theta_2$, Print[This is an isoclinic rotation], Print[This is a double rotation]]]

>A$_1$ = If[θ_1 == 0, DiagonalMatrix[$\{0, 0, 0, 0\}$], $\frac{1}{\theta_1(\theta_2^2-\theta_1^2)}$($\theta_2^2$A+MatrixPower[A,3])];

>A$_2$ = If[θ_1 == 0, $\frac{1}{\theta_2}$A, $\frac{1}{\theta_2(\theta_1^2-\theta_2^2)}$($\theta_1^2$A+MatrixPower[A,3])];

>A$_1$ // MatrixForm

>A$_2$ // MatrixForm

>R$_{rod}$=[IdentityMatrix[4]+Sin[θ_1]A$_1$+(1-Cos[θ_1])MatrixPower[A$_1$,2]+Sin[θ_2]A$_2$+(1-Cos[θ_2])MatrixPower[A$_2$,2])];

>R$_{rod}$ // MatrixForm

>R$_{cay}$=[IdentityMatrix[4]+($\frac{2\theta_1}{1+\theta_1^2}$)A$_1$+($\frac{2\theta_1^2}{1+\theta_1^2}$)MatrixPower[A$_1$,2]

+($\frac{2\theta_2}{1+\theta_2^2}$)A$_2$+($\frac{2\theta_2}{1+\theta_2^2}$)MatrixPower[A$_2$,2])];

>R$_{cay}$ // MatrixForm

Let us give a numerical example as an application. For given values $a_{12} = 1$, $a_{13} = -1$, $a_{14} = 1$, $a_{23} = 1$, $a_{24} = 0$ and $a_{34} = 1$, we obtain the following skew-symmetric matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

where the set of eigenvalues of A is $\{i, -i, 2i, -2i\}$ i.e. $\theta_1 = 1$ and $\theta_2 = 2$. We find

$$A_1 = \frac{1}{3}\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ -1 & -1 & 0 & 1 \\ -1 & -2 & -1 & 0 \end{bmatrix} \text{ and } A_2 = \frac{1}{3}\begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}.$$

Here, we easily see that skew-symmetric matrices A_1 and A_2 satisfying the properties given in (1). The rotation matrix, which is generated by Cayley rotation formula, is given as follows:

$$R_{cay} = \frac{1}{5}\begin{bmatrix} -2 & 4 & -1 & 2 \\ -2 & -1 & 4 & 2 \\ 1 & -2 & -2 & 4 \\ -4 & -2 & -2 & -1 \end{bmatrix}.$$

The eigenvectors of the rotation matrix corresponding to the eigenvalues $\eta_1 = i$, $\eta_2 = -i$, $\eta_3 = -\frac{3}{5} + \frac{4}{5}i$, $\eta_4 = \frac{3}{5} + \frac{4}{5}i$ are found as

$$\begin{aligned} \mathbf{v}_1 &= (-1-i,\ 2i,\ 1-i,\ 2), \\ \mathbf{v}_2 &= (-1+i,\ 2i, 1+i,\ 2), \\ \mathbf{v}_3 &= (1-i,\ i,\ -1-i,\ 1), \\ \mathbf{v}_4 &= (1+i,\ -i,\ -1-i,\ 1), \end{aligned}$$

respectively. Thus, the plane of rotations are

$$P_1 = span\{(-1,0,1,2),(1,2,-1,0)\},$$
$$P_2 = span\{(1,0,-1,1),(-1,1,-1,0)\}.$$

Note that the vectors, which are lying on the planes P_1 and P_2, transform to the other vectors lying on the planes P_1 and P_2 by the rotation, respectively. Here rotation angles are $\theta_1 = 90^\circ + 2k\pi$ and $\theta_2 = 127^\circ + 2k\pi$. Notice that, since θ_1, θ_2 are nonzero and $\theta_1 \neq \theta_2$, then there are two plane of rotations, the rotation matrix represent a double rotation.

References

[1] Gallier, J.H.: Geometric Methods and Applications, For Computer Science and Engineering. Texts in Applied Mathematics (2011).

[2] Gallier, J. H.: Notes on Differential Geometry and Lie Groups. University of Pennsylvania (2014).

[3] Gallier, J. H. and Xu, D.: *Computing Exponentials of Skew Symmetric Matrices and Logarithms of Orthogonal Matrices.* International Journal of Robotics and Automation, **18** (1), 10-20 (2000).

[4] Mebius, J. E.: *Derivation of Euler-Rodrigues Formula for three-dimensional rotations from the general formula for four dimensional rotations.* Preprint arxiv: math/0701759 (2007).

[5] Serre, D.: Matrices: Theory and Applications, Graduate text in Mathematics. Springer - Verlag, London (2002).

[6] Jadczyk, A. and Szulga, J.: *A Comment on "On the Rotation Matrix in Minkowski Space-time" by Özdemir and Erdoğdu.* Reports on Mathematical Physics. **74** (1), 39-44 (2014).

[7] Bükçü, B.: *On the Rotation Matrices in Semi-Euclidean Space.* Commun. Fac. Sci. Univ. Ank. Series A. **1** (55), 7-13 (2006).

[8] Özdemir, M., Erdoğdu, M.: *On the Rotation Matrix in Minkowski Space-time.* Reports on Mathematical Physics. **74** (1), 27-38 (2014).

[9] Erdoğdu, M., Özdemir, M. *Cayley formula in Minkowski space-time.* International Journal of Geometric Methods in Modern Physics. **12** (5), 1-11 (2015).

[10] Schmidt, J., Nieman, H.: *Using Quaternions for Parameterizing 3-D Rotations in Unconstrained Nonlinear Optimization.* Vision Modeling and Visualization. **1**, 399-406 (2001).

[11] Özdemir, M., Ergin, A.A.: *Rotations with unit timelike quaternions in Minkowski 3-space.* Journal of Geometry and Physics. **56** (2), 322-336 (2006).

[12] Özdemir, M., Erdoğdu, M., Şimşek, H.: *On the Eigenvalues and Eigenvectors of a Lorentzian Rotation Matrix by Using Split Quaternions.* Adv. Appl. Clifford Algebras. **24**(1), 179-192 (2014).

[13] Weiner, J. L., Wilkens, G.R.: *Quaternions and Rotations in* $\mathbb{E}^4$. The American Mathematical Monthly. **112** (1), 69-76 (2005).

[14] Murray, M.R., Li, Z., Sastry, S.S.: A Mathematical Introduction to Robotic Manipulation. Boca Raton F.L. CRC Press. (1994).

[15] Kula, L., Karacan, M.K. Yaylı, Y.: *Formulas for the Exponential of Semi Symmetric Matrix of order* 4. Mathematical and Computational Applications. **10**(1), 99-104 (2005).

[16] Politi, T.: *A Formula for the Exponential of a Real Skew-Symmetric Matrix of Order 4.* BIT Numerical Mathematics. **41** (4), 842-845 (2001).

[17] Selig, J.M.: *Cayley Maps for SE(3).* 12th International Federation for the Promotion of Mechanism and Machine Science World Congress. London South Bank University, Besancon. 18-21 (2007).

[18] Norris, A. N.: *Euler-Rodrigues and Cayley Formulae for Rotation of Elasticity Tensors.* Mathematics and Mechanics of Solids. **13** (16), 465-498 (2008).

[19] Özkaldı, S., Gündoğan, H.:*Cayley Formula, Euler Parameters and Rotations in 3- Dimensional Lorentzian Space.* Advances in Applied Clifford Algebras.**20** (2), 367-377 (2010).

[20] Eberly, D.: Constructing Rotation Matrices Using Power Series. Geometric Tools LLC. (2007).

[21] Pertti, L.: Clifford algebras and spinors. Cambridge University Press. (2001).

[22] Gallego, G., Yezzi, A.: *A compact Formula for Derivative of a 3-D Rotation in Exponential Coordinates.* Journal of Mathematical Imaging and Vision. **51** (3), 378-384 (2015).

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