MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
8 (1) 134-141 (2020) ©MSAEN
HTTPS://DOI.ORG/10.36753/MATHENOT.650271

Some Applications of the (p, q)-Lucas Polynomials to the bi-univalent Function Class Σ

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Abstract
In this present investigation, based on the (p, q)-Lucas polynomials, we want to build a bridge between the Theory of Geometric Functions and that of Special Functions, which are usually considered as very different fields.

Keywords: *(p, q)-Lucas polynomials; coefficient bounds; bi-univalent functions; Fekete-Szegö inequalities.*

AMS Subject Classification (2020): *Primary: 30C45; Secondary: 05A15; 33D15.*

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1. Introduction, preliminaries and known results

In modern science there is a huge interest in the theory and application of the the Dickson polynomials, Chebyshev polynomials, Fibonacci polynomials, Lucas polynomials and Lucas-Lehmer polynomials. These polynomials play a fundamental role in mathematics, and have numerous important applications in combinatorics, number theory, numerical analysis, etc. Therefore, they have been studied extensively, and various generalizations of them have been introduced (see, for example, [9, 12, 13, 18, 19]).

The classical Lucas polynomials $L_n(x)$ studied by M. Bicknell in 1970 are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x) \quad (n \geq 2),$$

with the initial condition $L_0(x) = 2$ and $L_1(x) = x$.

Since the above classical Lucas polynomials appeared, some authors have explored their different extensions. For example, the (p, q)-Lucas polynomials with some properties introduced by Lee and Aşçı [7] as follows.

Definition 1.1. (see [7]) Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The (p, q)-Lucas polynomials $L_{p,q,n}(x)$ are defined by the recurrence relation

$$L_{p,q,n}(x) = p(x)L_{p,q,n-1}(x) + q(x)L_{p,q,n-2}(x) \quad (n \geq 2),$$

from which the first few Lucas polynomials can be found as

$$\begin{aligned} L_{p,q,0}(x) &= 2, \\ L_{p,q,1}(x) &= p(x), \\ L_{p,q,2}(x) &= p^2(x) + 2q(x), \\ L_{p,q,3}(x) &= p^3(x) + 3p(x)q(x), \\ &\vdots \end{aligned} \tag{1.1}$$

Received : *24–01–2019,* ***Accepted :*** *27–09–2019*

For the special cases of $p(x)$ and $q(x)$, we can get the polynomials given in Table 1.

Table 1: Special cases of the $L_{p,q,n}(x)$ with given initial conditions are given.

$p(x)$	$q(x)$	$L_{p,q,n}(x)$
x	1	Lucas polynomials $L_n(x)$
2x	1	Pell-Lucas polynomials $D_n(x)$
1	2x	Jacobsthal-Lucas polynomials $j_n(x)$
3x	-2	Fermat-Lucas polynomials $f_n(x)$
2x	-1	Chebyshev polynomials first kind $T_n(x)$

Theorem 1.1. *(see [7]) Let $G_{\{L_{p,q,n}(x)\}}(z)$ be the generating function of the (p, q)-Lucas polynomial sequence $L_{p,q,n}(x)$. Then*

$$G_{\{L_{p,q,n}(x)\}}(z) = \sum_{n=0}^{\infty} L_{p,q,n}(x) z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}.$$

Let A be the class of functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{1.2}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized under the conditions

$$f(0) = 0,$$

$$f'(0) = 1.$$

Further, by S we shall denote the class of all functions in A which are univalent in Δ.

Here, we recall some definitions and concepts of classes of analytic functions. Denote by S^* the subclass of S of starlike functions, so that $f \in S^*$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \Delta).$$

Let f be given by (1.2). Then $f \in RT$ if it satisfies the inequality

$$\Re\left(f'(z)\right) > 0 \quad (z \in \Delta).$$

The subclass RT was studied systematically by MacGregor [10] who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

For $\alpha > 0$, let $B(\alpha)$ denote the class of Bazilevič functions defined in the open unit disk Δ such that

$$\Re\left(f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right) > 0 \quad (z \in \Delta).$$

This class of functions was studied first by Singh [15] and considered subsequently by London and Thomas [8].

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Given functions $f, g \in A$, f is subordinate to g if there exists a Schwarz function $w \in \Lambda$, where

$$\Lambda = \{w : w(0) = 0,\ |w(z)| < 1,\ z \in \Delta\},$$

such that

$$f(z) = g(w(z)) \qquad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \text{ or } f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\Delta) \subset g(\Delta).$$

According to the Koebe-One Quarter Theorem [4], it ensures that the image of Δ under every univalent function $f \in A$ contains a disk of radius $1/4$. Thus every univalent function $f \in A$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f\left(f^{-1}(w)\right) = w \ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$g(w) = f^{-1}(w) = w\ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.3}$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1.2). For a brief history and interesting examples in the class Σ, see [16] (see also [1, 3, 6, 11, 14]).

We want to remark explicitly that, in our article, by using the $L_{p,q,n}(x)$, functions, our methodology builds a bridge, to our knowledge not previously well known, between the Theory of Geometric Functions and that of Special Functions, which are usually considered as very different fields.

Definition 1.2. A function $f \in \Sigma$ is said to be in the class

$$B_\Sigma(\beta; x) \quad (0 \leq \beta \leq 1,\ z, w \in \Delta)$$

if the following subordinations are satisfied:

$$\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) \prec G_{\{L_{p,q,n}(x)\}}(z) - 1$$

and

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) \prec G_{\{L_{p,q,n}(x)\}}(w) - 1,$$

where the function g is given by (1.3).

Remark 1.1. Here, and in what follows, the argument $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$.

Example 1.1. Upon setting $\beta = 0$ in Definition 1.2, it is readily seen that a function $f \in \Sigma$ is in the class

$$B_\Sigma(x) \quad (z, w \in \Delta)$$

if the following conditions are satisfied:

$$\frac{zf'(z)}{f(z)} \prec G_{\{L_{p,q,n}(x)\}}(z) - 1$$

$$\frac{wg'(w)}{g(w)} \prec G_{\{L_{p,q,n}(x)\}}(w) - 1,$$

where the function g is given by (1.3).

Example 1.2. Upon setting $\beta = 1$ in Definition 1.2, it is readily seen that a function $f \in \Sigma$ is in the class

$$H_\Sigma(x) \quad (z, w \in \Delta)$$

if the following conditions are satisfied:

$$f'(z) \prec G_{\{L_{p,q,n}(x)\}}(z) - 1$$

and

$$g'(w) \prec G_{\{L_{p,q,n}(x)\}}(w) - 1,$$

where the function g is given by (1.3).

2. Coefficient Estimates

We begin this section by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $B_\Sigma(\beta; x)$ proposed by Definition 1.2.

Theorem 2.1. *Let f given by (1.2) be in the class $B_\Sigma(\beta; x)$. Then*

$$|a_2| \leq \frac{|p(x)|\sqrt{2|p(x)|}}{\sqrt{(\beta+1)|\beta p^2(x) + 4(\beta+1)q(x)|}}$$

and

$$|a_3| \leq \frac{p^2(x)}{(\beta+1)^2} + \frac{|p(x)|}{\beta+2}.$$

Proof. Let $f \in B_\Sigma(\beta; x)$. From Definition 1.2, for some analytic functions Φ, Ψ such that

$$\Phi(0) = 0, \ \ |\Phi(z)| = \left|t_1 z + t_2 z^2 + t_3 z^3 + \cdots\right| < 1 \ \ (z \in \Delta),$$

$$|t_k| \leq 1 \quad (k \in \mathbb{N})$$

and

$$\Psi(0) = 0, \ \ |\Psi(w)| = \left|s_1 w + s_2 w^2 + s_3 w^3 + \cdots\right| < 1 \ \ (w \in \Delta),$$

$$|s_k| \leq 1 \quad (k \in \mathbb{N}),$$

we can write

$$\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) = G_{\{L_{p,q,n}(x)\}}(\Phi(z)) - 1$$

and

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) = G_{\{L_{p,q,n}(x)\}}(\Psi(w)) - 1,$$

or equivalently

$$\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) = -1 + L_{p,q,0}(x) + L_{p,q,1}(x)\Phi(z) + L_{p,q,2}(x)\Phi^2(z) + \cdots \tag{2.1}$$

and

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) = -1 + L_{p,q,0}(x) + L_{p,q,1}(x)\Psi(w) + L_{p,q,2}(x)\Psi^2(w) + \cdots. \tag{2.2}$$

From the equalities (2.1) and (2.2), we obtain that

$$\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) = 1 + L_{p,q,1}(x)t_1 z + \left[L_{p,q,1}(x)t_2 + L_{p,q,2}(x)t_1^2\right] z^2 + \cdots \tag{2.3}$$

and

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) = 1 + L_{p,q,1}(x)s_1 w + \left[L_{p,q,1}(x)s_2 + L_{p,q,2}(x)s_1^2\right] w^2 + \cdots. \tag{2.4}$$

Thus, upon comparing the corresponding coefficients in (2.3) and (2.4), we have

$$(\beta+1)a_2 = L_{p,q,1}(x)t_1, \tag{2.5}$$

$$\frac{(\beta-1)(\beta+2)}{2}a_2^2 + (\beta+2)a_3 = L_{p,q,1}(x)t_2 + L_{p,q,2}(x)t_1^2, \tag{2.6}$$

$$-(\beta+1)a_2 = L_{p,q,1}(x)s_1 \tag{2.7}$$

and

$$\frac{(\beta+2)(\beta+3)}{2}a_2^2 - (\beta+2)a_3 = L_{p,q,1}(x)s_2 + L_{p,q,2}(x)s_1^2. \tag{2.8}$$

From the equations (2.5) and (2.7), we can easily see that

$$t_1 = -s_1, \tag{2.9}$$

$$2(\beta+1)^2 a_2^2 = L_{p,q,1}^2(x)\left(t_1^2+s_1^2\right). \tag{2.10}$$

If we add (2.6) to (2.8), we get

$$(\beta+1)(\beta+2)a_2^2 = L_{p,q,1}(x)\left(t_2+s_2\right) + L_{p,q,2}(x)\left(t_1^2+s_1^2\right). \tag{2.11}$$

By using (2.10) in the equality (2.11), we have

$$\left[(\beta+1)(\beta+2)L_{p,q,1}^2(x) - 2(\beta+1)^2 L_{p,q,2}(x)\right] a_2^2 = L_{p,q,1}^3(x)\left(t_2+s_2\right). \tag{2.12}$$

which gives

$$|a_2| \leq \frac{|p(x)|\sqrt{2\,|p(x)|}}{\sqrt{(\beta+1)\left|\beta p^2(x) + 4(\beta+1)q(x)\right|}}.$$

Moreover, if we subtract (2.8) from (2.6), we obtain

$$2(\beta+2)(a_3-a_2^2) = L_{p,q,1}(x)\left(t_2-s_2\right) + L_{p,q,2}(x)\left(t_1^2-s_1^2\right). \tag{2.13}$$

Then, in view of (2.9) and (2.10), (2.13) becomes

$$a_3 = \frac{L_{p,q,1}^2(x)}{2(\beta+1)^2}\left(t_1^2+s_1^2\right) + \frac{L_{p,q,1}(x)}{2(\beta+2)}\left(t_2-s_2\right).$$

Then, with the help of (1.1), we finally deduce

$$|a_3| \leq \frac{p^2(x)}{(\beta+1)^2} + \frac{|p(x)|}{\beta+2}.$$

□

Corollary 2.1. *Let f given by (1.2) be in the class $B_\Sigma(x)$. Then*

$$|a_2| \leq |p(x)|\sqrt{\left|\frac{p(x)}{2q(x)}\right|}$$

and

$$|a_3| \leq p^2(x) + \frac{|p(x)|}{2}.$$

Corollary 2.2. *Let f given by (1.2) be in the class $H_\Sigma(x)$. Then*

$$|a_2| \leq \frac{|p(x)|\sqrt{|p(x)|}}{\sqrt{|p^2(x)+8q(x)|}}$$

and

$$|a_3| \leq \frac{p^2(x)}{4} + \frac{|p(x)|}{3}.$$

3. Fekete-Szegö Problem

The classical Fekete-Szegö inequality, presented by means of Loewner's method, for the coefficients of $f \in S$ is

$$\left|a_3 - \vartheta a_2^2\right| \leq 1 + 2\exp(-2\vartheta/(1-\vartheta)) \text{ for } \vartheta \in [0,1).$$

As $\vartheta \to 1^-$, we have the elementary inequality $\left|a_3 - a_2^2\right| \leq 1$. Moreover, the coefficient functional

$$F_\vartheta(f) = a_3 - \vartheta a_2^2$$

on the normalized analytic functions f in the unit disk Δ plays an important role in function theory. The problem of maximizing the absolute value of the functional $F_\vartheta(f)$ is called the Fekete-Szegö problem, see [5]. Many other recent works on the Fekete-Szegö problem include, for example, [2, 17, 20].

In this section, we aim to provide Fekete-Szegö inequalities for functions in the class $B_\Sigma(\beta;x)$. These inequalities are given in the following theorem.

Theorem 3.1. *Let f given by (1.2) be in the class $B_\Sigma(\beta;x)$ and $\vartheta\in\mathbb{R}$. Then*

$$\left|a_3-\vartheta a_2^2\right|\le\begin{cases}\dfrac{|p(x)|}{\beta+2}, & |\vartheta-1|\le\dfrac{\beta+1}{2(\beta+2)}\left|\beta+4(\beta+1)\dfrac{q(x)}{p^2(x)}\right| \\[2ex] \dfrac{2\,|1-\vartheta|\,\left|p^3(x)\right|}{(\beta+1)\left|\beta p^2(x)+4(\beta+1)q(x)\right|}, & |\vartheta-1|\ge\dfrac{\beta+1}{2(\beta+2)}\left|\beta+4(\beta+1)\dfrac{q(x)}{p^2(x)}\right|\end{cases}.$$

Proof. From (2.12) and (2.13), we have

$$\begin{aligned}a_3-\vartheta a_2^2 &= \frac{L_{p,q,1}^3(x)\,(1-\vartheta)\,(t_2+s_2)}{(\beta+1)(\beta+2)L_{p,q,1}^2(x)-2(\beta+1)^2L_{p,q,2}(x)} \\ &+\frac{L_{p,q,1}(x)\,(t_2-s_2)}{2(\beta+2)} \\ &= L_{p,q,1}(x)\left[\left(h(\vartheta,x)+\frac{1}{2(\beta+2)}\right)t_2+\left(h(\vartheta,x)-\frac{1}{2(\beta+2)}\right)s_2\right],\end{aligned}$$

where

$$h(\vartheta,x)=\frac{L_{p,q,1}^2(x)\,(1-\vartheta)}{(\beta+1)(\beta+2)L_{p,q,1}^2(x)-2(\beta+1)^2L_{p,q,2}(x)}.$$

Then, in view of (1.1), we conclude that

$$\left|a_3-\vartheta a_2^2\right|\le\begin{cases}\dfrac{|p(x)|}{\beta+2}, & 0\le|h(\vartheta,x)|\le\dfrac{1}{2(\beta+2)} \\[2ex] 2\,|p(x)|\,|h(\vartheta,x)|, & |h(\vartheta,x)|\ge\dfrac{1}{2(\beta+2)}\end{cases}.$$

□

Corollary 3.1. *Let f given by (1.2) be in the class $B_\Sigma(x)$ and $\vartheta\in\mathbb{R}$. Then*

$$\left|a_3-\vartheta a_2^2\right|\le\begin{cases}\dfrac{|p(x)|}{2}, & |\vartheta-1|\le\dfrac{|q(x)|}{p^2(x)} \\[2ex] \dfrac{|1-\vartheta|\,\left|p^3(x)\right|}{2\,|q(x)|}, & |\vartheta-1|\ge\dfrac{|q(x)|}{p^2(x)}\end{cases}.$$

Corollary 3.2. *Let f given by (1.2) be in the class $H_\Sigma(x)$ and $\vartheta\in\mathbb{R}$. Then*

$$\left|a_3-\vartheta a_2^2\right|\le\begin{cases}\dfrac{|p(x)|}{3}, & |\vartheta-1|\le\dfrac{\left|p^2(x)+8q(x)\right|}{3p^2(x)} \\[2ex] \dfrac{|1-\vartheta|\,\left|p^3(x)\right|}{\left|p^2(x)+8q(x)\right|}, & |\vartheta-1|\ge\dfrac{\left|p^2(x)+8q(x)\right|}{3p^2(x)}\end{cases}.$$

If we choose $\vartheta=1$, we get the next corollaries.

Corollary 3.3. *If $f\in B_\Sigma(\beta;x)$, then*

$$\left|a_3-a_2^2\right|\le\frac{|p(x)|}{\beta+2}.$$

Corollary 3.4. *If $f\in B_\Sigma(x)$, then*

$$\left|a_3-a_2^2\right|\le\frac{|p(x)|}{2}.$$

Corollary 3.5. *If $f\in H_\Sigma(x)$, then*

$$\left|a_3-a_2^2\right|\le\frac{|p(x)|}{3}.$$

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