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On the Coherent Systems Subject to Marshall-Olkin Type Shocks

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Abstract
Coherent systems and Marshall-Olkin run shock models are combined. Coherent systems consisting of n components receive some kind of shocks from $n+1$ different sources similar to Marshall-Olkin type. More precisely, when the component j receive k consecutive fatal shocks from the source j or k consecutive fatal shocks from the source $n+1$, it fails, $j = 1, \ldots, n$. When the interarrival time of shocks has phase-type distribution, reliability, mean time to failure (MTTF) and mean residual life (MRL) function of the coherent systems are studied. Numerical examples and graphical representations are provided.

Keywords: *Phase-type distributions; Coherent system; Marshall-Olkin distribution; Reliability; Mean residual lifetime.*

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1. Introduction

In the studies of Marshall-Olkin shock models, n component system receive some shocks coming from $n+1$ different sources at a random time. According to this model, a shock coming from the jth source affects the jth component, $j = 1, \ldots, n$. On the other hand, a shock belonging the $n+1$th source affects all components in the system. If S_j and T_j denote respectively the lifetime of the jth component and the time of the jth shock, then $S_j = min(T_j, T_{n+1})$ [1]. In recent years, Marshall-Olkin shock models have been of great interest. A Marshall–Olkin type distribution including effect of shock magnitude was introduced by [2]. [3] considered Marshall–Olkin type shock model in Coherent systems. Marshall-Olkin type copulas produced by a common shock were researched by [4]. [5] generalizes the linear consecutive k-out-of-r-from-n: G system to multi-state case. Recently, [6] combined and studied run shock and Marshall–Olkin models.

Complex $(n-m+1)$-out-of-n: G systems are very important in the reliability of technical systems. An $(n-m+1)$-out-of-n: G system comprises of n components, functions if and only if at least $(n-m+1)$ components function. In the present paper, an $(n-m+1)$-out-of-n: G system, subject to Marshall-Olkin type shocks produced from $(n+1)$ sources at random time and affecting components of the system is studied. According to this model, run shock models are taken into account in the Marshall–Olkin shock models. A shock produced by source j only affects component j while the shock produced by source $(n+1)$ may affect all components. We assume that all shocks produced from sources are fatal shocks. If the component receive k consecutive fatal shocks produced from the same source then it fails. Because of being useful and suitable in terms of practical applications, interarrival times are supposed to have phase-type distributions [7-8]. There are numerous papers dealing with phase-type distributions such as [9-12].

Present paper is structured as follows. Some important properties of phase-type distributions are provided in Section 2. Also some examples with phase-type distributions and graphs are given. In Section 3 we study the computation of reliability and MTTF of the $(n-m+1)$-out-of-n: G system exposed to Marshall-Olkin type shocks. Coherent systems with given Samaniego's signatures and structure functions are considered, and the reliability and MTTF of the system are studied in Section 4. In Section 5, the formula for MRL of the mentioned system is derived.

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2. The Phase-type distributions

Consider a continuous random variable T. In a finite state continuous time Markov chain including one absorbing state and m transient states, if the random variable T is the distribution of the time til absorption, then its distribution can be considered as phase-type. If a Markov chain has at least one absorbing state and also in one or more steps it may go to an abosrbing state, the Markov chain is absorbing. For a continuous random variable T defined on [0,∞), cumulative distribution function (CDF) can be described as

$$P(T \leq t) = 1 - \alpha \exp(\mathbf{A}t)\mathbf{e}'$$

where the matrix $\mathbf{A}$ having $m \times m$ dimension with negative diagonal elements, non-negative off-diagonal entries, and all elements of the column vector $\mathbf{e}$ are one ($e_{1\times m}$). $\alpha = (\alpha_1, \ldots, \alpha_m)$ represents the row vector whose elements are non-negative , where $\sum_{i=1}^{m} \alpha_i = 1$. $T \sim PH_c(\alpha, \mathbf{A})$ will be used to represent having a continuous phase-type distribution. Some important continuous phase-type distributions are Erlang, exponential, Coxian and generalized Erlang distributions.

Similarly, a discrete Phase-type random variable N has the distribution of the time of absorbing Markov chain. The probability mass function (PMF) of phase-type random variable N is represented as

$$P(N = n) = \mathbf{a}\mathbf{Q}^{n-1}\mathbf{u}^{'}$$

for $n \in \mathbb{N}$, where the matrix $\mathbf{Q}$ =($\mathbf{q}_{ij})_{m\times m}$ includes the transition probabilities from m transient states, where the vector $\mathbf{u}' = (\mathbf{I} - \mathbf{Q})\mathrm{e}^{'}$ includes the transient probabilities from transient states to the absorbing state. In addtion, $\mathbf{I}$ denotes the identity matrix [7]. The matrix $\mathbf{I}$-$\mathbf{Q}$ must be non-singular. $N \sim PH_d(\mathbf{a}, \mathbf{Q})$ will be used to represent having a discrete phase-type distribution.

Under various operations, the phase-type distributions have closure property under some operations. Two phase-type closure properties are given in the following two propositions which are important in our study. For the proofs, other important properties and applications, we refer to [8].

Proposition 2.1. *Consider two independent phase-type random variables represented respectively as $X \sim PH_c(\alpha, \mathbf{A})$ and $Y \sim PH_c(\beta, \mathbf{B})$ Another random variable min(X, Y) has also phase-type distribution represented as $PH_c(\alpha \otimes \beta, \mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B})$, where $\otimes$ denotes Kronecker product.*

Proposition 2.2. *Assume that $X_1, X_2, \ldots$ are independent and $X_i \sim PH_c(\alpha, \mathbf{A})$ and $N \sim PH_d(\mathbf{a}, \mathbf{Q})$ with $\alpha\mathbf{e}' = 1$, $\mathbf{a}\mathbf{e}' = 1$, $i = 1, 2, \ldots$, then another random variable $\sum\limits_{i=1}^{N} X_i$ has a phase-type distribution with representation*

$$\sum_{i=1}^{N} X_i \sim PH_c(\alpha \otimes \mathbf{a}, \mathbf{A} \otimes \mathbf{I} + (\mathbf{a}^0\alpha) \otimes \mathbf{Q})$$

where $\mathbf{a}^0 = -\mathbf{A}\mathbf{e}'$.

3. Reliability of an $(n - m + 1)$-out-of-$n : G$ system

Consider an $(n - m + 1)$-out-of-n: G system that functions if and only if at least $(n - m + 1)$ components function. [3] proposed the reliability of the $(n - m + 1)$-out-of-n: G system subjected to classical Marshall-Olkin type shocks. Pursuant to the classical Marshall-Olkin type shock model, the shock arrived from the source j affects only component j and the shock arrived from the source $(n + 1)$ affects all components simultaneously. According to proposed model, failure of component j can be occured either k consecutive fatal shocks which are provided from same source or source $n + 1$, j = 1, . . . ,n. More precisely, if k consecutive fatal shocks are provided by the same source, then corresponding component(s) fail(s). If $S_1, S_2 \ldots, S_n$ denote the lifetime of the components of mentioned system, then the lifetimes of component j is defined as

$$S_j = \min(T_j(k), T_{n+1}(k)) \tag{3.1}$$

where

$$T_j(k) = \sum_{i=1}^{N_j(k)} X_{ji} \tag{3.2}$$

for $j = 1, 2, ...n$. $N_j(k)$ represents total number of shocks up to occurance of k consecutive fatal shocks. In 2015, [13] presented phase representation $N_j(k) \sim PH_d(\mathbf{a}, \mathbf{Q}_j)$ which is also known as the geometric distribution of order k with $\mathbf{a} = (1, 0, \ldots, 0)_{1\times k}$ and

$$\mathbf{Q}_j = \begin{bmatrix} 1-p_j & p_j & 0 & \ldots & 0 \\ 1-p_j & 0 & p_j & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-p_j & 0 & 0 & \ldots & 0 \end{bmatrix}_{k\times k}. \tag{3.3}$$

The matrix elements p_j denote the probability of the occurance of the shock coming from corresponding source is fatal, $j = 1, 2, ...n$.

When the interarrival times $X_{j1}, X_{j2}, \ldots$ have a phase-type distribution with $X_{ji} \sim PH_c(\alpha_j, \mathbf{A}_j)$, $j = 1, 2, ..., n$, and $i = 1, 2, ...$ [6] derived the individual lifetime random variable S_j as

$$S_j \sim PH_c\left(\mathbf{v}_j \otimes \mathbf{v}_{n+1}, \mathbf{Z}_j \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_{n+1}\right) \tag{3.4}$$

where $\mathbf{v}_j = (\alpha_j \otimes \mathbf{a})$ and $\mathbf{v}_{n+1} = (\alpha_{n+1} \otimes \mathbf{a})$, $j = 1, 2, ...n$.

The next Lemma will be useful in this study.

Lemma 3.1. *[Due to [14]] Let $T_1, T_2, ..., T_n$ be exchangeable random variables, and T_{n+1} is statistically independent of the random vector $(T_1, T_2, ..., T_n)$. The random variables $S_1 = \min(T_1, T_{n+1})$, $S_2 = \min(T_2, T_{n+1})$, and $S_n = \min(T_n, T_{n+1})$ are exchangeable, i.e., $(S_1, S_2, ..., S_n) \overset{d}{=} (S_{i_1}, S_{i_2}, ..., S_{i_n})$ for all $n!$ permutations $i_1, i_2, ..., i_n)$ of 1,2,..., n, where $\overset{d}{=}$ stands for equality in the distributions.*

In the following theorem, survival function of the related system is derived.

Theorem 3.1. *Let R be the lifetime of an $(n-m+1)$-out-of-$n : G$ system subject to random shocks as Marshall-Olkin run shock model. The survival function of the system is*

$$\begin{aligned} & P(R > u) = P(S_{m:n} > u) \\ = \; & (\alpha_{n+1} \otimes \mathbf{a}) \exp\left[\left((\mathbf{A}_{n+1} \otimes \mathbf{I}) + \left((-\mathbf{A}_{n+1}\mathbf{e}')\, \alpha_{n+1} \otimes \mathbf{Q}_{n+1}\right)\right) u\right] \mathbf{e}' \times \\ & \sum_{j=n-m+1}^{n} \binom{n}{j} \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} \left[(\alpha_1 \otimes \mathbf{a}) \exp\left[\left((\mathbf{A}_1 \otimes \mathbf{I}) + \left((-\mathbf{A}_1\mathbf{e}')\, \alpha_1 \otimes \mathbf{Q}_1\right)\right) u\right] \mathbf{e}'\right]^{j+i} \end{aligned}$$

Proof. It is a well-known fact that the lifetime of an $(n-m+1)$-out-of-$n : G$ system corresponds to mth order statistic $S_{m:n}$ constructed from the random variables $S_1, S_2, ..., S_n$, where $S_{1:n} \le S_{2:n} \le \cdots \le S_{n:n}$. The survival function of $S_{m:n}$ can be formulated as

$$P(S_{m:n} > u) = \sum_{j=n-m+1}^{n} \binom{n}{j} P(S_1 > u, ..., S_j > u, S_{j+1} \le u, ..., S_n \le u) \tag{3.5}$$

When random variables are exchangeable, from [15], $S_1, S_2, ..., S_n$

$$\begin{aligned} & P(S_1 > u, ..., S_j > u, S_{j+1} \le u, ..., S_n \le u) = \\ & \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} P(S_1 > u, S_2 > u, ..., S_{j+i} > u). \end{aligned} \tag{3.6}$$

Using (3.6) in (3.5), one has

$$\begin{aligned} & P(R > u) = P(S_{m:n} > u) \\ = \; & \sum_{j=n-m+1}^{n} \binom{n}{j} \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} P(S_1 > u, S_2 > u, ..., S_{j+i} > u) \\ = \; & \sum_{j=n-m+1}^{n} \binom{n}{j} \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} P(\min(T_1(k), T_{n+1}(k)) > u, \min(T_2(k),_{n+1}(k)) > u, ..., \min(T_{j+i}(k),_{n+1}(k)) > u) \\ = \; & \sum_{j=n-m+1}^{n} \binom{n}{j} \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} \prod_{l=1}^{j+i} P(T_l(k) > u) P(T_{n+1}(k) > u) \\ = \; & P(T_{n+1}(k) > u) \sum_{j=n-m+1}^{n} \binom{n}{j} \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} P(T_1(k) > u)^{j+i} \end{aligned}$$

Using Proposition 2, one can easiliy derive

$$P(R>u)=(\alpha_{n+1}\otimes\mathbf{a})\exp\left[\left((\mathbf{A}_{n+1}\otimes\mathbf{I})+((-\mathbf{A}_{n+1}\mathbf{e}')\,\alpha_{n+1}\otimes\mathbf{Q}_{n+1})\right)u\right]\mathbf{e}'\times$$
$$\sum_{j=n-m+1}^{n}\binom{n}{j}\sum_{i=0}^{n-j}(-1)^i\binom{n-j}{i}\left[(\alpha_1\otimes\mathbf{a})\exp\left[((\mathbf{A}_1\otimes\mathbf{I})+((-\mathbf{A}_1\mathbf{e}')\,\alpha_1\otimes\mathbf{Q}_1))\,u\right]\mathbf{e}'\right]^{j+i}$$

□

Example 3.1. Let X_{ji} denotes the times between fatal shocks coming from source j. Assume that X_{ji} has Erlang distribution with parameters $m_j=2,\ j=1,2,\ldots,(n+1)$, and $\lambda_j=1, j=1,2,\ldots n$,with $\lambda_{n+1}=2$. That is, $X_{ji}\sim PH_c(\alpha_j,\mathbf{A}_j)$, with $\alpha_j=(0,\ldots,0,1)$ and $A_j=\begin{bmatrix}-\lambda_j & 0 & \ldots & 0\\ \lambda_j & -\lambda_j & & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \ldots & \lambda_j & -\lambda_j\end{bmatrix}_{m_j\times m_j}$.

We plot the graph of the survival function of 3-out-of-5: G system under proposed model for different values of k in Fig.1. From the Fig.1, we observed that the survival function increases when k increases.

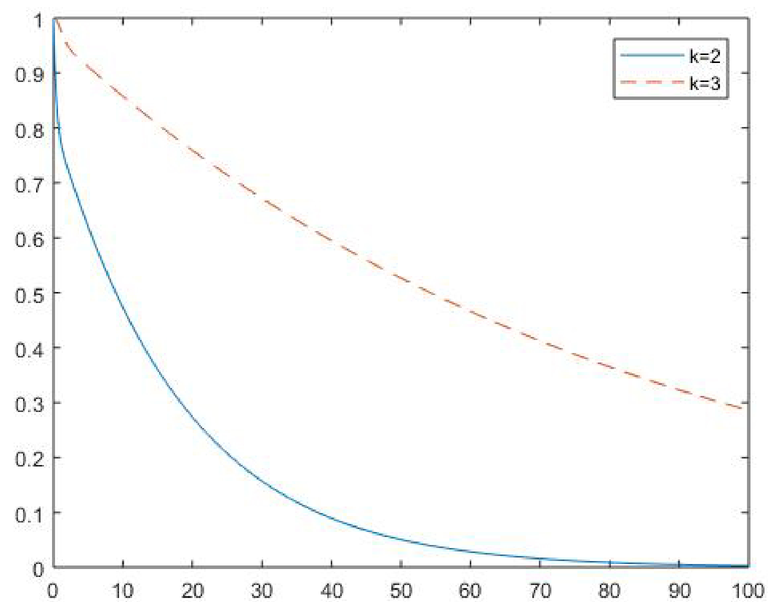

Figure 1. The graph of the survival function of 3-out-of-5: G system

In Table 1, the MTTF values of the 3-out-of-5 system are computed. An increase in k cause an increase in the MTTF values. As expected, when k is increased from 2 to 3, there is a significant difference between MTTF values. Also, if the probability of occurrence of fatal shocks increases, then MTTF values decreases.

k	System	$p_j,\, j=1,2,3,4,5$	p^*	MTTF
2	3-out-of-5	0.1	0.25	14.8339
		0.1	0.2	22.6380
		0.2	0.2	14.8454
		0.2	0.25	11.1149
3	3-out-of-5	0.1	0.25	79.3038
		0.1	0.2	147.9109
		0.2	0.2	104.7768
		0.2	0.25	68.0610

Table 1. MTTF values of the 3-out-of-5 systems

In Table 2, for particular points u, different values of k and probability of occurrence of fatal shocks p, some numerical values of reliability functions are provided . It is observed that for all value of u, k and p, 4-out-of-5: G system is more reliable than the 5-out-of-5: G system, 3-out-of-5: G system is more reliable than 4-out-of-5: G system. Similarly, one can easily see that 5-out-of-5: G series system has the lowest reliability. For all systems, the increase of the point u or in the rate of occurance p results in a reduction in the reliability. However, the expansion in k brings about an expansion in the reliability.

				$P(R>u)$				
k	$p_j,\, j=1,2,3,4,5$	p^*	u	5-out-of-5:G	4-out-of-5:G	3-out-of-5:G	2-out-of-5:G	1-out-of-5:G
2	0.1	0.15	10	0.3644	0.6297	0.7070	0.7183	0.7191
	0.1	0.15	15	0.2939	0.5471	0.6343	0.6493	0.6506
	0.2	0.15	15	0.0771	0.2821	0.5003	0.6165	0.6474
	0.1	0.25	15	0.1669	0.3106	0.3601	0.3686	0.3693
3	0.1	0.15	10	0.9010	0.9558	0.9571	0.9571	0.9571
	0.1	0.15	15	0.8780	0.9415	0.9433	0.9434	0.9434
	0.2	0.15	15	0.6577	0.9037	0.9405	0.9433	0.9434
	0.1	0.25	15	0.8069	0.8069	0.8069	0.8053	0.7510

Table 2. P(R>u) values of m-out-of-5 systems for m=1,...,5.

4. Reliability of Coherent Systems

Let R be the lifetime of a any coherent system having independent and identical n components whose lifetimes are S_1,S_2,...,S_n. [16] proved that the survival function of such a system can be written as

$$\bar{F}_R(u) \equiv P(R>u) = \sum_{i=1}^{n} p_i P(S_{i:n} > t), \tag{4.1}$$

where $S_{i:n}$ is the ith smallest among S_1,...,S_n. The vector $p = (p_1, p_2, ..., p_n)$ is called the system signature. More explicitly,

$$p_i = \frac{m_i}{n!},$$

where m_i is the number of orders for which failure of the ith component causes system failure, $i = 1, 2, ..., n$. Samaniego's signature vector of the system is represented by $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ [17]. Since the lifetimes $S_1, S_2, \ldots, S_n$ are exchangeable, the equality (4.1) also holds [18].
Consider the system with n components which is mentioned above. That is, component i is exposed to random shocks and fails when consecutively r fatal shocks produced by source i occur or consecutively r fatal shocks produced by source $(n+1)$ occur. Under this arrangement, the lifetime of the component i can be determined using

(3.1). According to [18] and also From Theorem 6, the survival function of a coherent system can be found as

$$\begin{aligned}\bar{F}_R(u) &\equiv P(R > u) = \sum_{i=1}^{n} p_i P(S_{i:n} > u) \\ &= \bar{F}_{S_{n+1}}(u) \sum_{i=1}^{n} p_i \sum_{j=n-i+1}^{n} \binom{n}{j} \sum_{z=0}^{n-j} (-1)^z \binom{n-j}{z} \bar{F}_{S_1}^{j+z}(u).\end{aligned} \tag{4.2}$$

k	Structure function	Signature	Reliability	MTTF
2	$R=\min(S_1,S_2,S_3,S_4,S_5)$	$p=[1,0,0,0,0]$	0.3644	7.1304
	$R=\max(S_1,S_2,S_3,S_4,S_5)$	$p=[0,0,0,0,1]$	0.7191	15.4917
	$R=\max(\min(S_1,S_2,S_3,\max(S_4,S_5)),\min(S_1,S_4,S_5,\max(S_2,S_3)),\max(S_2,S_3,S_4,S_5))$	$p=[0,1,0,0,0]$	0.6297	12.6346
	$R=\max(\min(S_1,S_2,\max(S_3,S_4)),\min(S_3,S_4,S_5,\max(S_1,S_2)))$	$p=[0,4/5,1/5,0,0]$	0.6452	13.0745
	$R=\max(\min(S_1,S_2,\max(S_3,S_4)),\min(S_3,S_4,S_5))$	$p=[0,7/10,3/10,0,0]$	0.6529	13.2944
	$R=\max(\min(S_1,S_2,\max(S_3,S_4,S_5)),\min(S_3,S_4,S_5))$	$p=[0,3/5,2/5,0,0]$	0.6606	13.5143
	$R=\max(\min(S_1,S_2,\max(S_3,S_4,S_5)),\min(S_3,S_4,\max(S_1,S_5)))$	$p=[0,1/2,1/2,0,0]$	0.6684	13.7342
	$R=\max(\min(S_1,S_2,\max(S_3,S_4,S_5)),\min(S_1,S_3,S_4),\min(S_2,S_3,S_5),\min(S_3,S_4,S_5))$	$p=[0,2/5,3/5,0,0]$	0.6761	13.9542
	$R=\max(\min(S_1,S_2,\max(S_3,S_4,S_5),\min(S_1,S_3,\max(S_4,S_5)),\min(S_2,S_3,\max(S_4,S_5)),\min(S_2,S_3,S_4,S_5))$	$p=[0,1/10,9/10,0,0]$	0.6993	14.6140
	$R=\max(\min(S_1,\max(S_2,S_3)),\min(S_4,S_5,\max(S_2,S_3))$	$p=[0,3/10,1/2,1/5,0]$	0.6861	14.2868
3	$R=\min(S_1,S_2,S_3,S_4,S_5)$	$p=[1,0,0,0,0]$	0.9010	64.7009
	$R=\max(S_1,S_2,S_3,S_4,S_5)$	$p=[0,0,0,0,1]$	0.9571	79.4995
	$R=\max(\min(S_1,S_2,S_3,\max(S_4,S_5)),\min(S_1,S_4,S_5,\max(S_2,S_3)),\max(S_2,S_3,S_4,S_5))$	$p=[0,1,0,0,0]$	0.9558	77.5418
	$R=\max(\min(S_1,S_2,\max(S_3,S_4)),\min(S_3,S_4,S_5,\max(S_1,S_2)))$	$p=[0,4/5,1/5,0,0]$	0.9560	77.8942
	$R=\max(\min(S_1,S_2,\max(S_3,S_4)),\min(S_3,S_4,S_5))$	$p=[0,7/10,3/10,0,0]$	0.9562	78.0704
	$R=\max(\min(S_1,S_2,\max(S_3,S_4,S_5)),\min(S_3,S_4,S_5))$	$p=[0,3/5,2/5,0,0]$	0.9563	78.2466
	$R=\max(\min(S_1,S_2,\max(S_3,S_4,S_5)),\min(S_3,S_4,\max(S_1,S_5)))$	$p=[0,1/2,1/2,0,0]$	0.9564	78.4228
	$R=\max(\min(S_1,S_2,\max(S_3,S_4,S_5)),\min(S_1,S_3,S_4),\min(S_2,S_3,S_5),\min(S_3,S_4,S_5))$	$p=[0,2/5,3/5,0,0]$	0.9566	78.5990
	$R=\max(\min(S_1,S_2,\max(S_3,S_4,S_5),\min(S_1,S_3,\max(S_4,S_5)),\min(S_2,S_3,\max(S_4,S_5)),\min(S_2,S_3,S_4,S_5))$	$p=[0,1/10,9/10,0,0]$	0.9570	79.1276
	$R=\max(\min(S_1,\max(S_2,S_3))),\min(S_4,S_5,\max(S_2,S_3))$	$p=[0,3/10,1/2,1/5,0]$	0.9567	78.8117)

Table 3. Signatures of some coherent systems

In Table 3, using (4.2) we calculate the reliability and MTTF of some systems for $k = 2,3$ and probability of occurrence of fatal shocks $p_i = 0.1, i = 1,2,\ldots,n$ and $p^* = 0.15$ when $u = 10$.

5. Mean residual lifetime

The mean residual life (MRL) function of the coherent system is of special importance in reliability engineering. Let T denotes the lifetime function then the function $E(T-t|T>t)$ is called MRL function of the system. For more details on MRL function, we refer to [19-22].

Consider an $(n-m+1)$-out-of-n system whose lifetimes are $S_1, S_2, ..., S_n$ and suppose at least $n-r+1$ of its components function at time t. MRL function of this system is defined as follows:

$$h_{m,r} = E(S_{m:n} - t|S_{r:n} > t), \tag{5.1}$$

where $S_{i:n}$, denotes ith order statistic among $S_1, S_2, \ldots, S_n$ and $1 \le r < m \le n$.

Since the lifetime random variable S_i is already defined as $S_i = \min(T_i, T_{n+1})$, $i = 1,2,...,n$, using Proposition 2 together with Lemma 3 of [23], one can easily derive the following Lemma.

Lemma 5.1.

$$\begin{aligned}P(S_{r:n} &> t, S_{m:n} > t+x) \\ &= (\alpha_{n+1}\otimes\mathbf{a})\exp\left[\left((\mathbf{A}_{n+1}\otimes\mathbf{I}) + ((-\mathbf{A}_{n+1}\mathbf{e}')\,\alpha_{n+1}\otimes\mathbf{Q}_{n+1})\right)(t+x)\right]\mathbf{e}' \times \\ &\quad \sum_{i=n-r+1}^{n}\sum_{j=n-m+1}^{i}\frac{n!}{j!(i-j)!(n-i)!}[\sum_{l=0}^{i-j}(-1)^l\binom{i-j}{l}\sum_{z=0}^{n-i}(-1)^z\binom{n-i}{z}\times \\ &\quad \bar{F}_{T_1}^{i-j+z-l}(t)] \\ &\quad ((\alpha_1\otimes\mathbf{a})\exp\left[((\mathbf{A}_1\otimes\mathbf{I}) + ((-\mathbf{A}_1\mathbf{e}')\,\alpha_1\otimes\mathbf{Q}_1))\,(t+x)\right]\mathbf{e}')^{j+l} \times \\ &\quad ((\alpha_1\otimes\mathbf{a})\exp\left[((\mathbf{A}_1\otimes\mathbf{I}) + ((-\mathbf{A}_1\mathbf{e}')\,\alpha_1\otimes\mathbf{Q}_1))\,(t)\right]\mathbf{e}')^{i-j+z-l}\end{aligned}$$

where $1 \le r < m \le n$.

Theorem 5.1. *MRL function of the above mentioned system can be defined as*

$$\begin{aligned} h_{m,r} &= E(S_{m:n} - t | S_{r:n} > t) = \\ & \frac{1}{\overline{F}_{S_{r:n}}(t)} \sum_{i=n-r+1}^{n} \sum_{j=n-m+1}^{i} \frac{n!}{j!(i-j)!(n-i)!} \sum_{l=0}^{i-j} (-1)^l \binom{i-j}{l} \times \\ & \sum_{z=0}^{n-i} (-1)^z \binom{n-i}{z} \overline{F}_{T_1(k)}^{i-j+z-l}(t) \int_t^\infty \overline{F}_{T_{n+1}(k)}(x) \overline{F}_{T_1(k)}^{j+l}(x) dx, \end{aligned}$$

where

$$\begin{aligned} \overline{F}_{S_{r:n}}(u) &= (\alpha_{n+1} \otimes \mathbf{a}) \exp\left[\left((\mathbf{A}_{n+1} \otimes \mathbf{I}) + \left((-\mathbf{A}_{n+1}\mathbf{e}')\, \alpha_{n+1} \otimes \mathbf{Q}_{n+1}\right)\right) u\right] \mathbf{e}' \times \\ & \sum_{j=n-m+1}^{n} \binom{n}{j} \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} \left[(\alpha_1 \otimes \mathbf{a}) \exp\left[\left((\mathbf{A}_1 \otimes \mathbf{I}) + \left((-\mathbf{A}_1 \mathbf{e}')\, \alpha_1 \otimes \mathbf{Q}_1\right)\right) u\right] \mathbf{e}'\right]^{j+i} \end{aligned}$$

and

$$\begin{aligned} \overline{F}_{T_{n+1}(k)}(u) &= (\alpha_{n+1} \otimes \mathbf{a}) \exp\left[\left((\mathbf{A}_{n+1} \otimes \mathbf{I}) + \left((-\mathbf{A}_{n+1}\mathbf{e}')\, \alpha_{n+1} \otimes \mathbf{Q}_{n+1}\right)\right) u\right] \mathbf{e}' \\ \overline{F}_{T_1(k)}(u) &= (\alpha_1 \otimes \mathbf{a}) \exp\left[\left((\mathbf{A}_1 \otimes \mathbf{I}) + \left((-\mathbf{A}_1 \mathbf{e}')\, \alpha_1 \otimes \mathbf{Q}_1\right)\right) u\right] \mathbf{e}'. \end{aligned}$$

Proof. From the definiton of the MRL function of the $(n-m+1)$-out-of-n :G system, for t>0 we have

$$\begin{aligned} h_{m,r} &= E(S_{m:n} - t | S_{r:n} > t), \; , \\ &= \frac{1}{P(S_{r:n} > t)} \int_t^\infty P(S_{r:n} > t, S_{m:n} > x) dx. \end{aligned}$$

After taking into account Theorem 6 and Lemma 8, we obtain the desired result. □

Conclusion. In this paper, we have combined Marshall–Olkin shock models and coherent systems. Reliability and some important characteristics of the $(n-m+1)$-out-of-n: G system subject to Marshall-Olkin type shocks have been investigated. We assumed that the interarrival times between shocks follow phase-type distributions which have several advantages such as simplicity in calculations. Coherent systems with given Samaniego's signatures or structure function are also considered.

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