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Summing Formulas for Generalized Tribonacci Numbers

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Abstract

In this paper, closed forms of the summation formulas for generalized Tribonacci numbers are presented. Then, some previous results are recovered as particular cases of the present results. As special cases, we give summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin, Narayana and some other third order linear recurrance sequences. All the summing formulas of well known recurrence sequences which we deal with are linear except the cases Pell-Padovan and Padovan-Perrin.

1. Introduction

In this work, we investigate linear summation formulas of generalized Tribonacci numbers. Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [11, 12], see also [9]. For linear sums of Fibonacci, Tribonacci, Tetranacci, Pentanacci and Hexanacci numbers, see [10,24], [8,16], [21, 31], [22], and [23] respectively. First, in this section, we present some background about generalized Tribonacci numbers. The generalized Tribonacci sequence $\{W_n(W_0,W_1,W_2;r,s,t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}, \qquad W_0=a, W_1=b, W_2=c, \;\; n\geq 3 \tag{1.1}$$

where W_0,W_1,W_2 are arbitrary complex numbers and r,s,t are real numbers. The generalized Tribonacci sequence has been studied by many authors, see for example [2,3,5,7,14,15,17,18,19,26,27,28,29,30].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n=1,2,3,...$ when $t\neq 0$. Therefore, recurrence (1.1) holds for all integer n.

If we set $r=s=t=1$ and $W_0=0, W_1=1, W_2=1$ then $\{W_n\}$ is the well-known Tribonacci sequence and if we set $r=s=t=1$ and $W_0=3, W_1=1, W_2=3$ then $\{W_n\}$ is the well-known Tribonacci-Lucas sequence.

In fact, the generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Tribonacci-Lucas, Padovan (Cordonnier), Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas. In literature, for example, the following names and notations (see Table 1) are used for the special case of r,s,t and initial values.

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Sequences (Numbers)	Notation	OEIS [20]
Tribonacci	$\{T_n\} = \{W_n(0,1,1;1,1,1)\}$	A000073, A057597
Tribonacci-Lucas	$\{K_n\} = \{W_n(3,1,3;1,1,1)\}$	A001644, A073145
third order Pell	$\{P_n^{(3)}\} = \{W_n(0,1,2;2,1,1)\}$	A077939, A077978
third order Pell-Lucas	$\{Q_n^{(3)}\} = \{W_n(3,2,6;2,1,1)\}$	A276225, A276228
third order modified Pell	$\{E_n^{(3)}\} = \{W_n(0,1,1;2,1,1)\}$	A077997, A078049
Padovan (Cordonnier)	$\{P_n\} = \{W_n(1,1,1;0,1,1)\}$	A000931
Perrin (Padovan-Lucas)	$\{E_n\} = \{W_n(3,0,2;0,1,1)\}$	A001608, A078712
Padovan-Perrin	$\{S_n\} = \{W_n(0,0,1;0,1,1)\}$	A000931, A176971
Pell-Padovan	$\{R_n\} = \{W_n(1,1,1;0,2,1)\}$	A066983, A128587
Pell-Perrin	$\{C_n\} = \{W_n(3,0,2;0,2,1)\}$	-
Jacobsthal-Padovan	$\{Q_n\} = \{W_n(1,1,1;0,1,2)\}$	A159284
Jacobsthal-Perrin (-Lucas)	$\{D_n\} = \{W_n(3,0,2;0,1,2)\}$	A072328
Narayana	$\{N_n\} = \{W_n(0,1,1;1,0,1)\}$	A078012
third order Jacobsthal	$\{J_n^{(3)}\} = \{W_n(0,1,1;1,1,2)\}$	A077947
third order Jacobsthal-Lucas	$\{j_n^{(3)}\} = \{W_n(2,1,5;1,1,2)\}$	A226308

Table 1: A few special case of generalized Tribonacci sequences

Note that the sequence $\{C_n\}$ is't in the database of http://oeis.org [20], yet.

2. Sum formulas of Generalized Tribonacci Numbers with Positive Subscripts

The following Theorem presents some linear summing formulas of generalized Tribonacci numbers with positive subscripts.

Theorem 2.1. *For $n \geq 0$, we have the following formulas:*

(a) *(Sum of the generalized Tribonacci numbers) If $r+s+t-1 \neq 0$, then*

$$\sum_{k=0}^{n} W_k = \frac{W_{n+3}+(1-r)W_{n+2}+(1-r-s)W_{n+1}-W_2+(r-1)W_1+(r+s-1)W_0}{r+s+t-1}.$$

(b) *If $2s+2rt+r^2-s^2+t^2-1 = (r+s+t-1)(r-s+t+1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{(-s+1)W_{2n+2}+(t+rs)W_{2n+1}+(t^2+rt)W_{2n}+(-1+s)W_2+(-t-rs)W_1+(-1+r^2-s^2+rt+2s)W_0}{(r+s+t-1)(r-s+t+1)}$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{(r+t)W_{2n+2}+(s-s^2+t^2+rt)W_{2n+1}+(t-st)W_{2n}+(-r-t)W_2+(-1+s+r^2+rt)W_1+(-t+st)W_0}{(r-s+t+1)(r+s+t-1)}.$$

(c) *If $r+t \neq 0$, $s=1$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{1}{r+t}\left(W_{2n+1}+tW_{2n}-W_1+rW_0\right)$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{1}{r+t}\left(W_{2n+2}+tW_{2n+1}-W_2+rW_1\right).$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}$$

i.e.

$$tW_{n-3} = W_n - rW_{n-1} - sW_{n-2}$$

we obtain

$$\begin{aligned}
tW_0 &= W_3 - rW_2 - sW_1 \\
tW_1 &= W_4 - rW_3 - sW_2 \\
tW_2 &= W_5 - rW_4 - sW_3 \\
&\vdots \\
tW_{n-1} &= W_{n+2} - rW_{n+1} - sW_n \\
tW_n &= W_{n+3} - rW_{n+2} - sW_{n+1}.
\end{aligned}$$

If we add the equations by side by, we get

$$\sum_{k=0}^{n} W_k = \frac{W_{n+3}+(1-r)W_{n+2}+(1-r-s)W_{n+1}-W_2+(r-1)W_1+(r+s-1)W_0}{r+s+t-1}.$$

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3}$$

we obtain

$$\begin{aligned}
rW_3 &= W_4 - sW_2 - tW_1 \\
rW_5 &= W_6 - sW_4 - tW_3 \\
&\vdots \\
rW_{2n+1} &= W_{2n+2} - sW_{2n} - tW_{2n-1}. \\
rW_{2n+3} &= W_{2n+4} - sW_{2n+2} - tW_{2n+1}
\end{aligned}$$

Now, if we add the above equations by side by, we get

$$r(-W_1+\sum_{k=0}^{n} W_{2k+1}) = (W_{2n+2}-W_2-W_0+\sum_{k=0}^{n} W_{2k}) - s(-W_0+\sum_{k=0}^{n} W_{2k}) - t(-W_{2n+1}+\sum_{k=0}^{n} W_{2k+1}). \tag{2.1}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3}$$

we write the following obvious equations;

$$\begin{aligned}
rW_2 &= W_3 - sW_1 - tW_0 \\
rW_4 &= W_5 - sW_3 - tW_2 \\
rW_6 &= W_7 - sW_5 - tW_4 \\
&\vdots \\
rW_{2n} &= W_{2n+1} - sW_{2n-1} - tW_{2n-2} \\
rW_{2n+2} &= W_{2n+3} - sW_{2n+1} - tW_{2n}.
\end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$r(-W_0+\sum_{k=0}^{n} W_{2k}) = (-W_1+\sum_{k=0}^{n} W_{2k+1}) - s(-W_{2n+1}+\sum_{k=0}^{n} W_{2k+1}) - t(-W_{2n}+\sum_{k=0}^{n} W_{2k}). \tag{2.2}$$

Then, solving the system (2.1)-(2.2), the required results of (b) and (c) follow.

For another proof (using mathematical induction) of the formula in Theorem 2.1 (a), see [4].
Taking $r = s = t = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.2. *If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+3} - W_{n+1} - W_2 + W_0)$.
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{2}(W_{2n+1} + W_{2n} - W_1 + W_0)$.
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}(W_{2n+2} + W_{2n+1} - W_2 + W_1)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 2.3. *[8,16]For $n \geq 0$, Tribonacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+3} - T_{n+1} - 1)$.
(b) $\sum_{k=0}^{n} T_{2k} = \frac{1}{2}(T_{2n+1} + T_{2n} - 1)$.
(c) $\sum_{k=0}^{n} T_{2k+1} = \frac{1}{2}(T_{2n+2} + T_{2n+1})$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Tribonacci-Lucas numbers.

Corollary 2.4. *[8,16]For $n \geq 0$, Tribonacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} K_k = \frac{1}{2}(K_{n+3} - K_{n+1})$.
(b) $\sum_{k=0}^{n} K_{2k} = \frac{1}{2}(K_{2n+1} + K_{2n} + 2)$.
(c) $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{2}(K_{2n+2} + K_{2n+1} - 2)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.5. *[25]If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{3}\left(W_{n+3} - W_{n+2} - 2W_{n+1} - W_2 + W_1 + 2W_0\right)$.
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}\left(W_{2n+1} + W_{2n} - W_1 + 2W_0\right)$.
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}\left(W_{2n+2} + W_{2n+1} - W_2 + 2W_1\right)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of third-order Pell numbers (take $W_n = P_n^{(3)}$ with $P_0^{(3)} = 0, P_1^{(3)} = 1, P_2^{(3)} = 2$).

Corollary 2.6. *[25]For $n \geq 0$, third-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} P_k^{(3)} = \frac{1}{3}(P_{n+3}^{(3)} - P_{n+2}^{(3)} - 2P_{n+1}^{(3)} - 1)$.
(b) $\sum_{k=0}^{n} P_{2k}^{(3)} = \frac{1}{3}(P_{2n+1}^{(3)} + P_{2n}^{(3)} - 1)$.
(c) $\sum_{k=0}^{n} P_{2k+1}^{(3)} = \frac{1}{3}(P_{2n+2}^{(3)} + P_{2n+1}^{(3)})$.

Taking $W_n = Q_n^{(3)}$ with $Q_0^{(3)} = 3, Q_1^{(3)} = 2, Q_2^{(3)} = 6$ in the last Proposition, we have the following Corollary which presents linear sum formulas of third-order Pell-Lucas numbers.

Corollary 2.7. *[25]For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} Q_k^{(3)} = \frac{1}{3}(Q_{n+3}^{(3)} - Q_{n+2}^{(3)} - 2Q_{n+1}^{(3)} + 2)$.
(b) $\sum_{k=0}^{n} Q_{2k}^{(3)} = \frac{1}{3}(Q_{2n+1}^{(3)} + Q_{2n}^{(3)} + 4)$.
(c) $\sum_{k=0}^{n} Q_{2k+1}^{(3)} = \frac{1}{3}(Q_{2n+2}^{(3)} + Q_{2n+1}^{(3)} - 2)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of third-order modified Pell numbers (take $W_n = E_n^{(3)}$ with $E_0^{(3)} = 0, E_1^{(3)} = 1, E_2^{(3)} = 1$).

Corollary 2.8. *[25]For $n \geq 0$, third-order modified Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} E_k^{(3)} = \frac{1}{3}(E_{n+3}^{(3)} - E_{n+2}^{(3)} - 2E_{n+1}^{(3)})$.
(b) $\sum_{k=0}^{n} E_{2k}^{(3)} = \frac{1}{3}(E_{2n+1}^{(3)} + E_{2n}^{(3)} - 1)$.
(c) $\sum_{k=0}^{n} E_{2k+1}^{(3)} = \frac{1}{3}(E_{2n+2}^{(3)} + E_{2n+1}^{(3)} + 1)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.9. *If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = W_{n+3} + W_{n+2} - W_2 - W_1$.
(b) $\sum_{k=0}^{n} W_{2k} = W_{2n+1} + W_{2n} - W_1$.
(c) $\sum_{k=0}^{n} W_{2k+1} = W_{2n+2} + W_{2n+1} - W_2$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P = 1, P_2 = 1$).

Corollary 2.10. *[1] For $n \geq 0$, Padovan numbers have the following properties.*

(a) $\sum_{k=0}^{n} P_k = P_{n+3} + P_{n+2} - 2$.
(b) $\sum_{k=0}^{n} P_{2k} = P_{2n+1} + P_{2n} - 1$.
(c) $\sum_{k=0}^{n} P_{2k+1} = P_{2n+2} + P_{2n+1} - 1$.

Taking $W_n = E_n$ with $E_0 = 3, E_2 = 0, E_2 = 2$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Perrin numbers.

Corollary 2.11. *[1] For $n \geq 0$, Perrin numbers have the following properties.*

(a) $\sum_{k=0}^{n} E_k = E_{n+3} + E_{n+2} - 2$.
(b) $\sum_{k=0}^{n} E_{2k} = E_{2n+1} + E_{2n}$.
(c) $\sum_{k=0}^{n} E_{2k+1} = E_{2n+2} + E_{2n+1} - 2$.

Taking $W_n = S_n$ with $S_0 = 0, S_2 = 0, S_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Padovan-Perrin numbers.

Corollary 2.12. *For $n \geq 0$, Padovan-Perrin numbers have the following properties.*

(a) $\sum_{k=0}^{n} S_k = S_{n+3} + S_{n+2} - 1.$
(b) $\sum_{k=0}^{n} S_{2k} = S_{2n+1} + S_{2n}.$
(c) $\sum_{k=0}^{n} S_{2k+1} = S_{2n+2} + S_{2n+1} - 1.$

If $r = 0, s = 2, t = 1$ then $(r - s + t + 1) = 0$ so we can't use Theorem 2.1 (b). In other words, the method of the proof Theorem 2.1 (b) can't be used to find $\sum_{k=0}^{n} W_{2k}$ and $\sum_{k=0}^{n} W_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 2.13. *If $r = 0, s = 2, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+3} + W_{n+2} - W_{n+1} - W_2 - W_1 + W_0).$
(b) $\sum_{k=0}^{n} W_{2k} = W_{2n+1} + (W_2 - W_1 - W_0)n + W_0 - W_1.$
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}(W_{2n+3} + W_{2n+2} - W_{2n+1} + 2n(-W_2 + W_1 + W_0) - W_2 + W_1 - W_0).$

Proof.

(a) Taking $r = 0, s = 2, t = 1$ in Theorem 2.1 (a) we obtain (a).
(b) and (c) Using the recurrence relation

$$W_n = 2W_{n-2} + W_{n-3}$$

we obtain

$$\sum_{k=0}^{0} W_{2k} = W_0$$

$$\sum_{k=0}^{1} W_{2k} = W_0 + W_2 = W_3 + W_2 - 2W_1$$

$$\sum_{k=0}^{2} W_{2k} = W_0 + W_2 + W_4 = W_5 + 2W_2 - 3W_1 - W_0$$

$$\vdots$$

$$\sum_{k=0}^{n} W_{2k} = W_{2n+1} + (W_2 - W_1 - W_0)n + W_0 - W_1.$$

This result can be also proved by mathematical induction. Note that from (a) we get

$$\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}(W_{2n+3} + W_{2n+2} + W_{2n+1} - W_2 - W_1 + W_0) - \sum_{k=0}^{n} W_{2k}.$$

Now, (c) follows from the last equation.

From the above Theorem we have the following Corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 2.14. *For $n \geq 0$, Pell-Padovan numbers have the following property:*

(a) $\sum_{k=0}^{n} R_k = \frac{1}{2}(R_{n+3} + R_{n+2} - R_{n+1} - 1).$
(b) $\sum_{k=0}^{n} R_{2k} = R_{2n+1} - n.$
(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{2}(R_{2n+3} + R_{2n+2} - R_{2n+1} + 2n - 1).$

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last Theorem, we have the following Corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 2.15. *For $n \geq 0$, Pell-Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n} C_k = \frac{1}{2}(C_{n+3} + C_{n+2} - C_{n+1} + 1).$
(b) $\sum_{k=0}^{n} C_{2k} = C_{2n+1} - n + 3.$
(c) $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{2}(C_{2n+3} + C_{2n+2} - C_{2n+1} + 2n - 5).$

Taking $r = 0, s = 1, t = 2$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.16. *If $r = 0, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+3} + W_{n+2} - W_2 - W_1).$
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{2}(W_{2n+1} + 2W_{2n} - W_1).$
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}(W_{2n+2} + 2W_{2n+1} - W_2).$

Taking $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Jacobsthal-Padovan numbers.

Corollary 2.17. *For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties.*

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{2}(Q_{n+3} + Q_{n+2} - 2).$

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{2}\left(Q_{2n+1} + 2Q_{2n} - 1\right).$
(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{2}\left(Q_{2n+2} + 2Q_{2n+1} - 1\right).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of Jacobsthal-Perrin numbers (take $W_n = D_n$ with $D_0 = 3, D_1 = 0, D_2 = 2$).

Corollary 2.18. *For $n \geq 0$, Jacobsthal-Perrin numbers have the following properties.*

(a) $\sum_{k=0}^{n} D_k = \frac{1}{2}\left(D_{n+3} + D_{n+2} - 2\right).$
(b) $\sum_{k=0}^{n} D_{2k} = \frac{1}{2}\left(D_{2n+1} + 2D_{2n}\right).$
(c) $\sum_{k=0}^{n} D_{2k+1} = \frac{1}{2}\left(D_{2n+2} + 2D_{2n+1} - 2\right).$

Taking $r = 1, s = 0, t = 1$ in Theorem 2.1 (a) and (c), we obtain the following Proposition.

Proposition 2.19. *If $r = 1, s = 0, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = W_{n+3} - W_2.$
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(W_{2n+2} + W_{2n+1} + 2W_{2n} - W_2 - W_1 + W_0).$
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(2W_{2n+2} + 2W_{2n+1} + W_{2n} - 2W_2 + W_1 - W_0).$

From the last Proposition, we have the following Corollary which presents linear sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 2.20. *For $n \geq 0$, Narayana numbers have the following properties.*

(a) $\sum_{k=0}^{n} N_k = N_{n+3} - 1.$
(b) $\sum_{k=0}^{n} N_{2k} = \frac{1}{3}(N_{2n+2} + N_{2n+1} + 2N_{2n} - 2).$
(c) $\sum_{k=0}^{n} N_{2k+1} = \frac{1}{3}(2N_{2n+2} + 2N_{2n+1} + N_{2n} - 1).$

Taking $r = 1, s = 1, t = 2$ in Theorem 2.1 (a) and (c), we obtain the following Proposition.

Proposition 2.21. *If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{3}(W_{n+3} - W_{n+1} - W_2 + W_0).$
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(W_{2n+1} + 2W_{2n} - W_1 + W_0).$
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(W_{2n+2} + 2W_{2n+1} - W_2 + W_1).$

Taking $W_n = J_n^{(3)}$ with $J_0^{(3)} = 0, J_1^{(3)} = 1, J_2^{(3)} = 1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of third order Jacobsthal numbers.

Corollary 2.22. *For $n \geq 0$, third order Jacobsthal numbers have the following properties.*

(a) *[6]* $\sum_{k=0}^{n} J_k^{(3)} = \frac{1}{3}(J_{n+3}^{(3)} - J_{n+1}^{(3)} - 1).$
(b) $\sum_{k=0}^{n} J_{2k}^{(3)} = \frac{1}{3}(J_{2n+1}^{(3)} + 2J_{2n}^{(3)} - 1).$
(c) $\sum_{k=0}^{n} J_{2k+1}^{(3)} = \frac{1}{3}(J_{2n+2}^{(3)} + 2J_{2n+1}^{(3)}).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of third order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0^{(3)} = 2, j_1^{(3)} = 1, j_2^{(3)} = 5$).

Corollary 2.23. *For $n \geq 0$, third order Jacobsthal-Lucas numbers have the following properties.*

(a) *[6]* $\sum_{k=0}^{n} j_k^{(3)} = \frac{1}{3}(j_{n+3}^{(3)} - j_{n+1}^{(3)} - 3).$
(b) $\sum_{k=0}^{n} j_{2k}^{(3)} = \frac{1}{3}(j_{2n+1}^{(3)} + 2j_{2n}^{(3)} + 1).$
(c) $\sum_{k=0}^{n} j_{2k+1}^{(3)} = \frac{1}{3}(j_{2n+2}^{(3)} + 2j_{2n+1}^{(3)} - 4).$

3. Sum formulas of Generalized Tribonacci Numbers with Negative Subscripts

The following Theorem presents some linear summing formulas (identities) of generalized Tribonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$, we have the following formulas:*

(a) *(Sum of the generalized Tribonacci numbers with negative indices) If $r + s + t - 1 \neq 0$, then*

$$\sum_{k=1}^{n} W_{-k} = \frac{-(r+s+t)W_{-n-1} - (s+t)W_{-n-2} - tW_{-n-3} + W_2 + (1-r)W_1 + (1-r-s)W_0}{r+s+t-1}.$$

(b) *If $(r+s+t-1)(r-s+t+1) \neq 0$ then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{-(r+t)W_{-2n+1} + (r^2+rt+s-1)W_{-2n} + (st-t)W_{-2n-1} + (1-s)W_2 + (t+rs)W_1 + (1-rt-2s-r^2+s^2)W_0}{(r+s+t-1)(r-s+t+1)}$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{(s-1)W_{-2n+1} - (t+rs)W_{-2n} - (t^2+rt)W_{-2n-1} + (r+t)W_2 + (1-r^2-rt-s)W_1 + (t-st)W_0}{(r+s+t-1)(r-s+t+1)}.$$

(c) *If* $(r+s+t-1)(r-s+t+1) \neq 0 \wedge r+t=0 \wedge s \neq 1$ *then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{-W_{-2n} - tW_{-2n-1} + W_2 + tW_1 + (1-s)W_0}{s-1}$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{s-1}\left(-W_{-2n+1} - tW_{-2n} + W_1 + tW_0\right).$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_{-n+3} = rW_{-n+2} + sW_{-n+1} + tW_{-n} \Rightarrow W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

i.e.

$$tW_{-n} = W_{-n+3} - rW_{-n+2} - sW_{-n+1}$$

or

$$W_{-n} = \frac{1}{t}W_{-n+3} - \frac{r}{t}W_{-n+2} - \frac{s}{t}W_{-n+1}$$

we obtain

$$\begin{aligned}
tW_{-n} &= W_{-n+3} - rW_{-n+2} - sW_{-n+1} \\
tW_{-n+1} &= W_{-n+4} - rW_{-n+3} - sW_{-n+2} \\
tW_{-n+2} &= W_{-n+5} - rW_{-n+4} - sW_{-n+3} \\
&\vdots \\
tW_{-2} &= W_1 - r \times W_0 - s \times W_{-1} \\
tW_{-1} &= W_2 - r \times W_1 - s \times W_0.
\end{aligned}$$

If we add the above equations by side by, we get

$$\sum_{k=1}^{n} W_{-k} = \frac{-(rW_{-n-1} + s(W_{-n-1} + W_{-n-2}) + t(W_{-n-1} + W_{-n-2} + W_{-n-3}) - W_2 + (r-1)W_1 + (r+s-1)W_0)}{r+s+t-1}.$$

(b) and (c) Using the recurrence relation

$$W_{-n+3} = rW_{-n+2} + sW_{-n+1} + tW_{-n}$$

i.e.

$$sW_{-n+1} = W_{-n+3} - rW_{-n+2} - tW_{-n}$$

we obtain

$$\begin{aligned}
sW_{-2n+1} &= W_{-2n+3} - rW_{-2n+2} - tW_{-2n} \\
sW_{-2n+3} &= W_{-2n+5} - rW_{-2n+4} - tW_{-2n+2} \\
&\vdots \\
sW_{-3} &= W_{-1} - rW_{-2} - tW_{-4} \\
sW_{-1} &= W_1 - rW_0 - tW_{-2}.
\end{aligned}$$

If we add the equations by side by, we get

$$s\sum_{k=1}^{n} W_{-2k+1} = (-W_{-2n+1} + W_1 + \sum_{k=1}^{n} W_{-2k+1}) - r(-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) - t(\sum_{k=1}^{n} W_{-2k}). \tag{3.1}$$

Similarly, using the recurrence relation

$$W_{-n+3} = rW_{-n+2} + sW_{-n+1} + tW_{-n}$$

i.e.

$$sW_{-n+1} = W_{-n+3} - rW_{-n+2} - tW_{-n}$$

we obtain

$$\begin{aligned} sW_{-2n} &= W_{-2n+2} - rW_{-2n+1} - tW_{-2n-1} \\ sW_{-2n+2} &= W_{-2n+4} - rW_{-2n+3} - tW_{-2n+1} \\ &\vdots \\ sW_{-6} &= W_{-4} - rW_{-5} - tW_{-7} \\ sW_{-4} &= W_{-2} - rW_{-3} - tW_{-5} \\ sW_{-2} &= W_0 - rW_{-1} - tW_{-3}. \end{aligned}$$

If we add the above equations by side by, we get

$$s\sum_{k=1}^{n} W_{-2k} = (-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) - r(\sum_{k=1}^{n} W_{-2k+1}) - t(W_{-2n-1} - W_{-1} + \sum_{k=1}^{n} W_{-2k+1}).$$

Since

$$W_{-1} = (-\frac{s}{t}W_0 - \frac{r}{t}W_1 + \frac{1}{t}W_2).$$

it follows that

$$s\sum_{k=1}^{n} W_{-2k} = (-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) - r(\sum_{k=1}^{n} W_{-2k+1}) - t(W_{-2n-1} - (-\frac{s}{t}W_0 - \frac{r}{t}W_1 + \frac{1}{t}W_2) + \sum_{k=1}^{n} W_{-2k+1}). \tag{3.2}$$

Then, solving system (3.1)-(3.2) the required results of (b) and (c) follow.

Note that (c) of the above theorem can be written as follows: If $r+t=0 \wedge s \neq 1$ then

$$\sum_{k=1}^{n} W_{-2k} = \frac{-W_{-2n} + rW_{-2n-1} + W_2 - rW_1 + (1-s)W_0}{s-1}$$

and

$$\sum_{k=1}^{n} W_{-2k} = \frac{-W_{-2n} + rW_{-2n-1} + W_2 - rW_1 + (1-s)W_0}{s-1}.$$

Next, we present several sum formulas (identities).
Taking $r=s=t=1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.2. *If $r=s=t=1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}\left(-3W_{-n-1} - 2W_{-n-2} - W_{-n-3} + W_2 - W_0\right)$.
(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{2}\left(-W_{-2n+1} + W_{-2n} + W_1 - W_0\right)$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{2}\left(-W_{-2n} - W_{-2n-1} + W_2 - W_1\right)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 3.3. *For $n \geq 1$, Tribonacci numbers have the following properties.*

(a) *[13]* $\sum_{k=1}^{n} T_{-k} = \frac{1}{2}(-3T_{-n-1} - 2T_{-n-2} - T_{-n-3} + 1)$.
(b) $\sum_{k=1}^{n} T_{-2k} = \frac{1}{2}(-T_{-2n+1} + T_{-2n} + 1)$.
(c) $\sum_{k=1}^{n} T_{-2k+1} = \frac{1}{2}(-T_{-2n} - T_{-2n-1})$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above Proposition, we have the following Corollary which gives linear sum formulas of Tribonacci-Lucas numbers.

Corollary 3.4. *For $n \geq 1$, Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} K_{-k} = \frac{1}{2}(-3K_{-n-1} - 2K_{-n-2} - K_{-n-3})$.
(b) $\sum_{k=1}^{n} K_{-2k} = \frac{1}{2}(-K_{-2n+1} + K_{-2n} - 2)$.
(c) $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{2}(-K_{-2n} - K_{-2n-1} + 2)$.

Taking $r=2, s=1, t=1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.5. *If $r=2, s=1, t=1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{3}\left(-4W_{-n-1} - 2W_{-n-2} - W_{-n-3} + W_2 - W_1 - 2W_0\right)$.
(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}\left(-W_{-2n+1} + 2W_{-2n} + W_1 - 2W_0\right)$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}\left(-W_{-2n} - W_{-2n-1} + W_2 - 2W_1\right)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of third-order Pell numbers (take $W_n = P_n^{(3)}$ with $P_0^{(3)} = 0, P_1^{(3)} = 1, P_2^{(3)} = 2$).

Corollary 3.6. *For $n \geq 1$, third-order Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} P_{-k}^{(3)} = \frac{1}{3}(-4P_{-n-1}^{(3)} - 2P_{-n-2}^{(3)} - P_{-n-3}^{(3)} + 1)$.
(b) $\sum_{k=1}^{n} P_{-2k}^{(3)} = \frac{1}{3}(-P_{-2n+1}^{(3)} + 2P_{-2n}^{(3)} + 1)$.
(c) $\sum_{k=1}^{n} P_{-2k+1}^{(3)} = \frac{1}{3}(-P_{-2n}^{(3)} - P_{-2n-1}^{(3)})$.

Taking $W_n = Q_n^{(3)}$ with $Q_0^{(3)} = 3, Q_1^{(3)} = 2, Q_2^{(3)} = 6$ in the last Proposition, we have the following Corollary which gives linear sum formulas of third-order Pell-Lucas numbers.

Corollary 3.7. *For $n \geq 1$, third-order Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} Q_{-k}^{(3)} = \frac{1}{3}(-4Q_{-n-1}^{(3)} - 2Q_{-n-2}^{(3)} - Q_{-n-3}^{(3)} - 2)$.
(b) $\sum_{k=1}^{n} Q_{-2k}^{(3)} = \frac{1}{3}(-Q_{-2n+1}^{(3)} + 2Q_{-2n}^{(3)} - 4)$.
(c) $\sum_{k=1}^{n} Q_{-2k+1}^{(3)} = \frac{1}{3}(-Q_{-2n}^{(3)} - Q_{-2n-1}^{(3)} + 2)$.

From the last Proposition, we have the following Corollary which presents linear sum formulas of third-order modified Pell numbers (take $W_n = E_n^{(3)}$ with $E_0^{(3)} = 0, E_1^{(3)} = 1, E_2^{(3)} = 1$).

Corollary 3.8. *For $n \geq 1$, third-order modified Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} E_{-k}^{(3)} = \frac{1}{3}(-4E_{-n-1}^{(3)} - 2E_{-n-2}^{(3)} - E_{-n-3}^{(3)})$.
(b) $\sum_{k=1}^{n} E_{-2k}^{(3)} = \frac{1}{3}(-E_{-2n+1}^{(3)} + 2E_{-2n}^{(3)} + 1)$.
(c) $\sum_{k=1}^{n} E_{-2k+1}^{(3)} = \frac{1}{3}(-E_{-2n}^{(3)} - E_{-2n-1}^{(3)} - 1)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.9. *If $r = 0, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = -2W_{-n-1} - 2W_{-n-2} - W_{-n-3} + W_2 + W_1$.
(b) $\sum_{k=1}^{n} W_{-2k} = -W_{-2n+1} + W_1$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = -W_{-2n} - W_{-2n-1} + W_2$.

Taking $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Padovan numbers.

Corollary 3.10. *For $n \geq 1$, Padovan numbers have the following properties.*

(a) $\sum_{k=1}^{n} P_{-k} = -2P_{-n-1} - 2P_{-n-2} - P_{-n-3} + 2$.
(b) $\sum_{k=1}^{n} P_{-2k} = -P_{-2n+1} + 1$.
(c) $\sum_{k=1}^{n} P_{-2k+1} = -P_{-2n} - P_{-2n-1} + 1$.

From the last Proposition, we have the following Corollary which presents linear sum formulas of Perrin numbers (take $W_n = E_n$ with $E_0 = 3, E = 0, E_2 = 2$).

Corollary 3.11. *For $n \geq 1$, Perrin numbers have the following properties.*

(a) $\sum_{k=1}^{n} E_{-k} = -2E_{-n-1} - 2E_{-n-2} - E_{-n-3} + 2$.
(b) $\sum_{k=1}^{n} E_{-2k} = -E_{-2n+1}$.
(c) $\sum_{k=1}^{n} E_{-2k+1} = -E_{-2n} - E_{-2n-1} + 2$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Padovan-Perrin numbers.

Corollary 3.12. *For $n \geq 1$, Padovan-Perrin numbers have the following properties.*

(a) $\sum_{k=1}^{n} S_{-k} = -2S_{-n-1} - 2S_{-n-2} - S_{-n-3} + 1$.
(b) $\sum_{k=1}^{n} S_{-2k} = -S_{-2n+1}$.
(c) $\sum_{k=1}^{n} S_{-2k+1} = -S_{-2n} - S_{-2n-1} + 1$.

If $r = 0, s = 2, t = 1$ then $(r+s+t-1)(r-s+t+1) = 0$ so we can't use Theorem 3.1 (b) and (c). In other words, the method of the proof Theorem 3.1 (b) and (c) can't be used to find $\sum_{k=0}^{n} W_{2k}$ and $\sum_{k=0}^{n} W_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 3.13. *If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-3W_{-n-1} - 3W_{-n-2} - W_{-n-3} + W_2 + W_1 - W_0)$.
(b) $\sum_{k=1}^{n} W_{-2k} = -W_{-2n+1} + W_{-2n} + (W_1 - W_0) + (W_2 - W_1 - W_0)n$.
(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{2}(W_{-2n+1} - 3W_{-2n} - W_{-2n-1} + (W_2 - W_1 + W_0) + 2(-W_2 + W_1 + W_0)n)$.

Proof.

(a) Taking $r=0,s=2,t=1$ in Theorem 3.1 (a) we obtain (a).
(b) and (c) Proof can be done as in the proof of Theorem 2.13. Induction also can be used for the proof.

From the last Theorem, we have the following Corollary which gives sum formula of Pell-Padovan numbers (take $W_n=R_n$ with $R_0=1,R=1,R_2=1$).

Corollary 3.14. *For $n\geq 1$, Pell-Padovan numbers have the following property:*

(a) $\sum_{k=1}^{n} R_{-k}=\frac{1}{2}\left(-3R_{-n-1}-3R_{-n-2}-R_{-n-3}+1\right).$
(b) $\sum_{k=1}^{n} R_{-2k}=-R_{-2n+1}+R_{-2n}-n.$
(c) $\sum_{k=1}^{n} R_{-2k+1}=\frac{1}{2}(R_{-2n+1}-3R_{-2n}-R_{-2n-1}+1+2n).$

Taking $W_n=C_n$ with $C_0=3,C=0,C_2=2$ in the last Theorem, we have the following Corollary which gives sum formulas of Pell-Perrin numbers.

Corollary 3.15. *For $n\geq 1$, Pell-Perrin numbers have the following property:*

(a) $\sum_{k=1}^{n} C_{-k}=\frac{1}{2}\left(-3C_{-n-1}-3C_{-n-2}-C_{-n-3}-1\right)$
(b) $\sum_{k=1}^{n} C_{-2k}=-C_{-2n+1}+C_{-2n}-3-n$
(c) $\sum_{k=1}^{n} C_{-2k+1}=\frac{1}{2}(C_{-2n+1}-3C_{-2n}-C_{-2n-1}+5+2n)$

Taking $r=0,s=1,t=2$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.16. *If $r=0,s=1,t=2$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k}=\frac{1}{2}\left(-3W_{-n-1}-3W_{-n-2}-2W_{-n-3}+W_2+W_1\right).$
(b) $\sum_{k=1}^{n} W_{-2k}=\frac{1}{2}\left(-W_{-2n+1}+W_1\right).$
(c) $\sum_{k=1}^{n} W_{-2k+1}=\frac{1}{2}\left(-W_{-2n}-2W_{-2n-1}+W_2\right).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of Jacobsthal-Padovan numbers (take $W_n=Q_n$ with $Q_0=1,Q_1=1,Q_2=1$).

Corollary 3.17. *For $n\geq 1$, Jacobsthal-Padovan numbers have the following properties.*

(a) $\sum_{k=1}^{n} Q_{-k}=\frac{1}{2}\left(-3Q_{-n-1}-3Q_{-n-2}-2Q_{-n-3}+2\right).$
(b) $\sum_{k=1}^{n} Q_{-2k}=\frac{1}{2}\left(-Q_{-2n+1}+1\right).$
(c) $\sum_{k=1}^{n} Q_{-2k+1}=\frac{1}{2}\left(-Q_{-2n}-2Q_{-2n-1}+1\right).$

Taking $W_n=D_n$ with $D_0=3,D_1=0,D_2=2$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Jacobsthal-Perrin numbers.

Corollary 3.18. *For $n\geq 1$, Jacobsthal-Perrin numbers have the following properties.*

(a) $\sum_{k=1}^{n} D_{-k}=\frac{1}{2}\left(-3D_{-n-1}-3D_{-n-2}-2D_{-n-3}+2\right).$
(b) $\sum_{k=1}^{n} D_{-2k}=\frac{-1}{2}D_{-2n+1}.$
(c) $\sum_{k=1}^{n} D_{-2k+1}=\frac{1}{2}\left(-D_{-2n}-2D_{-2n-1}+2\right).$

Taking $r=1,s=0,t=1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 3.19. *If $r=1,s=0,t=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k}=-2W_{-n-1}-W_{-n-2}-W_{-n-3}+W_2.$
(b) $\sum_{k=1}^{n} W_{-2k}=\frac{1}{3}\left(-2W_{-2n+1}+W_{-2n}-W_{-2n-1}+W_2+W_1-W_0\right).$
(c) $\sum_{k=1}^{n} W_{-2k+1}=\frac{1}{3}\left(-W_{-2n+1}-W_{-2n}-2W_{-2n-1}+2W_2-W_1+W_0\right).$

From the above Proposition, we have the following Corollary which gives linear sum formulas of Narayana numbers (take $W_n=N_n$ with $N_0=0,N_1=1,N_2=1$).

Corollary 3.20. *For $n\geq 1$, Narayana numbers have the following properties.*

(a) $\sum_{k=1}^{n} N_{-k}=-2N_{-n-1}-N_{-n-2}-N_{-n-3}+1.$
(b) $\sum_{k=1}^{n} N_{-2k}=\frac{1}{3}\left(-2N_{-2n+1}+N_{-2n}-N_{-2n-1}+2\right).$
(c) $\sum_{k=1}^{n} N_{-2k+1}=\frac{1}{3}\left(-N_{-2n+1}-N_{-2n}-2N_{-2n-1}+1\right).$

Taking $r=1,s=1,t=2$ in Theorem 3.1, we obtain the following Proposition.

Proposition 3.21. *If $r=1,s=1,t=2$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k}=\frac{1}{3}(-4W_{-n-1}-3W_{-n-2}-2W_{-n-3}+W_2-W_0).$
(b) $\sum_{k=1}^{n} W_{-2k}=\frac{1}{3}\left(-W_{-2n+1}+W_{-2n}+W_1-W_0\right).$
(c) $\sum_{k=1}^{n} W_{-2k+1}=\frac{1}{3}\left(-W_{-2n}-2W_{-2n-1}+W_2-W_1\right).$

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of third order Jacobsthal numbers.

Corollary 3.22. *For $n \geq 1$, third order Jacobsthal numbers have the following properties.*

(a) $\sum_{k=1}^{n} J_{-k}^{(3)} = \frac{1}{3}(-4J_{-n-1}^{(3)} - 3J_{-n-2}^{(3)} - 2J_{-n-3}^{(3)} + 1).$
(b) $\sum_{k=1}^{n} J_{-2k}^{(3)} = \frac{1}{3}(-J_{-2n+1}^{(3)} + J_{-2n}^{(3)} + 1).$
(c) $\sum_{k=1}^{n} J_{-2k+1}^{(3)} = \frac{1}{3}(-J_{-2n}^{(3)} - 2J_{-2n-1}^{(3)}).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of third order Jacobsthal-Lucas numbers (take $W_n = j_n^{(3)}$ with $j_0^{(3)} = 2, j_1^{(3)} = 1, j_2^{(3)} = 5$).

Corollary 3.23. *For $n \geq 1$, third order Jacobsthal-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} j_{-k}^{(3)} = \frac{1}{3}(-4j_{-n-1}^{(3)} - 3j_{-n-2}^{(3)} - 2j_{-n-3}^{(3)} + 3).$
(b) $\sum_{k=1}^{n} j_{-2k}^{(3)} = \frac{1}{3}(-j_{-2n+1}^{(3)} + j_{-2n}^{(3)} - 1).$
(c) $\sum_{k=1}^{n} j_{-2k+1}^{(3)} = \frac{1}{3}(-j_{-2n}^{(3)} - 2j_{-2n-1}^{(3)} + 4).$

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