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Discussion on Some Common Properties of Hilbert Space and Topological Space

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Keywords

Hilbert Space
Topological Space
Hausdorff Space
Inner product space
Γ- Hilbert space

Abstract

Theory of elliptic equations and undergone considerable progress that has created the concept of the topological character and eventually has created interesting connection with the analysis and Hilbert Space. Connection of Hilbert space and Topological space has been a matter of curiosity for many. Although there is no much work done on this topic, the answer to this question is Hilbert space is a part of the Topological space only when there is a need of functional analysis. In other words topology is induced to the Hilbert space but in real they are not a part of each other. In algebra concept, topology is involved in the Hilbert spaces to support the idea of metric space. Topological space consists of abstract sets of points that includes specific collection open sets of subsets that need to satisfy the axioms. Hausdorff space is one of the types of topological space .Lot of properties are satisfied by the Hausdorff space which are not satisfied by other way. Unlike the Hilbert space, the topological space is not highly complicated and forms the basis of the functional analysis. Two things, a topological space and one special type of vector are present at once in the Hilbert space. Thus, in Hilbert space more topological structure is given by the topological space while the special type of vector would help in giving some algebraic space.

1. Introduction

For the number system, the Hilbert space can use only the associative Division rings. Inner product space is Hilbert space and primarily is a concept of linear algebra. It holds the abstract algebraic approach where the bounded linear operator approach can be observed. Concept of the Hilbert space emerged from the notion of the Euclidean space. It includes the concept of calculus from the two-dimensional space to spaces with both finite and infinite number of dimensions.

Considering the definition of Topological space in mathematical term, it is generalization of notion of an object present in three-dimension space. This work focus on identifying the common properties of Hilbert space and Topological space. This work also includes the definition and analysis of Hausdroff space and how it is related to Topological space [1-12].

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2. Preliminaries

2.1. Hilbert space

Let E be a complex vector space. A mapping $\langle .,. \rangle: E \times E \to \mathbb{C}$ is called an *inner product in E* if for any $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$ the following conditions are satisfied:

(a) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (the bar denotes the complex conjugate)
(b) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;
(c) $\langle x, x \rangle \geq 0$;
(d) $\langle x, x \rangle = 0$ implies $x = 0$.

A vector space with an inner product is called an inner product space. A complete inner product space is called a Hilbert Space.

2.2. Topological space

A topology on a set X is a collection τ of subsets of X having the following properties:

a) Empty set $\emptyset$ and X are in τ.
b) The union of elements of any sub collection of τ is in τ.
c) The intersection of the elements of any finite sub collection of τ is in τ.

X is a topological space with topology τ.

A topological space X is said to satisfy the Hausdorff separation property or simply Hausdorff space ,if for all distinct points $x, y \in X$ there exist disjoint open sets $U \; and \; V$ such that $x \in U \; and \; y \in V$. Such open sets are said to separate $x \; and \; y$.

In mathematical concept, Hausdorff space is one of the types of topological space. Lot of properties are satisfied by the Hausdorff space which are not by topological space.

Hilbert space is isomorphic to the space i.e. L^2.Structure of the Hilbert space are having higher structure than the topological spaces.

3. Theorem

Hilbert space is topological and vectors are part of Hilbert space.

Two things, a topological space and one special type of vector are present at once in the Hilbert space. Thus, in Hilbert space more topological structure is given by the topological space while the special type of vector would help in giving some algebraic space. A vector space is defined as a bunch of vectors that could be multiplied or added by scalars. Operations of them are highly compatible where scalar multiplication obeys the distributive law and the vector addition follows the associative law. Therefore, Hilbert space is a vector space that does have an inner product. Inner product space of the vector is operation in various ways.

■ Lets prove the above theorem –

Consider $U, V \; and \; W$ are the vectors. Their vector space operation could be
$(U + V, W) = (U, W) + (V, W)$.

Now considering the scalars, conjugate symmetric could be presented as $(x, y) = \overline{(y, x)}$, the complex conjugation is indicated by the bar above. We have already defined inner product and this can be used to specify the notion of size of the vector X and define $||X||^2 = \langle X, X \rangle$ and this can be operated with the help of the Pythagorean theorem.

Considering the vector size, for defining the convergence a bunch of x_n vectors converge to a specific point that is p (say) only if $\|x_n - p\|$. The result will decrease to 0. This indicates that topology can be done with respect to the vectors. This indicates that the algebraic tool here come from the inner product and it is related to limits of sequences and continuous functions. Eventually indicates that topological space is a part of the topological space only when there is a need of functional analysis.

Ideally, a quadratic constraint helps in defining the curve of the Hilbert space which could be gamma systems. Γ-Hilbert space was first introduced by D.K. Bhattacharya along with T.E. Aman in 2003.However, the theory goes like very group denoted by Γ Having a left invariant metric d i.e. (Γ,d) is regarded as a bounded geometry. This confirms that it satisfies of getting embedded into a Hilbert space. Γ-Hilbert space is uniformly embedded into the Hilbert space.

4. Result

From the above theory it can be concluded that the vector space is complete and is a part of the Hilbert space. Both the topological structure and the algebraic structure are highly compatible. A Hilbert space is a vector space that has an inner product space which is complete nth topology and which is induced by the inner product. Hilbert spaces are denoted as R_n or C_n.

Now we see the Relationship between various types of abstract spaces in Figure 1.

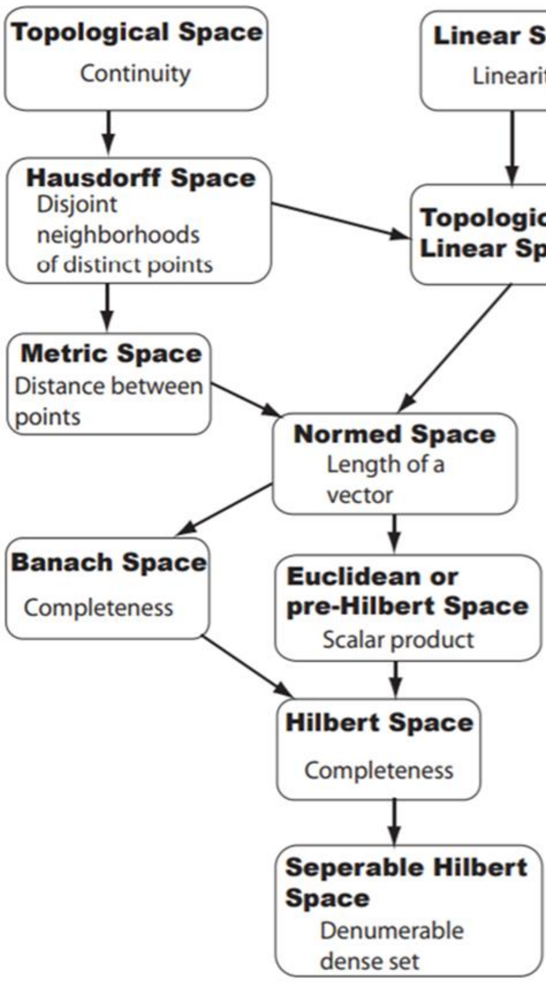

Figure 1. Relationship between various types of abstract spaces.

5. Discussion

Hilbert space is ideally denoted as H which is a complex or real inner product space. Also which is a complete metric space in consideration to the distance that is induced by the inner product function. Complex vector space is one of the properties of Hilbert space. Again three main properties of Hilbert space include,

In its first argument the inner product is linear i.e.

$$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$$

Conjugate symmetry can be seen for the inner product i.e.

$$\langle x, y\rangle = \overline{\langle y, x\rangle}$$

For an element, its inner product is positive definite which is,

$$\begin{cases} \langle x, x\rangle > 0 & x \neq 0 \\ \langle x, x\rangle = 0 & x = 0 \end{cases}.$$

For sequencing in the Hilbert space (H) , Cauchy criterion form is used that would lead to the completeness of Hilbert space while the special type of vector would help in giving some algebraic space.

A vector space operation is highly compatible where scalar multiplication obeys the distributive law and the vector addition follows associative law.

Topological space or linear topological space is a form of basic structure that comes under the functional analysis. It is a form of vector space which have a notion of continuity that allows a notion uniform convergence. Topological structure is a form of a commutative group that are responsible for operation of two group elements which does not depend each other. This group is also known as Abelian group. Uniform convergence, completeness and uniform continuity are the features of the topological vector space. Completeness helps in doing the real analysis as it is the fundamental property of the real numbers.

Ideally, there should be no gaps in the space and thus helps in completing the sequence.

Γ-Hilbert space is composed with a metric d which is left-invariant and has bounded geometry, the embedding in the Hilbert space would be highly uniform. This indicates that its property is independent and its coarse assembly could define Γ as an isomorphic.

6. Conclusions

It can be said that concept of the Hilbert space emerged from the notion of the Euclidean space. It includes the concept of calculus from two-dimensional spaces to spaces with both finite and infinite number of dimensions. The Hilbert space occurs frequently and naturally bit is without having an angle. Topological space is a generalization of notion of an object present in three-dimensional space. There is a huge scope for further work in this topic and we will extend our work.

Declaration of Competing Interest

The authors declare that there is no competing financial interests or personal relationships that influences the work in this paper.

Authorship Contribution Statement

Ashoke Das: Supervision, Reviewing and Editing, Conceptualization, Methodology.
Sahin Injamamul Islam: Data creation, Writing, Reviewing, Draft Preparation, Investigation.

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