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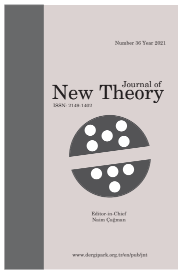

Relative Gol'dberg Order and Type of Multiple Entire Dirichlet Series in Terms of Coefficients and Exponents

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Abstract — This paper aims to define and characterize the relative Gol'dberg order and type of a multiple entire Dirichlet series with respect to another multiple entire Dirichlet series in terms of their coefficients and exponents. By using the definition, we study the growth properties of the Hadamard product between two such series.

Keywords — *Multiple entire Dirichlet series, Gol'dberg order and type of multiple entire Dirichlet series, Relative Gol'dberg order and type of multiple entire Dirichlet series*

Mathematics Subject Classification (2020) — 30B50, 30D99

1. Introduction

Relative Gol'dberg order and type of a multiple entire Dirichlet series with respect to another multiple entire Dirichlet series in terms of their maximum modulus function, has been defined in [1] and [2] respectively. Those definitions have been used to study about growth properties of sum functions and asymptotically equivalent multiple entire Dirichlet series. Hadamard product between two such series involves coefficients and exponents of them. Therefore, use of the definition which involves maximum modulus function, may not be useful to study about growth property of Hadamard product. So, it is necessary to find an expression of relative Gol'dberg order and type in terms of coefficients and exponents of the two multiple entire Dirichlet series.

We now briefly discuss about entire Dirichlet series in one complex variable and the reasons due to which, the series satisfies the condition to be an entire function.

A Series of the form

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n s} \tag{1}$$

where $s = \sigma + it, a_n \in \mathcal{C}$ and $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ is called a Dirichlet series in one complex variable. If the Dirichlet series is convergent at $s_0 = \sigma_0 + it_0 \in \mathcal{C}$, then it is convergent at any point in the set $D = \{s \in \mathcal{C} : Re(s) < Re(s_0)\}$ and uniformly convergent in the domain $D_1 = \{s \in \mathcal{C} : |arg(s - s_0)| \leq \theta < \frac{\pi}{2}\}$. The abscissa of convergence of (1) is $\sigma_c = \sup\{\sigma \in \mathcal{R} :$ series(1) converges for all $s \in \mathcal{C}$ where $Re(s) < \sigma\}$.

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The series (1) is absolutely convergent if the series

$$\sum_{n=1}^{\infty} |a_n e^{\lambda_n s}| = \sum_{n=1}^{\infty} |a_n| e^{\lambda_n \sigma} \tag{2}$$

is convergent. Let σ_A be the abscissa of absolute convergence of the series (1).

The following theorem gives a general relationship between the abscissa of convergence and the abscissa of absolute convergence.

Theorem 1.1. ([3], p-31) If the exponents λ_n in (1) satisfy the condition $L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n} < \infty$, then $0 \le \sigma_c - \sigma_A \le L$.

If $L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0$, then $\sigma_A = \sigma_c = -\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n}$.

Therefore, the series (1) will represent an entire function $g(s)$ if and only if [3]

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty \tag{3}$$

In this paper, we have studied about multiple entire Dirichlet series. Next, we write the notations which have been used throughout this paper and then briefly discuss about the series in multiple complex variables.

1.1. Notations

For $s = (s_1, s_2, \ldots, s_n), w = (w_1, w_2, \ldots, w_n) \in \mathcal{C}^n$, and $\alpha \in \mathcal{C}$, we define $s = w$ if and only if $s_i = w_i$, $s + w = (s_1 + w_1, s_2 + w_2, \ldots, s_n + w_n)$, $\alpha s = (\alpha s_1, \alpha s_2, \ldots, \alpha s_n)$, $s.w = s_1 w_1 + s_2 w_2 + \cdots + s_n w_n$, $\mid s \mid = (|s_1|^2 + |s_2|^2 + \cdots + |s_n|^2)^{\frac{1}{2}}$. $s + R = (s_1 + R, s_2 + R, \ldots, s_n + R)$, for $R \in \mathcal{R}$. $\lambda_{n,m} = (\lambda_{1m_1}, \lambda_{2m_2}, \ldots, \lambda_{nm_n}) \in \mathcal{R}^{+^n}$ where $\mathcal{R}^{+^n} = \{x : x \in \mathcal{R}^n, x_i \ge 0\}$. $s.\lambda_{n,m} = s_1\lambda_{1m_1} + s_2\lambda_{2m_2} + \cdots + s_n\lambda_{nm_n}$. $\lambda_{n,m}^k = (\lambda_{1m_1}^k, \lambda_{2m_2}^k, \ldots, \lambda_{nm_n}^k) \in \mathcal{R}^{+^n}$, for $k = 0, 1, 2, \ldots$ $\|\lambda_{n,m}\| = \lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n}$ For $r, t \in \mathcal{R}^{+^n}$, we define $r \le t$ if and only if $r_i \le t_i$ and $r < t$ if and only if $r_i < t_i$ for $i = 1, 2, \ldots, n$.

Definition 1.2. A multiple entire Dirichlet Series is of the form

$$f(s) = \sum_{\|m\|=1}^{\infty} a_{m_1,\ldots,m_n} e^{s.\lambda_{n,m}} \tag{4}$$

where $a_{m_1,\ldots,m_n} \in \mathcal{C}, s = (s_1, s_2, \ldots, s_n) \in \mathcal{C}^n$, $s_j = \sigma_j + it_j, j = 1, 2, \ldots, n$, and $\{\lambda_{j,m_j}\}_{m_j=1}^{\infty}$, $j = 1, \ldots, n$ are n sequences of exponents satisfying the conditions $0 \le \lambda_{jm_1} < \lambda_{jm_2} < \cdots < \lambda_{jm_k} \to \infty$ as $k \to \infty$, $j = 1, \ldots, n$, and $\lim_{m_j\to\infty} \frac{\log m_j}{\lambda_{jm_j}} = 0, j = 1, 2, \ldots, n$.

As described in Equation (3), series (4) must satisfy the following condition in order to represent an entire function

$$\limsup_{\|m\|\to\infty} \frac{\log |a_{m_1 \cdots n}|}{\|\lambda_{n,m}\|} = -\infty \tag{5}$$

Let $D \subset \mathcal{C}^n$ be an arbitrary complete n-half-plane defined by $D = \{s : s \in \mathcal{C}^n, Re(s_i) \le r_i\}$ where $r = (r_1, r_2, \ldots, r_n) \in \mathcal{R}^n$. Consider a parameter $R \in \mathcal{R}$, define $R + D = D + R = \{s + R : s \in D\}$. Then, for the multiple Dirichlet entire function f, the maximum modulus function $M_{f,D}(R)$ with respect to the region D and $R \in \mathcal{R}$ is defined as

$$M_{f,D}(R) = \sup\{\mid f(s) \mid : s \in D + R\} \tag{6}$$

$M_{f,D}(R)$ is strictly increasing, increases to ∞ and continuous functions of R. The inverse function is $M_{f,D}^{-1} : (L, \infty) \to (-\infty, \infty)$ where $0 \le L = \lim_{R\to-\infty} M_{f,D}(R)$.

Throughout this paper, we have considered a class of multiple entire Dirichlet series with same sequence of exponents.

For $k = 0, 1, 2, 3, \ldots$, we define

$$f^k(s) = \sum_{\|m\|=1}^{\infty} \|\lambda_{n,m}^k\| a_m e^{s.\lambda_{n,m}} \tag{7}$$

Therefore, for two multiple entire Dirichlet series $f(s)$ and $g(s)$, we have $(f+g)^k = f^k + g^k, k = 0, 1, 2, \ldots$

2. Preliminary

In this section, we write a few preliminary definitions which have been used to prove the theorems and results in the next section.

Definition 2.1. [4] The Gol'dberg order of a multiple entire Dirichlet Series f with respect to the domain D is defined by

$$\rho_f(D) = \limsup_{R\to\infty} \frac{\log\log M_{f,D}(R)}{R} \tag{8}$$

Definition 2.2. [4] The lower Gol'dberg order of a multiple entire Dirichlet Series f with respect to the domain D is defined by

$$\lambda_f(D) = \liminf_{R\to\infty} \frac{\log\log M_{f,D}(R)}{R} \tag{9}$$

f is said to be of regular growth if $\rho_f(D) = \lambda_f(D)$.

Definition 2.3. [4] The Gol'dberg type of a multiple entire Dirichlet Series f with Gol'dberg order $\rho_f(D)$, $(0 < \rho_f(D) < \infty)$ with respect to the domain D, is defined by

$$\sigma_f(D) = \limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{R\rho_f(D)}} \tag{10}$$

Definition 2.4. [4] The lower Gol'dberg type of a multiple entire Dirichlet Series f with Gol'dberg order $\rho_f(D)$, $(0 < \rho_f(D) < \infty)$ with respect to the domain D, is defined by

$$\tau_f(D) = \liminf_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{R\rho_f(D)}} \tag{11}$$

f is said to be of perfectly regular growth if $\rho_f(D) = \lambda_f(D)$ and $\sigma_f(D) = \tau_f(D)$.

Definition 2.5. [1] Let f and g be two multiple entire Dirichlet series. The relative Gol'debrg order of f with respect to g, denoted by $\rho_{g,D}(f)$, is defined as

$$\rho_{g,D}(f) = \limsup_{R\to\infty} \frac{M_{g,D}^{-1}(M_{f,D}(R))}{R} \tag{12}$$

Definition 2.6. [2] The relative Gol'dberg type of a multiple entire Dirichlet series f with respect to another multiple entire Dirichlet series g, with $0 < \rho_{g,D}(f) < \infty$, denoted by $\sigma_{g,D}(f)$, is defined as

$$\sigma_{g,D}(f) = \limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{\log M_{g,D}(\rho_{g,D}(f)R)} \tag{13}$$

From the definition(2.5) and (2.6) it follows that for $k = 0, 1, 2, \ldots$

$$\rho_{g,D}(f^k) = \limsup_{R\to\infty} \frac{M_{g,D}^{-1}(M_{f^k,D}(R))}{R} \tag{14}$$

and

$$\sigma_{g,D}(f^k) = \limsup_{R\to\infty} \frac{\log M_{f^k,D}(R)}{\log M_{g,D}(\rho_g(f^k)R)} \tag{15}$$

where f^k is defined in Equation (7).

Definition 2.7. [5] P. K. Sarkar defined the Gol'dberg order $\rho_f(D)$ of a multiple entire Dirichlet series f in terms of coefficients and exponents as

$$\rho_f(D) = \limsup_{\|m\|\to\infty} \frac{\|\lambda_{n,m}\| \log \|\lambda_{n,m}\|}{-\log |a_m|} \tag{16}$$

Definition 2.8. [5] P. K. Sarkar also defined the Gol'dberg type $\sigma_f(D)$ of a multiple entire Dirichlet series f in terms of coefficients and exponents as

$$\sigma_f(D) = \frac{1}{e\rho_f(D)} \limsup_{\|m\|\to\infty} \|\lambda_{n,m}\| \Big\{ |a_m| \phi_D(m) \Big\}^{\frac{\rho_f(D)}{\|\lambda_{n,m}\|}} \tag{17}$$

where $\phi_D(m) = \sup_{s\in D} |\exp\{s.\lambda_{n,m}\}|$

Definition 2.9. For $f(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $g(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$, Hadamard product $f * g$ is defined as

$$(f * g)(s) = f(s) * g(s) = \sum_{\|m\|=1}^{\infty} a_m b_m e^{s.\lambda_{n,m}} \tag{18}$$

Then, for $k = 0, 1, 2, 3, \ldots,$

$$(f(s) * g(s))^k = \sum_{\|m\|=1}^{\infty} \|\lambda_{n,m}^k\| a_m b_m e^{s.\lambda_{n,m}} \tag{19}$$

$$f^k(s) * g^k(s) = \sum_{\|m\|=1}^{\infty} \|\lambda_{n,m}^{2k}\| a_m b_m e^{s.\lambda_{n,m}} \tag{20}$$

where f^k is defined in Equation (7).

We know that Gol'dberg order $\rho_f(D)$ and relative Gol'dberg order $\rho_{g,D}(f)$ does not depend on the choice of domain D while Gol'dberg type $\sigma_f(D)$ and relative Gol'dberg type $\sigma_{g,D}(f)$ does ([5]). Henceforth we may write ρ_f and $\rho_g(f)$ instead of writing $\rho_f(D)$ and $\rho_{g,D}(f)$.

3. Theorems and Results

In this section, we have proved the theorems which establish relative Gol'dberg order and type in terms of coefficients and exponents of a multiple entire Dirichlet series. Before proving the theorem, we write the statement of Theorem (4.2.2) ([6], Chapter 4) and Theorem (5.2.5) ([6], Chapter 5) which has been used to prove the main results.

Lemma 3.1. Let f and g be two multiple entire Dirichlet series of finite Gol'dberg orders ρ_f and ρ_g such that $\rho_g \neq 0$. Then the relative Gol'dberg order of f with respect to g satisfies the inequality $\rho_g(f) \geq \frac{\rho_f}{\rho_g}$. If g is of regular growth then $\rho_g(f) = \frac{\rho_f}{\rho_g}$.

Lemma 3.2. Let f and g be two multiple entire Dirichlet series of finite Gol'dberg orders ρ_f , ρ_g and Gol'dberg types $\sigma_f(D)$, $\sigma_g(D)$ respectively such that $\sigma_g(D) \neq 0$ and g is of regular growth. Then the relative Gol'dberg type of f with respect to g satisfies the inequality $\sigma_{g,D}(f) \geq \frac{\sigma_f(D)}{\sigma_g(D)}$. Moreover, if g is of perfectly regular growth then $\sigma_{g,D}(f) = \frac{\sigma_f(D)}{\sigma_g(D)}$.

Theorem 3.3. Let $f(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $g(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ be two multiple entire Dirichlet series of finite Gol'dberg orders ρ_f and ρ_g such that $\rho_g \neq 0$ and g is of regular growth. Then, the relative Gol'dberg order of f with respect to g is given by $\rho_g(f) = \limsup_{\|m\|\to\infty} \frac{\log |b_m|}{\log |a_m|}$.

Proof. Since g is of regular growth, by Lemma (3.1),

$$\begin{aligned}
\rho_g(f) &= \frac{\rho_f}{\rho_g} \\
&= \limsup_{\|m\|\to\infty} \frac{\| \lambda_{n,m} \| \log \| \lambda_{n,m} \|}{\log |\frac{1}{a_m}|} \liminf_{\|m\|\to\infty} \frac{\log |\frac{1}{b_m}|}{\| \lambda_{n,m} \| \log \| \lambda_{n,m} \|}, \qquad \text{by Equation (16)} \\
&\leq \limsup_{\|m\|\to\infty} \frac{\log |b_m|}{\log |a_m|} = \mu \ \text{(say)} \qquad (21)
\end{aligned}$$

Then, for any $\varepsilon > 0$ there is an increasing sequence $\{m_k\}$ of positive integers, increasing to ∞, such that $\frac{\log |\frac{1}{b_{m_k}}|}{\log |\frac{1}{a_{m_k}}|} > \mu - \varepsilon$. Hence,

$$\frac{\log |\frac{1}{b_{m_k}}|}{\| \lambda_{n,m_k} \| \log \| \lambda_{n,m_k} \|} . \frac{\| \lambda_{n,m_k} \| \log \| \lambda_{n,m_k} \|}{\log |\frac{1}{a_{m_k}}|} > \mu - \varepsilon \qquad (22)$$

This implies

$$\begin{aligned}
\rho_f &= \limsup_{\|m\|\to\infty} \frac{\| \lambda_{n,m} \| \log \| \lambda_{n,m} \|}{\log |\frac{1}{a_m}|} \\
&\geq \limsup_{\|m_k\|\to\infty} \frac{\| \lambda_{n,m_k} \| \log \| \lambda_{n,m_k} \|}{\log |\frac{1}{a_{m_k}}|} \\
&\geq (\mu - \varepsilon) \limsup_{\|m_k\|\to\infty} \frac{\| \lambda_{n,m_k} \| \log \| \lambda_{n,m_k} \|}{\log |\frac{1}{b_{m_k}}|}, \qquad \text{by Equation (22)} \\
&\geq (\mu - \varepsilon) \liminf_{\|m\|\to\infty} \frac{\| \lambda_{n,m} \| \log \| \lambda_{n,m} \|}{\log |\frac{1}{b_m}|} \\
&= (\mu - \varepsilon)\rho_g, \qquad [\text{Since } g \text{ is of regular growth}]
\end{aligned}$$

Therefore,

$$\frac{\rho_f}{\rho_g} > \mu - \varepsilon \qquad (23)$$

Since $\varepsilon > 0$ is arbitrary, combining (21) and (23)

$$\rho_g(f) = \limsup_{\|m\|\to\infty} \frac{\log |b_m|}{\log |a_m|}$$

□

In the next theorem, we have established relative Gol'dberg type in terms of coefficients and exponents of a multiple entire Dirichlet series.

Theorem 3.4. Let $f(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $g(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ be two non constant multiple entire Dirichlet series of finite Gol'dberg orders ρ_f , ρ_g and Gol'dberg types $\sigma_f(D)$, $\sigma_g(D)$ respectively such that $\sigma_g(D) \neq 0$ and g is of perfectly regular growth. Then, the relative Gol'dberg type of f with respect to g is given by

$$\begin{aligned}
\sigma_{g,D}(f) &= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \left[\frac{\{|a_m|\phi_D(m)\}^{\rho_f}}{\{|b_m|\phi_D(m)\}^{\rho_g}} \right]^{\frac{1}{\|\lambda_{n,m}\|}} \\
&= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \left[\frac{\{|a_m|\phi_D(m)\}^{\rho_g(f)}}{|b_m|\phi_D(m)} \right]^{\frac{\rho_g}{\|\lambda_{n,m}\|}}
\end{aligned}$$

where $\phi_D(m) = \sup_{s\in D} |\exp\{s.\lambda_{n,m}\}|$.

PROOF. Since $\sigma_g(D) \neq 0$ and g is of perfectly regular growth, then by Lemma (3.2) ,

$$\begin{aligned}
\sigma_{g,D}(f) &= \frac{\sigma_f(D)}{\sigma_g(D)} \\
&= \frac{\frac{1}{e\rho_f} \limsup\limits_{\|m\|\to\infty} \parallel \lambda_{n,m} \parallel \{|a_m|\phi_D(m)\}^{\frac{\rho_f}{\|\lambda_{n,m}\|}}}{\frac{1}{e\rho_g} \limsup\limits_{\|m\|\to\infty} \parallel \lambda_{n,m} \parallel \{|b_m|\phi_D(m)\}^{\frac{\rho_g}{\|\lambda_{n,m}\|}}}, \quad \text{by Definition (2.7)} \\
&\leq \frac{\rho_g}{\rho_f} \limsup_{\|m\|\to\infty} \Big[\frac{\{|a_m|\phi_D(m)\}^{\rho_f}}{\{|b_m|\phi_D(m)\}^{\rho_g}}\Big]^{\frac{1}{\|\lambda_{n,m}\|}} \qquad (24) \\
&= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \Big[\frac{\{|a_m|\phi_D(m)\}^{\rho_g(f)}}{|b_m|\phi_D(m)}\Big]^{\frac{\rho_g}{\|\lambda_{n,m}\|}} = \mu(\text{say}) \qquad (25)
\end{aligned}$$

Therefore, for any $\varepsilon > 0$ there is an increasing sequence $\{m_k\}$ of positive integers, increasing to infinity such that

$$\frac{1}{\rho_g(f)} \Big[\frac{\{|a_{m_k}|\phi_D(m_k)\}^{\rho_f}}{\{|b_{m_k}|\phi_D(m_k)\}^{\rho_g}}\Big]^{\frac{1}{\|\lambda_{n,m_k}\|}} > \mu - \varepsilon$$

Hence,

$$\frac{\{|a_{m_k}|\phi_D(m_k)\}^{\frac{\rho_f}{\|\lambda_{n,m_k}\|}}}{e.\rho_f} \geq (\mu - \varepsilon) \frac{\{|b_{m_k}|\phi_D(m_k)\}^{\frac{\rho_g}{\|\lambda_{n,m_k}\|}}}{e.\rho_g} \qquad (26)$$

Therefore,

$$\begin{aligned}
\sigma_f(D) &= \frac{1}{e\rho_f} \limsup_{\|m\|\to\infty} \parallel \lambda_{n,m} \parallel \{|a_m|\phi_D(m)\}^{\frac{\rho_f}{\|\lambda_{n,m}\|}} \\
&\geq \limsup_{\|m_k\|\to\infty} (\mu - \varepsilon) \frac{\parallel \lambda_{n,m_k} \parallel \{|b_{m_k}|\phi_D(m_k)\}^{\frac{\rho_g}{\|\lambda_{n,m_k}\|}}}{e.\rho_g}, \quad \text{by Equation (26)} \\
&\geq \liminf_{\|m_k\|\to\infty} (\mu - \varepsilon) \frac{\parallel \lambda_{n,m_k} \parallel \{|b_{m_k}|\phi_D(m_k)\}^{\frac{\rho_g}{\|\lambda_{n,m_k}\|}}}{e.\rho_g} \\
&\geq \liminf_{\|m\|\to\infty} (\mu - \varepsilon) \frac{\|\lambda_{n,m}\| \{|b_m|\phi_D(m)\}^{\frac{\rho_g}{\|\lambda_{n,m}\|}}}{e.\rho_g} \\
&= (\mu - \varepsilon)\sigma_g(D), \quad [\text{Since } g \text{ is of perfectly regular growth.}]
\end{aligned}$$

Hence,

$$\sigma_{g,D}(f) = \frac{\sigma_f(D)}{\sigma_g(D)} \geq (\mu - \varepsilon) \qquad (27)$$

Since $\varepsilon > 0$ is arbitrarily small, combining (25) and (27) we get

$$\begin{aligned}
\sigma_{g,D}(f) &= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \Big[\frac{\{|a_m|\phi_D(m)\}^{\rho_f}}{\{|b_m|\phi_D(m)\}^{\rho_g}}\Big]^{\frac{1}{\|\lambda_{n,m}\|}} \\
&= \frac{1}{\rho_g(f)} \limsup_{\|m\|\to\infty} \Big[\frac{\{|a_m|\phi_D(m)\}^{\rho_g(f)}}{|b_m|\phi_D(m)}\Big]^{\frac{\rho_g}{\|\lambda_{n,m}\|}}
\end{aligned}$$

□

Now, by using the above definitions we discuss about relative growth of Hadamard product of two multiple entire Dirichlet series.

Theorem 3.5. Let f_1, f_2, and g be three multiple entire Dirichlet series of finite order such that g is of regular growth. If $\rho_g(f)$ denotes the relative Gol'dberg order of f with respect to g, then $\rho_g((f_1 * f_2)^k) = \rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)$, for $k = 0, 1, 2, \ldots$

PROOF. Let $f_1(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$, $f_2(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$, and $g(s) = \sum_{\|m\|=1}^{\infty} c_m e^{s.\lambda_{n,m}}$. Then, by Theorem (3.3)

$$\begin{aligned}
\frac{1}{\rho_g((f_1 * f_2)^k)} &= \liminf_{\|m\|\to\infty} \frac{\log|(\|\lambda_{n,m}^k\|)a_m b_m|}{\log|c_m|} \\
&= \lim_{\|m\|\to\infty} \frac{\log \| \lambda_{n,m}^k \|}{\log|c_m|} + \liminf_{\|m\|\to\infty} \frac{\log|a_m b_m|}{\log|c_m|} \\
&= \liminf_{\|m\|\to\infty} \frac{\log|a_m b_m|}{\log|c_m|}, \quad \left[\text{Since } g \text{ is entire, by Equation (5), } \limsup_{\|m\|\to\infty} \frac{\log|c_m|}{\|\lambda_{n,m}\|} = -\infty.\right] \\
&= \frac{1}{\limsup_{\|m\|\to\infty} \frac{\log|c_m|}{\log|a_m b_m|}} = \frac{1}{\rho_g(f_1 * f_2)}
\end{aligned}$$

Hence, $\rho_g((f_1 * f_2)^k) = \rho_g(f_1 * f_2)$. Similarly it can be proved that $\rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)$. Therefore, $\rho_g((f_1 * f_2)^k) = \rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)$, for $k = 0, 1, 2, \ldots$ □

Theorem 3.6. Let f_1,f_2 and g be three multiple entire Dirichlet series, then

$$\sigma_{g,D}((f_1 * f_2)^k) = \sigma_{g,D}(f_1^k * f_2^k) = \sigma_{g,D}(f_1 * f_2)$$

PROOF. Let $f_1(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$, $f_2(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ and $g(s) = \sum_{\|m\|=1}^{\infty} c_m e^{s.\lambda_{n,m}}$. Then, by Theorem (3.4),

$$\begin{aligned}
\sigma_{g,D}(f_1 * f_2)^k &= \frac{1}{\rho_g(f_1 * f_2)^k} \limsup_{\|m\|\to\infty} \left[\frac{\{|(\|\lambda_{n,m}^k\|)a_m b_m|\phi_D(m)\}^{\rho_g(f_1 * f_2)^k}}{|c_m|\phi_D(m)}\right]^{\frac{\rho_g}{\|\lambda_{n,m}\|}} \\
&\text{where } \phi_D(m) = \sup_{s\in D} |\exp\{s.\lambda_{n,m}\}| \\
&= \frac{1}{\rho_g(f_1 * f_2)} \limsup_{\|m\|\to\infty} \left[\frac{\{|(\|\lambda_{n,m}^k\|)a_m b_m|\phi_D(m)\}^{\rho_g(f_1 * f_2)}}{|c_m|\phi_D(m)}\right]^{\frac{\rho_g}{\|\lambda_{n,m}\|}} \\
&[\text{Since } \rho_g((f_1 * f_2)^k) = \rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)] \\
&= \frac{1}{\rho_g(f_1 * f_2)} \limsup_{\|m\|\to\infty} \left[\frac{\{|a_m b_m|\phi_D(m)\}^{\rho_g(f_1 * f_2)}}{|c_m|\phi_D(m)}\right]^{\frac{\rho_g}{\|\lambda_{n,m}\|}} \\
&\left[\text{Since } \limsup_{\|m\|\to\infty} \left[\lambda_{1m_1}^{k_1} + \cdots + \lambda_{nm_n}^{k_n}\right]^{\frac{1}{\|\lambda_{n,m}\|}} = 1\right] \\
&= \sigma_{g,D}(f_1 * f_2)
\end{aligned}$$

Similarly, it can be proved that $\sigma_{g,D}(f_1^k * f_2^k) = \sigma_{g,D}(f_1 * f_2)$. Therefore,

$$\sigma_{g,D}((f_1 * f_2)^k) = \sigma_{g,D}(f_1^k * f_2^k) = \sigma_{g,D}(f_1 * f_2)$$

□

Theorem 3.7. Let $f_1(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $f_2(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ be two multiple entire Dirichlet series of relative Gol'dberg order $\rho_g(f_1)$ and $\rho_g(f_2)$ with respect to another multiple entire Dirichlet series $g = \sum_{\|m\|=1}^{\infty} c_m e^{s.\lambda_{n,m}}$ respectively. Then, for $k = 0, 1, 2, \ldots$

$$\frac{1}{\rho_g(f_1^k * f_2^k)} = \frac{1}{\rho_g(f_1 * f_2)^k} \geq \frac{1}{\rho_g(f_1)} + \frac{1}{\rho_g(f_2)}$$

.

Proof. Using Theorem (3.3) and Equation (20), we have

$$\begin{aligned}
\frac{1}{\rho_g(f_1 * f_2)^k} &= \liminf_{\|m\|\to\infty} \frac{\log|(\|\lambda_{n,m}^k\|)a_m b_m|}{\log|c_m|} \\
&\geq \liminf_{\|m\|\to\infty} \frac{\log\|\lambda_{n,m}^k\|}{\log|c_m|} + \liminf_{\|m\|\to\infty} \frac{\log|a_m|}{\log|c_m|} + \liminf_{\|m\|\to\infty} \frac{\log|b_m|}{\log|c_m|} \\
&= \frac{1}{\rho_g(f_1)} + \frac{1}{\rho_g(f_2)}
\end{aligned}$$

Therefore,

$$\frac{1}{\rho_g(f_1 * f_2)^k} \geq \frac{1}{\rho_g(f_1)} + \frac{1}{\rho_g(f_2)}$$

By Theorem (3.5), $\rho_g(f_1^k * f_2^k) = \rho_g(f_1 * f_2)^k$. Therefore,

$$\frac{1}{\rho_g(f_1^k * f_2^k)} = \frac{1}{\rho_g(f_1 * f_2)^k} \geq \frac{1}{\rho_g(f_1)} + \frac{1}{\rho_g(f_2)}$$

□

Theorem 3.8. Let $f_1(s) = \sum_{\|m\|=1}^{\infty} a_m e^{s.\lambda_{n,m}}$ and $f_2(s) = \sum_{\|m\|=1}^{\infty} b_m e^{s.\lambda_{n,m}}$ be two multiple entire Dirichlet series of relative Gol'dberg order $\rho_g(f_1)$ and $\rho_g(f_2)$ with respect to another multiple entire Dirichlet series $g = \sum_{\|m\|=1}^{\infty} c_m e^{s.\lambda_{n,m}}$ respectively. Then, for $k = 0, 1, 2, \ldots,$

$$\rho_g(f_1^k * f_2^k) \leq \left[\rho_g(f_1^k).\rho_g(f_2^k)\right]^{\frac{1}{2}}$$

provided

$$\log|(\|\lambda_{n,m}^{2k}\|)a_m b_m| \sim \left\{\log|(\|\lambda_{n,m}^k\|)a_m|\log|(\|\lambda_{n,m}^k\|)b_m|\right\}^{\frac{1}{2}}$$

Proof. We have

$$\frac{1}{\rho_g(f_1^k)} = \liminf_{\|m\|\to\infty} \frac{\log|(\|\lambda_{n,m}^k\|)a_m|}{\log|c_m|}$$

and

$$\frac{1}{\rho_g(f_2^k)} = \liminf_{\|m\|\to\infty} \frac{\log|(\|\lambda_{n,m}^k\|)b_m|}{\log|c_m|}$$

Therefore, for any arbitrary $\varepsilon > 0$, and for all sufficiently large $\| m \|$

$$\frac{1}{\rho_g(f_1^k)} - \frac{\varepsilon}{2} < \frac{\log|(\|\lambda_{n,m}^k\|)a_m|}{\log|c_m|}$$

and

$$\frac{1}{\rho_g(f_2^k)} - \frac{\varepsilon}{2} < \frac{\log|(\|\lambda_{n,m}^k\|)b_m|}{\log|c_m|}$$

Thus,

$$\Big(\frac{1}{\rho_g(f_1^k)}-\frac{\varepsilon}{2}\Big)\Big(\frac{1}{\rho_g(f_2^k)}-\frac{\varepsilon}{2}\Big)<\frac{\log|(\|\lambda_{n,m}^k\|)a_m|\log|(\|\lambda_{n,m}^k\|)b_m|}{(\log|c_m|)^2}$$

or

$$\Big\{\Big(\frac{1}{\rho_g(f_1^k)}-\frac{\varepsilon}{2}\Big)\Big(\frac{1}{\rho_g(f_2^k)}-\frac{\varepsilon}{2}\Big)\Big\}^{\frac{1}{2}}<\frac{\Big\{\log|(\|\lambda_{n,m}^k\|)a_m|\log|(\|\lambda_{n,m}^k\|)b_m|\Big\}^{\frac{1}{2}}}{\log|c_m|}$$

for all sufficiently large $\| m \|$. Since $\log|(\|\lambda_{n,m}^{2k}\|)a_mb_m|\sim\big\{\log|(\|\lambda_{n,m}^k\|)a_m|\log|(\|\lambda_{n,m}^k\|)b_m|\big\}^{\frac{1}{2}}$, for all sufficiently large $\| m \|$, we have

$$\Big\{\Big(\frac{1}{\rho_g(f_1^k)}-\frac{\varepsilon}{2}\Big)\Big(\frac{1}{\rho_g(f_2^k)}-\frac{\varepsilon}{2}\Big)\Big\}^{\frac{1}{2}}<\frac{\log|(\|\lambda_{n,m}^{2k}\|)a_mb_m|}{\log|c_m|}$$

Therefore,

$$\Big(\frac{1}{\rho_g(f_1^k)\rho_g(f_2^k)}\Big)^{\frac{1}{2}}\leq\liminf_{\|m\|\to\infty}\frac{\log|(\|\lambda_{n,m}^{2k}\|)a_mb_m|}{\log|c_m|}$$

or

$$\Big(\frac{1}{\rho_g(f_1^k)\rho_g(f_2^k)}\Big)^{\frac{1}{2}}\leq\frac{1}{\rho_g(f_1^k*f_2^k)}$$

Hence,

$$\rho_g(f_1^k*f_2^k)\leq\big[\rho_g(f_1^k).\rho_g(f_2^k)\big]^{\frac{1}{2}}$$

□

4. Conclusion

Thus, we understand that, in the study of growth properties of Hadamard product between two multiple entire Dirichlet series, the method of using coefficients and exponents is easy and useful. However, not only in the case of Hadamard product, but also for any study of growth property which involves exponents and coefficients of the series, our result will be useful and an easy method to prove them.

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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