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Some Results of f-Harmonic and Bi-f-Harmonic Maps with Potential

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

Abstract

In this note, we characterize the f-harmonic maps and bi-f-harmonic maps with potential. We prove that every bi-f-harmonic map with potential from complete Riemannian manifold, satisfying some conditions is a f-harmonic map with potential. More, we study the case of conformal maps between equidimensional manifolds.

Keywords: *f-harmonic maps with potential, bi-f-harmonic maps with potential, H-f-energy.*
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1. Introduction

f-harmonic maps between two Riemannian manifolds, which generalize harmonic maps, were first introduced by Lichnerowicz [1] in 1970, and were studied by N. Course [9] recently. f-harmonic maps relate to the equations of the motion of a continuous system of spins with inhomogeneous neighbor Heisenberg interaction in mathematical physics. Moreover, F-harmonic maps between Riemannian manifolds were first introduced by Ara [6] in 1999, which could be considered as the special cases of f-harmonic maps. f-biharmonic maps between Riemannian manifolds were studied by Ouakkas, Nasri and Djaa [13] in 2010, which generalized biharmonic maps. The concept of harmonic maps with potential, was initially suggested by Ratto in [3] and recently developed by several authors : V. Branding [14], Jiang [12] and others. The notion of biharmonic maps with potential was studied by A. Mohammed Cherif and M. Djaa in 2017 [2], and by A. Zagane and S. Ouakass [4] in 2018.

In this paper we establish the first and second variation of the H-f-energy functional (Theorem2.2), we introduce the notion of bi-f-harmonic maps with potential and we characterize the bi-f-harmonic maps with potential (Corollary 3.1), moreover we construct some examples. Also we prove that every bi-f- harmonic map with potential from complete Riemannian manifold satisfying some conditions is a f-harmonic map with potential (Theorem3.2). Finally we study the case of conformal maps between equidimensional manifolds of the same dimension $n \geq 3$.

2. f-HARMONIC MAPS WITH POTENTIAL

Consider a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds, let H be a smooth function on N and let f be a smooth positive function on M. For any compact domain D of M the H-f-energy functional of φ is defined by

$$E_{H,f}(\varphi) = \int_D [fe(\varphi) - H(\varphi)]v_g.$$

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where, $e(\varphi)$ is the energy density of φ defined by $e(\varphi) = \frac{1}{2}\sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i))$, v_g is the volume element and $\{e_i\}_{i=1,m}$ is an orthonormal frame on (M^m, g).

Definition 2.1. A map φ is called f-harmonic with potential H if it is a critical point of the H-f-energy functional over any compact subset D of M, i.e

$$\frac{d}{dt}E_{H,f}(\varphi_t)\Big|_{t=0} = 0$$

where $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D.

Theorem 2.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a ma between Riemannian manifolds, H be a smooth function on N and let f be a smooth positive function on M. Then*

$$\frac{d}{dt}E_{H,f}(\varphi_t)\Big|_{t=0} = -\int_D h(\tau_{H,f}(\varphi), \upsilon)v_g,$$

such that:

$$\tau_{H,f}(\varphi) = \tau_f(\varphi) + (\mathrm{grad}^N H)\circ\varphi, \tag{2.1}$$

where $\tau_f(\varphi) = f\tau(\varphi) + d\varphi(\mathrm{grad}^M f)$ is the f-tension field of φ (see [13]), $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D and $\upsilon = \frac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ.

Proof.
Let $\phi : M\times(-\epsilon,\epsilon) \longrightarrow N$ define by $\phi(x,t) = \varphi_t(x)$, ∇^φ denote the pull-back connection on $\varphi^{-1}(TN)$. Note that for any vector field X on M considered as a vector field on $M\times(-\epsilon,\epsilon)$, we have $[\partial t, X] = 0$. Let $\{e_i\}_{i=1,..,m}$ be an orthonormal frame on M, such that $\nabla^M_{e_i}e_j = 0$ at the fixed point $x \in M$.
At $x \in M$ we have:

$$\frac{d}{dt}E_{H,f}(\varphi_t)\Big|_{t=0} = \int_D \Big[\frac{\partial}{\partial t}fe(\varphi_t) - \frac{\partial}{\partial t}H(\varphi_t)\Big]\Big|_{t=0} v_g, \tag{2.2}$$

for the first term in the right hand of (2.2), we have

$$\begin{aligned}
\frac{\partial}{\partial t}fe(\varphi_t) &= \frac{1}{2}\frac{\partial}{\partial t}f\sum_{i=1}^{m} h(d\varphi_t(e_i), d\varphi_t(e_i)) \\
&= f\sum_{i=1}^{m} h(\nabla^\phi_{\frac{\partial}{\partial t}} d\varphi_t(e_i), d\varphi_t(e_i)) \\
&= f\sum_{i=1}^{m} h(\nabla^\phi_{e_i} d\varphi_t(\frac{\partial}{\partial t}), d\varphi_t(e_i)) \\
&= \sum_{i=1}^{m} e_i h(d\varphi_t(\frac{\partial}{\partial t}), fd\varphi_t(e_i)) - \sum_{i=1}^{m} h(d\varphi_t(\frac{\partial}{\partial t}), \nabla^\phi_{e_i} fd\varphi_t(e_i)).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t}fe(\varphi_t)\Big|_{t=0} &= \sum_{i=1}^{m} e_i h(d\varphi_t(\frac{\partial}{\partial t}), fd\varphi_t(e_i))\Big|_{t=0} - \sum_{i=1}^{m} h(d\varphi_t(\frac{\partial}{\partial t}), \nabla^\phi_{e_i} fd\varphi_t(e_i))\Big|_{t=0} \\
&= \mathrm{div}(\omega) - h(\upsilon, \tau_f(\varphi)),
\end{aligned} \tag{2.3}$$

where $\omega(.) = \sum_{i=1}^{m} h(d\varphi_t(\frac{\partial}{\partial t}), fd\varphi_t(.))\Big|_{t=0}$.

For the second term in the right hand of (2.2), we have

$$\begin{aligned}
\frac{\partial}{\partial t}H(\varphi_t)\Big|_{t=0} &= h(d\varphi_t(\frac{\partial}{\partial t}), (\mathrm{grad}H)\circ\varphi)\Big|_{t=0} \\
&= h(\upsilon, (\mathrm{grad}H)\circ\varphi).
\end{aligned} \tag{2.4}$$

By replacing (2.4) and (2.3) in (2.2) and using the divergence theorem, we obtain

$$\frac{d}{dt}E_{H,f}(\varphi_t)\Big|_{t=0} = -\int_D h(\tau_f(\varphi) + (\mathrm{grad}H)\circ\varphi), \upsilon)v_g,$$

Corollary 2.1. *A smooth map $\varphi : (M^m; g) \longrightarrow (N^n; h)$ between Riemannian manifolds is f-harmonic with potential H if and only if $\tau_{H,f}(\varphi) = 0$.*

Example 2.1. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $t \longmapsto \varphi(t)$,
$f : \mathbb{R} \longrightarrow \mathbb{R}$ and $H : \mathbb{R} \longrightarrow \mathbb{R}$.

$$\tau_{H,f}(\varphi) = f\varphi^{''} + f^{'}\varphi^{'} + H^{'}.$$

we consider $\varphi(t) = t^2$ and $f(t) = e^t$, then a map φ is f-harmonic with potential H, for $H(t) = -2te^t$.

Remark 2.1. Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. If the potential H is constant, then φ is f-harmonic with potential H if and only if it is f-harmonic map. One can refer to ([13]) for background on harmonic maps and generalized harmonic maps.

2.1. The second variation of the H-f-energy functional

We consider $\{\varphi_{s,t}\}_{s,t\in(-\epsilon,\epsilon)}$ a two-parameter variation with compact support in D. Let $v = \frac{\partial\varphi_{s,t}}{\partial t}\Big|_{s=t=0}$, $W = \frac{\partial\varphi_{s,t}}{\partial s}\Big|_{s=t=0}$. Under the notation above we have the following

Theorem 2.2. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a f-harmonic map with potential H, where H is a smooth function on N and f be a smooth positive function on M. Then*

$$\frac{\partial^2}{\partial s\partial t}E_{H,f}(\varphi_{s,t})\Big|_{s=t=0} = -\int_D h(J^{\varphi}_{H,f}(v), W)v_g, \tag{2.5}$$

where $J^{\varphi}_{H,f}(v) \in \Gamma(\varphi^{-1}(TN))$ is the Jacobi operator given by

$$J^{\varphi}_{H,f}(v) \quad = \quad f\mathrm{trace}_g R^N(v, d\varphi)d\varphi + \mathrm{trace}_g\nabla^{\varphi} f\nabla^{\varphi}v + (\nabla^N_v \mathrm{grad}H) \circ \varphi,$$

here R^N is the curvature tensor of (N^n, h).

Proof:

Define $\phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \longrightarrow N$ by $\phi(x, t, s) = \varphi_{t,s}(x)$, let ∇^{ϕ} denote the pull-back connection on $\varphi^{-1}(TN)$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$, we have $[\partial t, X] = 0$, $[\partial s, X] = 0$, $[\partial t, \partial s] = 0$. Then

$$\frac{\partial^2}{\partial s\partial t}E_{H,f}(\varphi_{s,t})\Big|_{s=t=0} = -\int_D \frac{\partial^2}{\partial s\partial t}\big[fe(\varphi_{s,t}) - H(\varphi_{s,t})\big]\Big|_{s=t=0} v_g. \tag{2.6}$$

We calculate the first term in the right hand of (2.6):

$$\begin{aligned}\int_D \frac{\partial^2}{\partial s\partial t}[fe(\varphi_{s,t})]v_g &= \sum_{i=1}^m \int_D \frac{\partial}{\partial s} h(\nabla^{\phi}_{\partial t} d\varphi_{s,t}(e_i), fd\varphi_{s,t}(e_i))\\ &= \sum_{i=1}^m \Big[\int_D h(\nabla^{\phi}_{\partial s}\nabla^{\phi}_{\partial t} d\varphi_{s,t}(e_i), fd\varphi_{s,t}(e_i))v_g + \int_D h(\nabla^{\phi}_{\partial t} d\varphi_{s,t}(e_i), \nabla^{\phi}_{\partial s} fd\varphi_{s,t}(e_i))v_g\Big] \qquad (2.7)\\ &= \sum_{i=1}^m \Big[\int_D h(\nabla^{\phi}_{\partial s}\nabla^{\phi}_{e_i} d\varphi_{s,t}(\partial t), fd\varphi_{s,t}(e_i))v_g + \int_D h(f\nabla^{\phi}_{e_i} d\varphi_{s,t}(\partial t), \nabla^{\phi}_{e_i} d\varphi_{s,t}(\partial s))v_g\Big].\end{aligned}$$

By using the divergence theorem, the first term in the right hand of (2.7), became

$$\begin{aligned}
\sum_{i=1}^{m}\int_D h(\nabla^{\phi}_{\partial s}\nabla^{\phi}_{e_i}d\varphi_{s,t}(\partial t), fd\varphi_{s,t}(e_i))v_g &= \sum_{i=1}^{m}\Big[\int_D h(fR^N(d\phi_{s,t}(\partial s), d\phi_{t,s}(e_i))d\varphi_{s,t}(\partial t), d\varphi_{s,t}(e_i))\Big|_{t=s=0}v_g \\
&+ \int_D h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial s}d\varphi_{s,t}(\partial t), fd\varphi_{s,t}(e_i))\Big|_{t=s=0}v_g\Big] \\
&= \sum_{i=1}^{m}\Big[\int_D h(fR^N(w, d\varphi(e_i))v, d\varphi(e_i))v_g \\
&+ \int_D e_i h(\nabla^{\phi}_{\partial s}d\varphi_{s,t}(\partial t), fd\varphi_{s,t}(e_i))\Big|_{t=s=0}v_g \\
&- \int_D h(\nabla^{\phi}_{\partial s}d\varphi_{s,t}(\partial t), \nabla^{\phi}_{e_i}fd\varphi_{s,t}(e_i))\Big|_{t=s=0}v_g\Big] \\
&= \sum_{i=1}^{m}\Big[-\int_D h(fR^N(v, d\varphi(e_i))d\varphi(e_i), w)v_g \\
&- \int_D h(\nabla^{\phi}_{\partial s}d\varphi_{s,t}(\partial t), \nabla^{\phi}_{e_i}fd\varphi_{s,t}(e_i))\Big|_{t=s=0}v_g\Big]. \qquad (2.8)
\end{aligned}$$

For the second term in the right hand of (2.7), we get

$$\begin{aligned}
\int_D h(f\nabla^{\phi}_{e_i}d\varphi_{s,t}(\partial t), \nabla^{\phi}_{e_i}d\varphi_{s,t}(\partial s))\Big|_{t=s=0}v_g &= \int_D e_i h(d\varphi_{s,t}(\partial t), f\nabla^{\phi}_{e_i}d\varphi_{s,t}(\partial s))\Big|_{t=s=0}v_g \\
&- \int_D h(d\varphi_{s,t}(\partial t), \nabla^{\phi}_{e_i}f\nabla^{\phi}_{e_i}d\varphi_{s,t}(\partial s))\Big|_{t=s=0}v_g \\
&= -\int_D h(w, \nabla^{\varphi}_{e_i}f\nabla^{\varphi}_{e_i}v). \qquad (2.9)
\end{aligned}$$

Now, we calculate the second term of (2.6)

$$\begin{aligned}
\int_D \frac{\partial^2}{\partial s\partial t}H(\varphi_{t,s})\Big|_{t=s=0}v_g &= \int_D \frac{\partial}{\partial s}h\big(d\varphi_{t,s}(\partial t), (\operatorname{grad} H)\circ\varphi\big)\Big|_{t=s=0}v_g \\
&= \int_D h\big(\nabla^{\phi}_{\partial s}d\varphi_{t,s}(\partial t), (\operatorname{grad} H)\circ\varphi\big)\Big|_{t=s=0}v_g \\
&+ \int_D h\big(d\varphi_{t,s}(\partial t), \nabla^{\phi}_{\partial s}(\operatorname{grad} H)\circ\varphi\big)\Big|_{t=s=0}v_g \qquad (2.10) \\
&= \int_D h\big(\nabla^{\phi}_{\partial s}d\varphi_{t,s}(\partial t), (\operatorname{grad} H)\circ\varphi\big)\Big|_{t=s=0}v_g \\
&+ \int_D h\big(v, \nabla^N_w(\operatorname{grad} H)\circ\varphi\big)v_g.
\end{aligned}$$

By substituting (2.10), (2.9) and (2.8) in (2.6), and using that φ is f-harmonic with potential H, we get

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}E_{H,f}(\varphi_{s,t})\Big|_{s=t=0} &= \sum_{i=1}^{m}\int_D h\Big(-fR^N(v, d\varphi(e_i))d\varphi(e_i) - \nabla^{\varphi}_{e_i}f\nabla^{\varphi}_{e_i}v - \nabla^N_v(\operatorname{grad} H)\circ\varphi, w\Big)v_g \\
&= -\int_D h\Big(f\operatorname{trace}_g R^N(v, d\varphi)d\varphi + \operatorname{trace}_g \nabla^{\varphi}f\nabla^{\varphi}v - \nabla^N_v(\operatorname{grad} H)\circ\varphi, w\Big)v_g
\end{aligned}$$

3. Bi-f-harmonic Maps with potential.

Consider a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds, let H be a smooth function on N and $f \in \mathcal{C}^{\infty}(M)$ be a positive function. A natural generalization of f-harmonic maps with potential is given by integrating the square of the norm of $\tau_{H,f}(\varphi)$. More precisely, the H-bi-f-energy functional of φ is defined by

$$E^2_{H,f}(\varphi) = \frac{1}{2}\int_D |\tau_{H,f}(\varphi)|^2 v_g$$

Definition 3.1. A map φ is called bi-f-harmonic with potential H, if it is critical point of the H-bi-f-energy functional over any compact subset D of M.

3.1. The first variation of H-bi-f-energy functional

Theorem 3.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, H a smooth function on N and $f \in \mathcal{C}^\infty(M)$ be a positive function. D a compact subset of M and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ with compact support in D. Then*

$$\frac{d}{dt}E^2_{H,f}(\varphi_t)\Big|_{t=0} = -\int_D h(\tau^2_{H,f}(\varphi), \upsilon)v_g, \tag{3.1}$$

where $\tau^2_{H,f}(\varphi) \in \Gamma(\varphi^{-1}TN)$ is given by

$$\tau^2_{H,f}(\varphi) = f\,\mathrm{trace}_g\, R^N(\tau_{H,f}(\varphi), d\varphi)d\varphi + \mathrm{trace}_g\, \nabla^\varphi f \nabla^\varphi \tau_{H,f}(\varphi) + (\nabla^N_{\tau_{H,f}(\varphi)} \mathrm{grad}^N H) \circ \varphi.$$

Proof:
Recall that $\phi : M \times (-\epsilon, \epsilon) \longrightarrow N$ with $\phi(x,t) = \varphi_t(x)$, ∇^φ the pull-back connection on $\varphi^{-1}(TN)$ and $\{e_i\}_{i=1,..,m}$ be an orthonormal frame on M, such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$. First note that

$$\frac{d}{dt}E^2_{H,f}(\varphi_t)\Big|_{t=0} = -\int_D h(\nabla^\varphi_{\partial t}\tau_{H,f}(\varphi_t), \tau_{H,f}(\varphi_t))\Big|_{t=0} v_g, \tag{3.2}$$

Calculating in a normal frame at $x \in M$, we have

$$\begin{aligned} \nabla^\varphi_{\partial t}\tau_{H,f}(\varphi_t) &= \nabla^\varphi_{\partial t}\big[\tau_f(\varphi_t) + (\mathrm{grad}^N H) \circ \varphi_t\big] \\ &= \nabla^\phi_{\partial t}\nabla^\phi_{e_i} f d\varphi_t(e_i) + \nabla^\phi_{\partial t}(\mathrm{grad}^N H) \circ \varphi_t, \end{aligned} \tag{3.3}$$

by the definition of the curvature tensor of (N, h) we have:

$$\nabla^\phi_{\partial t}\nabla^\phi_{e_i} f d\varphi_t(e_i) = \nabla^\phi_{e_i}\nabla^\phi_{\partial t} f d\varphi_t(e_i) + f R^N(d\phi(\partial t), d\varphi_t(e_i))d\varphi_t(e_i). \tag{3.4}$$

By using $[\partial t, e_i] = 0$ and the compatibility of ∇^ϕ with h we have

$$\begin{aligned} h(\nabla^\phi_{e_i}\nabla^\phi_{\partial t} f d\varphi_t(e_i), \tau_{H,f}(\varphi_t)) &= e_i h(\nabla^\phi_{\partial t} f d\varphi_t(e_i), \tau_{H,f}(\varphi_t)) - h(\nabla^\phi_{e_i} f d\varphi_t(\partial t), \nabla^\phi_{e_i}\tau_{H,f}(\varphi_t)) \\ &= e_i h(\nabla^\phi_{\partial t} f d\varphi_t(e_i), \tau_{H,f}(\varphi_t)) - e_i h(f d\varphi_t(\partial t), \nabla^\phi_{e_i}\tau_{H,f}(\varphi_t)) \\ &\quad + h(d\varphi_t(\partial t), \nabla^\phi_{e_i} f \nabla^\phi_{e_i}\tau_{H,f}(\varphi_t)) \end{aligned} \tag{3.5}$$

From the definition of ∇^ϕ and the symmetry of the Hessian tensor
$(i.e\ \mathrm{Hess}_H(X,Y) = h(\nabla^\phi_X \mathrm{grad}\, H, Y) = \mathrm{Hess}_H(Y,X)$, we have

$$\begin{aligned} h(\nabla^\phi_{\partial t}(\mathrm{grad}\, H \circ \varphi_t), \tau_{H,f}(\varphi_t)) &= h(\nabla^N_{d\phi(\partial t)}(\mathrm{grad}\, H \circ \varphi_t), \tau_{H,f}(\varphi_t)) \\ &= h(\nabla^N_{\tau_{H,f}(\varphi_t)}(\mathrm{grad}\, H \circ \varphi_t), d\varphi_t(\partial t)) \end{aligned} \tag{3.6}$$

By (3.5), (3.4), (3.3), (3.2), (3.1), $\upsilon = \frac{\partial\varphi_t}{\partial t}$ when $t = 0$ and the divergence theorem, the Theorem (3.1) follows.
From the Theorem (3.1), we deduce the following

Corollary 3.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, H a smooth function on N and $f \in \mathcal{C}^\infty(M)$ be a positive function, then φ is bi-f-harmonic with potential H if and only if:*

$$\tau^2_{H,f}(\varphi) = f\,\mathrm{trace}\, R^N(\tau_{H,f}(\varphi), d\varphi)d\varphi + \mathrm{trace}\, \nabla^\varphi f \nabla^\varphi \tau_{H,f}(\varphi) + (\nabla^N_{\tau_{H,f}(\varphi)} \mathrm{grad}^N H) \circ \varphi = 0. \tag{3.7}$$

Remark 3.1. Let $\varphi : (M^m g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, H a smooth function on N and $f \in \mathcal{C}^\infty(M)$ be a positive function, then

$$\tau^2_{H,f}(\varphi) = \tau_{2,f}(\varphi) + J_{f,\varphi}(\mathrm{grad}^N H) \circ \varphi + (\nabla^N_{\tau_f(\varphi)} \mathrm{grad}^N H) \circ \varphi + (\nabla^N_{(\mathrm{grad}^N H)\circ\varphi} \mathrm{grad}^N H) \circ \varphi, \tag{3.8}$$

where

$$J_{f,\varphi}(\mathrm{grad}^N H) \circ \varphi = f\,\mathrm{trace}\, R^N(\mathrm{grad}^N H, d\varphi)d\varphi + \mathrm{trace}\, \nabla^\varphi f \nabla^\varphi \mathrm{grad}^N H$$

is the Jacobi operator of φ and

$$\tau_{2,f}(\varphi) = f\,\mathrm{trace}_g\, R^N(\tau_f(\varphi), d\varphi)d\varphi + \mathrm{trace}_g\, \nabla^\varphi f \nabla^\varphi \tau_f(\varphi)$$

is the bi-f-tension field of φ. In the case where φ is f-harmonic, we obtain the following corollary.

Corollary 3.2. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a f-harmonic map, H a smooth function on N and $f \in \mathcal{C}^\infty(M)$ be a smooth positive function. Then φ is bi-f-harmonic with potential H if and only if*

$$J_{f,\varphi}(\operatorname{grad}^N H) \circ \varphi + (\nabla^N_{(\operatorname{grad}^N H)\circ\varphi} \operatorname{grad}^N H) \circ \varphi = 0.$$

Example 3.1. Let $\varphi : \mathbb{R}^* \times \mathbb{R} \longrightarrow \mathbb{R}$, with $(t, x) \longmapsto \varphi(t, x)$ be a smooth function and $f \in \mathcal{C}^\infty(\mathbb{R}^* \times \mathbb{R})$ be a positive function. We have

$$\begin{aligned} \tau_f(\varphi) &= f\tau(\varphi) + d\varphi(\operatorname{grad} f) \\ &= f\big[\tau(\varphi) + d\varphi(\operatorname{grad} \ln(f))\big], \end{aligned}$$

then φ is f-harmonic if and only if

$$\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial \ln(f)}{\partial t}.\frac{\partial\varphi}{\partial t} + \frac{\partial \ln(f)}{\partial x}.\frac{\partial\varphi}{\partial x} = 0.$$

If the map φ depends only on t, then φ is f-harmonic if and only if

$$\varphi^{''} + \frac{\partial \ln(f)}{\partial t}.\varphi^{'} = 0,$$

we obtain $f(x, t) = \frac{\alpha(x)}{|\varphi'(t)|}$, where α is a positive function on $\mathbb{R}$.

Application:

If we put $\varphi(t, x) = t^2$, then φ is f-harmonic for $f(t, x) = \frac{\alpha(x)}{2t}$. We can take for example $f(t, x) = \frac{x^2+1}{2t}$. By using the corollary (3.2) we conclude that:
φ is bi-f-harmonic with potential H (H is a smooth function on $\mathbb{R}$), if and only if

$$\operatorname{trace} \nabla^\varphi f \nabla^\varphi (\operatorname{grad}^N H) \circ \varphi + (\nabla^N_{(\operatorname{grad}^N H)\circ\varphi} \operatorname{grad}^N H) \circ \varphi = 0.$$

Suppose that the function $\psi = H \circ \varphi$ depends only on t, then φ is bi-f-harmonic with potential H if and only if

$$f\psi^{'''}(t) + \frac{\partial f}{\partial t}\psi^{''}(t) + \psi^{''}(t)\psi^{'}(t) = 0.$$

A particular solution is given by : $\psi(t) = (H \circ \varphi)(t) = at + b$, $(a, b) \in \mathbb{R}^* \times \mathbb{R}$.

Corollary 3.3. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and $f \in \mathcal{C}^\infty(M)$ be a positive function. If the potential H is constant, then φ is bi-f-harmonic with potential H if and only if it is bi-f-harmonic.*

From Theorem 3.1, we have the following

Corollary 3.4. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and $f \in \mathcal{C}^\infty(M)$ be a positive function. If φ is f-harmonic with potential H, then φ is bi-f-harmonic with potential H.*

Definition 3.2. Let H a smooth function on N and $f \in \mathcal{C}^\infty(M)$ be a positive function.. A map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is called a proper bi-f-harmonic map with potential H if and only if φ is a bi-f-harmonic map with potential H which is not a f-harmonic map with potential H.

Corollary 3.5. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a f-harmonic map, H be a non constant function on N and $f \in \mathcal{C}^\infty(M)$ be a smooth positive function. Then φ is a proper bi-f-harmonic with potential H if and only if*

$$J_{f,\varphi}(\operatorname{grad}^N H) \circ \varphi + (\nabla^N_{(\operatorname{grad}^N H)\circ\varphi} \operatorname{grad}^N H) \circ \varphi = 0.$$

Example 3.2. Let $\varphi : \mathbb{R}^* \times \mathbb{R}^*_+ \longrightarrow \mathbb{R}$, with $(t, x) \longmapsto \varphi(t, x)$ be a smooth function and $f \in \mathcal{C}^\infty(\mathbb{R}^* \times \mathbb{R})$ be a positive function. We have

$$\begin{aligned} \tau_f(\varphi) &= f\tau(\varphi) + d\varphi(\operatorname{grad} f) \\ &= f\big[\tau(\varphi) + d\varphi(\operatorname{grad} \ln(f))\big], \end{aligned}$$

then φ is f-harmonic if and only if

$$\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial \ln(f)}{\partial t}.\frac{\partial\varphi}{\partial t} + \frac{\partial \ln(f)}{\partial x}.\frac{\partial\varphi}{\partial x} = 0.$$

If we put $\varphi(t,x) = e^{t+x}$, then φ is f-harmonic for $f(t,x) = e^{-t-x}$.
By the corollary (3.5), then φ is a proper bi-f-harmonic with potential H if and only if

$$\Big[f\Delta\varphi + \frac{\partial f}{\partial t}\frac{\partial\varphi}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial\varphi}{\partial x}\Big]H''\circ\varphi + f\Big[(\frac{\partial\varphi}{\partial t})^2 + (\frac{\partial\varphi}{\partial x})^2\Big]H'''\circ\varphi + H'H''\circ\varphi = 0, \tag{3.9}$$

a particular solution of (3.9) is given by : $H(y) = ay + b$, $(a,b) \in \mathbb{R}^* \times \mathbb{R}$.
by replacing f and φ in (3.9), we have

$$2e^{t+x}H'''\circ\varphi + H'H''\circ\varphi = 0, \tag{3.10}$$

we can put $e^{t+x} = y\circ\varphi$, then the equation (3.10) became

$$2yH''' + H'H'' = 0, \tag{3.11}$$

the general solution of (3.11) is given by:

$$H(y) = C_3 + \int \frac{2C_1 + \tanh(\frac{-C_2+\ln(y)}{4C_1})}{C_1}dy,$$

where $(C_1, C_2, C_3) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}$.

Now, we investigate sufficient conditions for bi-f-harmonic map with potential to be f-harmonic map with potential.

Theorem 3.2. *Let (M^m, g) be a complete Riemannian manifold with infinite volume, (N^n, h) a Riemannian manifold with non-positive sectional curvature, $f \in \mathcal{C}^\infty(M)$ a positive function satisfying $h(\nabla^\varphi_{\text{grad } f}\tau_{H,f}(\varphi), \tau_{H,f}(\varphi)) \leq 0$ and H a smooth function on N with $\text{Hess}(H) \leq 0$. Then, every bi-f-harmonic map φ with potential H from (M^m, g) to (N^n, h), satisfying*

$$\int_M |\tau_{H,f}(\varphi)|^2 v_g < \infty, \tag{3.12}$$

is f-harmonic with potential H.

Proof
Assume that $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is a bi-f-harmonic map with potential H, let's fixe a point x in M and let $\{e_1, e_2, .., e_m\}$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_i}e_j = 0$, at x for all $i, j = 1, .., m$. By formula (3.6) we have

$$f \operatorname{trace} R^N(\tau_{H,f}(\varphi), d\varphi)d\varphi + \operatorname{trace}\nabla^\varphi f\nabla^\varphi\tau_{H,f}(\varphi) + (\nabla^N_{\tau_{H,f}(\varphi)}\operatorname{grad}^N H)\circ\varphi = 0,$$

and then

$$-f\sum_{i=1}^m h\big(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau_{H,f}(\varphi), \tau_{H,f}(\varphi)\big) = h(\nabla^\varphi_{\text{grad } f}\tau_{H,f}(\varphi), \tau_{H,f}(\varphi)) + f\sum_{i=1}^m h\big(R^N(\tau_{H,f}(\varphi), d\varphi(ei))d\varphi(e_i), \tau_{H,f}(\varphi)\big)$$
$$+ \text{Hess}_H(\tau_{H,f}(\varphi), \tau_{H,f}(\varphi)).$$

Since the sectional curvature of N is non-positive, $\text{Hess}(H) \leq 0$ and f is positive such that $h(\nabla^\varphi_{\text{grad } f}\tau_{H,f}(\varphi), \tau_{H,f}(\varphi)) \leq 0$, we conclude that:

$$-\sum_{i=1}^m h\big(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau_{H,f}(\varphi), \tau_{H,f}(\varphi)\big) \leq 0. \tag{3.13}$$

Let ρ be a smooth function with compact support on M. By (3.13) we have:

$$-\sum_{i=1}^m h\big(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau_{H,f}(\varphi), \rho^2\tau_{H,f}(\varphi)\big) \leq 0,$$

which is equivalent to

$$-\sum_{i=1}^m e_i h\big(\nabla^\varphi_{e_i}\tau_{H,f}(\varphi), \rho^2\tau_{H,f}(\varphi)\big) + \sum_{i=1}^m h\big(\nabla^\varphi_{e_i}\tau_{H,f}(\varphi), \nabla^\varphi_{e_i}\rho^2\tau_{H,f}(\varphi)\big) \leq 0. \tag{3.14}$$

Formula (3.14) is equivalent to

$$-\sum_{i=1}^{m} e_i h\big(\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi),\rho^2\tau_{H,f}(\varphi)\big)+\sum_{i=1}^{m}\rho^2|\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)|^2+\sum_{i=1}^{m}2\rho e_i(\rho)h\big(\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi),\tau_{H,f}(\varphi)\big)\leq 0. \tag{3.15}$$

If we denote by $\omega(X)=h\big(\nabla^{\varphi}_{X}\tau_{H,f}(\varphi),\rho^2\tau_{H,f}(\varphi)\big)$, then the inequality (3.15) becomes

$$-\operatorname{div}^M\omega+\sum_{i=1}^{m}\rho^2|\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)|^2+\sum_{i=1}^{m}2\rho e_i(\rho)h\big(\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi),\rho\tau_{H,f}(\varphi)\big)\leq 0. \tag{3.16}$$

By integrating the formula (3.16) over M and Using the divergence theorem, we have

$$\sum_{i=1}^{m}\int_M\rho^2|\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)|^2v_g+\sum_{i=1}^{m}\int_M 2\rho e_i(\rho)h\big(\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi),\rho\tau_{H,f}(\varphi)\big)v_g\leq 0. \tag{3.17}$$

By the Young inequality

$$-2\rho e_i(\rho)h\big(\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi),\tau_{H,f}(\varphi)\big)\leq\epsilon\rho^2|\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)|^2+\frac{1}{\epsilon}e_i^2(\rho)|\tau_{H,f}(\varphi)|^2,$$

we obtain

$$\sum_{i=1}^{m}\int_M\rho^2|\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)|^2v_g\leq\epsilon\sum_{i=1}^{m}\int_M\rho^2|\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)|^2v_g+\frac{1}{\epsilon}\sum_{i=1}^{m}\int_M e_i^2(\rho)|\tau_{H,f}(\varphi)|^2v_g. \tag{3.18}$$

When $\epsilon=1$, the inequality (3.18) became

$$\frac{1}{2}\sum_{i=1}^{m}\int_M\rho^2|\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)|^2v_g\leq 2\sum_{i=1}^{m}\int_M e_i^2(\rho)|\tau_{H,f}(\varphi)|^2v_g. \tag{3.19}$$

Choose the smooth cut-of function $\rho=\rho_R$, i.e

$$\begin{cases}\rho\leq 1 & \text{on M}\\ \rho=1 & \text{on the ball B(x,R)}\\ \rho=0 & on\ M\setminus B(x,R)\\ |\operatorname{grad}\rho|\leq\frac{2}{R}, & \text{on M.}\end{cases}$$

Replacing $\rho=\rho_R$, in (3.19) we obtain

$$\frac{1}{2}\sum_{i=1}^{m}\int_M|\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)|^2v_g\leq\frac{2}{R}\sum_{i=1}^{m}\int_M|\tau_{H,f}(\varphi)|^2v_g. \tag{3.20}$$

Since, $\int_M|\tau_{H,f}(\varphi)|^2v_g<\infty$, when $R\to\infty$ we have

$$\frac{1}{2}\sum_{i=1}^{m}\int_M|\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)|^2v_g=0$$

In this way $\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi)=0$ and $h(\nabla^{\varphi}_{e_i}\tau_{H,f}(\varphi),\tau_{H,f}(\varphi))=\frac{1}{2}e_i|\tau_{H,f}(\varphi)|^2=0$
for all $i=1,...,m$ i.e the function $|\tau_{H,f}(\varphi)|^2$ is constant on M. Finally, since the volume of M is infinite ($V(M)=\int_M v_g=+\infty$), from the formula (3.12), we conclude that $|\tau_{H,f}(\varphi)|^2=constant=0$, i.e φ is f-harmonic with potential H.

4. THE CASE OF CONFORMAL MAPS

We study conformal maps between equidimensional manifolds of the same dimension $n\geq 3$. Recall that a mapping $\varphi:(M^n,g)\longrightarrow(N^n,h)$ is called conformal if there exists a $\mathcal{C}^\infty$ function $\lambda:M\longrightarrow\mathbb{R}^*_+$. such that for any $X,Y\in\Gamma(TM)$:

$$h(d\varphi(X),d\varphi(Y))=\lambda^2g(X,Y).$$

The function λ is called the dilation for the map φ.
Then the f-tension field of the map φ is given by (see [13]).

$$\tau_f(\varphi) = (2-n) f d\varphi(\operatorname{grad} ln\lambda) + d\varphi(\operatorname{grad} f). \tag{4.1}$$

Where $f \in \mathcal{C}^\infty(M)$ is a positive function.

Note that, if $n = 2$ or the dilation λ is constant, the conformal map $\varphi : (M^n, g) \longrightarrow (N^n, h)$ of dilation λ is f-harmonic if and only if $\operatorname{grad} f \in \ker(d\varphi)$.
The f-bitension field of the conformal map is given by the following equation (see [13])

$$\begin{aligned}
\tau_{2,f}(\varphi) &= (n-2)f^2 d\varphi(\operatorname{grad}\Delta ln\lambda) + (n-2)\big[f(\Delta f) + |\operatorname{grad} f|^2\big] d\varphi(\operatorname{grad} ln\lambda) - (n-2)f^2 \nabla^\varphi_{\operatorname{grad} ln\lambda} d\varphi(\operatorname{grad} ln\lambda)\\
&+ 4(n-2)f \nabla^\varphi_{\operatorname{grad} f} d\varphi(\operatorname{grad} ln\lambda) - f d\varphi(\operatorname{grad}\Delta f)\\
&+ 2(n-2)f^2 < \nabla d\varphi, \nabla dln\lambda > -2f < \nabla d\varphi, \nabla df > -\nabla^\varphi_{\operatorname{grad} f} d\varphi(\operatorname{grad} f)\\
&+ 2(n-2)f^2 d\varphi(\operatorname{Ricci}^M(\operatorname{grad} ln\lambda)) - 2f d\varphi(\operatorname{Ricci}^M(\operatorname{grad} f)),
\end{aligned} \tag{4.2}$$

where $< \nabla d\varphi, \nabla dln\lambda >= \nabla d\varphi(e_i, e_j)\nabla dln\lambda(e_i, e_j)$ and $\{e_i\}_{i=1,..,n}$ being an ortonormal basis on M.
From the Theoreme (2.1) and the equation (4.1), we deduce the following

Proposition 4.1. *Let $\varphi : (M^n, g) \longrightarrow (N^n, h)$, $(n \geq 3)$ be a conformal map of dilation λ, H be a smooth function on N and f be a smooth positive function on M. Then φ is f-harmonic with potential H if and only if*

$$(2-n) f d\varphi(\operatorname{grad} ln\lambda) + d\varphi(\operatorname{grad} f) + (\operatorname{grad}^N H) \circ \varphi = 0. \tag{4.3}$$

From the formulas (3.8) and (4.2), we deduce the following

Proposition 4.2. *Let $\varphi : (M^n, g) \longrightarrow (N^n, h)$, $(n \geq 3)$ be a conformal map of dilation λ, H be a smooth function on N and f be a smooth positive function on M. Then φ is bi-f-harmonic with potential H if and only if*

$$\begin{aligned}
& (n-2)f^2 d\varphi(\operatorname{grad}\Delta ln\lambda) + (n-2)\big[f(\Delta f) + |\operatorname{grad} f|^2\big] d\varphi(\operatorname{grad} ln\lambda) - (n-2)f^2 \nabla^\varphi_{\operatorname{grad} ln\lambda} d\varphi(\operatorname{grad} ln\lambda)\\
+ & 4(n-2)f \nabla^\varphi_{\operatorname{grad} f} d\varphi(\operatorname{grad} ln\lambda) - f d\varphi(\operatorname{grad}\Delta f)\\
+ & 2(n-2)f^2 < \nabla d\varphi, \nabla dln\lambda > -2f < \nabla d\varphi, \nabla df > -\nabla^\varphi_{\operatorname{grad} f} d\varphi(\operatorname{grad} f)\\
+ & 2(n-2)f^2 d\varphi(\operatorname{Ricci}^M(\operatorname{grad} ln\lambda)) - 2f d\varphi(\operatorname{Ricci}^M(\operatorname{grad} f)) + J_{f,\varphi}(\operatorname{grad}^N H) \circ \varphi\\
+ & (n-2)f(\nabla^N_{d\varphi(\operatorname{grad} ln\lambda)} \operatorname{grad}^N H) \circ \varphi + (\nabla^N_{d\varphi(\operatorname{grad} f)} \operatorname{grad}^N H) \circ \varphi + (\nabla^N_{(\operatorname{grad}^N H)\circ\varphi} \operatorname{grad}^N H) \circ \varphi = 0.
\end{aligned} \tag{4.4}$$

Remark 4.1. If $n = 2$, then φ is bi-f-harmonic with potential H if and only if

$$\begin{aligned}
& f d\varphi(\operatorname{grad}\Delta f) + 2f < \nabla d\varphi, \nabla df > + \nabla^\varphi_{\operatorname{grad} f} d\varphi(\operatorname{grad} f) + 2f d\varphi(\operatorname{Ricci}^M(\operatorname{grad} f))\\
- & J_{f,\varphi}(\operatorname{grad}^N H) \circ \varphi - (\nabla^N_{d\varphi(\operatorname{grad} f)} \operatorname{grad}^N H) \circ \varphi - (\nabla^N_{(\operatorname{grad}^N H)\circ\varphi} \operatorname{grad}^N H) \circ \varphi = 0
\end{aligned}$$

In particular, if we consider the identity map, we obtain the following result

Corollary 4.1. *$Id_M : (M^n, g) \longrightarrow (M^n, g)$ is bi-f-harmonicwith potential H if and only if*

$$\begin{aligned}
& f \operatorname{grad}(\Delta H - \Delta f) + \frac{1}{2}\operatorname{grad}(|\operatorname{grad} H|^2 - |\operatorname{grad} f|^2)\\
+ & 2f \operatorname{Ricci}(\operatorname{grad} H - \operatorname{grad} f) + 2\nabla_{\operatorname{grad} f} \operatorname{grad} H = 0
\end{aligned} \tag{4.5}$$

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