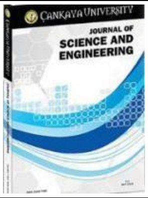
e-ISSN: 2564-7954 CUJSE 17(2): 101-107 (2020) Research Article



Çankaya University

Journal of Science and Engineering

https://dergipark.org.tr/cankujse

CUJSE

Convergence of the Associated Filters via Set-Operators

Takashi Noiri [1], Sk Selim [2], Shyamapada Modak [2*]

[1] 2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi Kumomoto-ken, 869-5142 Japan
[2] Department of Mathematics, University of Gour Banga, India

Keywords

Ideal
Filter
Associated filter
Limit point of a filter
Local function

Abstract

Let (X, τ) be a topological space. For a proper ideal $\mathcal{I}$ on (X, τ), the associated filter $\mathcal{F}_I$ is defined and investigated in [1]. In this paper, we define several set-operators on an ideal topological space $(X, \tau, \mathcal{I})$ and investigate the relationship between the set-operators and limit points of the associated filter $\mathcal{F}_I$.

1. Introduction and Preliminaries

Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X, then for $A \subseteq X$, the local function is defined in [2] as $A^*(\mathcal{I}, \tau) = \{x \in X : U_x \cap A \notin \mathcal{I}$ for every $U_x \in \tau(x)\}$, where $\tau(x)$ is the collection of all open sets containing x. $A^*(\mathcal{I}, \tau)$ is simply denoted as $A^*(\mathcal{I})$ or A^*. For the simplest ideals $\{\emptyset\}$ and $\mathcal{P}(X)$, we observe that $A^*(\{\emptyset\}) = cl(A)$ ($cl(A)$ denotes the closure of A) and $A^*(\mathcal{P}(X)) = \emptyset$ for every $A \subseteq X$. Thus the study of the local function is interesting when the ideal $\mathcal{I}$ is a proper ideal (an ideal $\mathcal{I}$ does not contain whole set X) on the topological space. Otherwise, when an ideal contains the hole set X, then it contains all subsets of X and the value of local function on any set is always empty.

The complementary set-operator ψ of the set-operator $()^*$ is defined in [3] as $\psi(A) = X \setminus (X \setminus A)^*$. It is notable that $()^*$ is not a closure operator and ψ is not an interior operator. However, the set operator $C: \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $C(A) = A \cup A^*$ makes a closure operator [2, 4] and it is denoted as 'cl^*', that is $cl^*(A) = A \cup A^*$. This closure operator induces a topology on X and it is called the $*$-topology [5-12]. This topology is denoted as τ^*.

Let X be a nonempty set and $\mathcal{F} \subseteq \mathcal{P}(X)$. Then $\mathcal{F}$ is called a filter [13, 14] on X if it satisfies the following:

1. $\emptyset \notin \mathcal{F}$
2. $B \in \mathcal{F}$ and $B \subseteq A$ implies $A \in \mathcal{F}$,
3. $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

Let $\mathcal{I}$ be a proper ideal on a topological space (X, τ). Then $F = \{A \subseteq X : X \setminus A \in I\}$ forms a filter on X. This filter is called the associated filter on X and denoted as $\mathcal{F}_I$.

Definition 1. [1] An ideal $\mathcal{I}_u$ on a set X is called a universal ideal if for any $A \subseteq X$, either $A \in \mathcal{I}_u$ or $X \setminus A \in \mathcal{I}_u$.

* Corresponding Author: spmodak2000@yahoo.co.in
Received: April 11, 2020, Accepted: May 5, 2020

2. Associated filters

We define the operator $\wedge$ on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way: for a subset A of X, $\wedge(A) = \psi(A) \setminus A$.

Lemma 1. *Let $\mathcal{I}$ be a proper ideal on a topological space (X, τ) and $A \subseteq X$. Then for $x \in \psi(A) \in \mathcal{I}$, x is not a limit point of the associated filter $\mathcal{F}_{\mathrm{I}}$.*

Proof. Given that $x \in \psi(A) \in \mathcal{I}$. Thus for $\psi(A) \in \tau(x)$, $X \setminus \psi(A) \in \mathcal{F}_{\mathrm{I}}$. Hence $x \in \psi(A) \not\in \mathcal{F}_{\mathrm{I}}$. Therefore, x is not a limit point of $\mathcal{F}_{\mathrm{I}}$.

Corollary 1. *Let $\mathcal{I}$ be a proper ideal on a topological space (X, τ) and $A \subseteq X$. Then for $x \in \wedge(A)$ and $\psi(A) \in \mathcal{I}$, x is not a limit point of the associated filter $\mathcal{F}_{\mathrm{I}}$.*

Corollary 2. *Let $\mathcal{I}$ be a proper ideal on a topological space (X, τ) and $A \subseteq X$. Then for $x \in \wedge(A)$ and $\psi(A) \in \mathcal{I}$, x is not a cluster point of the associated filter $\mathcal{F}_{\mathrm{I}}$.*

For converse of the above corollary and lemma, we have followings:

Theorem 1. *Let $\mathcal{I}_{\mathrm{u}}$ be a universal ideal on a topological space X. If x is not a limit point of the associated filter* U*, then $x \in \psi(X \setminus A)$ for any $A \in \mathcal{P}(X)$.*

Proof. Let $x \in X$ and x be not a limit point of U. Then there exists $U_x \in \tau(x)$ such that $U_x \not\in$ U and hence $U_x \cap A \not\in$ U, for any $A \in \mathcal{P}(X)$. Therefore, $X \setminus (U_x \cap A) \not\in \mathcal{I}_{\mathrm{u}}$ and hence $A \cap U_x \in \mathcal{I}_{\mathrm{u}}$. Thus $x \not\in A^*$ and hence $x \in (X \setminus A^*) = \psi(X \setminus A)$.

We define the operator Δ_1 on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way: for a subset A of X, $\Delta_1(A) = \underline{\vee}(A) \cup \wedge(A) = (\psi(A) \setminus A^*) \cup (\psi(A) \setminus A)$.

Theorem 2. *Let $\mathcal{I}$ be a proper ideal on a topological space (X, τ) and $A \subseteq X$. Then for $x \in \Delta_1(A)$ and $\psi(A) \in \mathcal{I}$, x is not a limit (resp. cluster) point of the associated filter $\mathcal{F}_{\mathrm{I}}$.*

Proof. Case (i): Suppose $x \in (\psi(A) \setminus A^*)$. Thus $x \in X \setminus (X \setminus A)^*$ and $x \not\in A^*$.Thus $x \not\in (X \setminus A)^* \cup A^*$ and hence $x \not\in X^*$. Thus there exists $U_x \in \tau(x)$ such that $U_x \cap X \in \mathcal{I}$, hence $U_x \in \mathcal{I}$. Thus $X \setminus U_x \in \mathcal{F}_{\mathrm{I}}$. Therefore, x is not a limit point of $\mathcal{F}_{\mathrm{I}}$.

Case (ii): Suppose $x \in (\psi(A) \setminus A)$. Then from Lemma 1, x is not a limit point of $\mathcal{F}_{\mathrm{I}}$.

We define the operator Δ_2 on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way:

$\Delta_2(A) = \underline{\vee}(A) \cup \overline{\wedge}(A) = (\psi(A) \setminus A^*) \cup (A \setminus A^*)$.

For convergence of the associated filter to a point $x \in \Delta_2(A)$, we get followings:

Theorem 3. *Let $\mathcal{I}$ be a proper ideal on a topological space (X, τ). Then for $x \in (\psi(A) \setminus A^*)$ but $x \not\in (A \setminus A^*)$, x is not a limit point of the associated filter $\mathcal{F}_{\mathrm{I}}$.*

Proof. Similar to Theorem 2 Case (i).

Theorem 4. *Let* I *be a proper ideal on a topological space (X, τ). Then for $x \not\in (\psi(A) \setminus A^*)$ but $x \in (A \setminus A^*)$, x is not a limit point of the associated filter $\mathcal{F}_{\mathrm{I}}$.*

Proof. Given that $x \not\in (\psi(A) \setminus A^*)$ and $x \in (A \setminus A^*)$, then $x \not\in A^*$, $x \in A$ and $x \not\in \psi(A)$

(since $x \not\in A^*$, if $x \in \psi(A)$ then $x \in \psi(A) \setminus A^*$, contradiction). Therefore, $x \not\in A^*$, $x \in A$ and $x \in (X \setminus A)^*$. Then there exists $U_x \in \tau(x)$ such that $U_x \cap A \in \mathcal{I}$, and for all $W_x \in \tau(x)$, $W_x \cap (X \setminus A) \not\in \mathcal{I}$ and $x \in A$. Thus, for the particular U_x, $U_x \cap A \in \mathcal{I}$ and $U_x \setminus A \not\in \mathcal{I}$. Thus for the particular U_x, $U_x \not\in \mathcal{I}$ and hence $U_x \not\in \mathcal{F}_I$. Therefore, x is not a limit point of $\mathcal{F}_I$.

Moreover, if $x \in (\psi(A) \setminus A^*)$ and $x \in (A \setminus A^*)$, then x is not a limit of the associated filter

We define the operator Δ_3 on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way: for any subset A of X, $\Delta_3(A) = \overline{\wedge}(A) \cup \wedge(A) = (A \setminus A^*) \cup (\psi(A) \setminus A)$.

For convergence of the associated filter to a point of $\Delta_3(A)$, we shall take the help of next section.

3. Complementary set-functions

In this section we consider the complementary function to the earlier section and discuss the convergence of the associated filter in terms of the complementary function.

We define the operator $\vee$ and ∇_i on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way:

for any subset A of X, $\vee(A) = X \setminus \wedge(X \setminus A)$ and $\nabla_i(A) = X \setminus \Delta_i(X \setminus A)$ for $i = 1, 2, 3$.

Theorem 5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then for $A \in \mathcal{P}(X)$,*

1. $\vee(A) = A^* \cup (X \setminus A)$.
2. $\vee(X \setminus A) = (X \setminus A)^* \cup A$.
3. $\vee(X \setminus A) = (X \setminus \psi(A)) \cup A$
4. $\vee(A) \cup A = X$.

Proof. 1. $\vee(A) = X \setminus (\psi(X \setminus A) \setminus (X \setminus A)) = X \setminus ((X \setminus A^*) \setminus (X \setminus A)) = X \setminus [(X \setminus A^*) \cap A] = A^* \cup (X \setminus A)$.

Proof of 2,3 and 4 follows from 1.

If $x \in \vee(A)$, then x may be a limit point of the associated filter $\mathcal{F}_I$.

Example 3.1. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Let $A = \{a, c\}$. Then $c, b \in \vee(A) = A^* \cup (X \setminus A) = \{c\} \cup (X \setminus \{a, c\}) = \{b, c\}$. But the associated filter $\mathcal{F}_I = \{\{b, c\}, X\}$ does not converges to b. Moreover, $\mathcal{F}_I \longrightarrow c$.*

Theorem 6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then for $A \in \mathcal{P}(X)$,*

1. $\Delta_1(A) = \psi(A) \setminus (A^* \cap A)$.
2. $\Delta_1(X \setminus A) = (\psi(A) \cup A) \setminus A^*$.
3. $\nabla_1(A) = (X^* \setminus A) \cup A^*$.
4. $\nabla_1(A) \cup A = X$, *if the space is Hayashi-Samuel.*
5. $\Delta_1(A) \subseteq \psi(A)$.
6. $\Delta_1(A) \subseteq A^*$, *if the space is Hayashi-Samuel.*

Proof. 1. $\Delta_1(A) = (\psi(A) \setminus A^*) \cup (\psi(A) \setminus A) = \psi(A) \cap [(X \setminus A^*) \cup (X \setminus A)] = \psi(A) \cap [X \setminus (A^* \cap A)] = \psi(A) \setminus (A^* \cap A)$.

2. $\Delta_1(X \setminus A) = (\psi(X \setminus A) \setminus ((X \setminus A)^* \cap (X \setminus A)) = (X \setminus A^*) \setminus ((X \setminus A)^* \setminus A) = [(X \setminus A^*) \setminus (X \setminus A)^*] \cup A = [(X \setminus A^*) \setminus (X \setminus A)^*] \cap (X \setminus A) = [(X \setminus A^*) \setminus (X \setminus \psi(A))] \cap (X \setminus A) = [(X \setminus A^*) \cap (\psi(A) \cup A) = (\psi(A) \cup A) \setminus A^*$.

3. $\nabla_1(A) = X \setminus [(\psi(A) \cup A) \setminus A^*] = X \setminus [(\psi(A) \cup A) \cap (X \setminus A^*)] = [X \setminus (\psi(A) \cup A)] \cup A^* =$

$[(X \setminus \psi(A)) \cap (X \setminus A)] \cup A^* = [(X \setminus A)^* \cap (X \setminus A)] \cup A^* = [(X \setminus A)^* \cup A^*] \cap ((X \setminus A)] \cup A^*) = X^* \cap ((X \setminus A)] \cup A^*)$ $= [X^* \cap (X \setminus A)] \cup (X^* \cap A^*) = [X^* \cap (X \setminus A)] \cup A^* = (X^* \setminus A) \cup A^*$.

4. $\nabla_1(A) = (X^* \setminus A) \cup A^* = (X \setminus A) \cup A^*$. Therefore, $\nabla_1(A) \cup A = X$.

5. By 1., the proof is obvious.

6. From 5., $\Delta_1(A) \subseteq \psi(A)$. Then $\Delta_1(A) \subseteq X \setminus (X \setminus A)^* \subseteq X \setminus (X^* \setminus A^*)$ [7] $= X \setminus (X \setminus A^*) = A^*$.

If $x \in \nabla_1(A)$, then x may be a limit point of the associated filter $\mathcal{F}_I$ and it is followed by the following example.

Example 3.2. *(i). Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, *and* $I = \{\emptyset, \{a\}\}$. *Suppose* $A = \{a\}$.*Then for* $b \in \nabla_1(A)$, $\mathcal{F}_I \rightarrow b$.

(ii). Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{c, b\}\}$ *and* $I = \{\emptyset, \{a\}\}$. *Then* $\mathcal{F}_I = \{\{b, c\}, X\}$. *Consider* $B = \{a\}$, *then* $c \in \nabla_1(B) = X^* \setminus A = \{a, b, c\} \setminus \{a\} = \{b, c\}$,*but* $\mathcal{F}_I \nrightarrow c$.

Theorem 7. *Let* (X, τ, I) *be an ideal topological space. Then for* $A \in \mathcal{P}(X)$,

1. $\Delta_2(A) = (\psi(A) \cup (A)) \setminus A^*$.

2. $\Delta_2(A) = \psi(A) \setminus A^*$ *if* A *is open.*

3. $\nabla_2(A) = (X^* \cap A) \cup (X \setminus A)^*$.

4. $\nabla_2(A) = A \cup (X \setminus A)^*$, *when the space* (X, τ, I) *is Hayashi-Samuel.*

Proof. 1. $\Delta_2(A) = (\psi(A) \setminus A^*) \cup (A \setminus A^*) = (\psi(A) \cap (X \setminus A^*)) \cup (A \cap (X \setminus A^*)) = (\psi(A) \cup A) \cap (X \setminus A^*) = (\psi(A) \cup A) \setminus A^*$.

2. Since A is open, $X \setminus A$ is closed and it is τ^*-closed. Hence $(X \setminus A)^* \subset X \setminus A$ and $\psi(A) \supseteq A$. Therefore, $\psi(A) \cup A = \psi(A)$. Hence $\Delta_2(A) = \psi(A) \setminus A^*$.

3. $\nabla_2(A) = X \setminus \Delta_2(X \setminus A) = X \setminus [(\psi(X \setminus A) \setminus (X \setminus A)^*) \cup ((X \setminus A) \setminus (X \setminus A)^*)] =$

$X \setminus [((X \setminus A^*) \setminus (X \setminus \psi(A)) \cup (X \setminus A) \setminus (X \setminus \psi(A)))] = X \setminus [((X \setminus A^*)) \cap \psi(A)] \cup ((X \setminus A) \cap \psi(A))] =$

$X \setminus [((X \setminus A^*) \cup (X \setminus A)) \cap \psi(A)] = X \setminus [(X \setminus (A^* \cap A)) \cap \psi(A)] = (A^* \cap A) \cup (X \setminus \psi(A)) =$

$(A^* \cap A) \cup (X \setminus A)^* = (A^* \cup (X \setminus A)^*) \cap (A \cup (X \setminus A)^*) = X^* \cap (A \cup (X \setminus A)^*) = (X^* \cap A) \cup$

$(X^* \cap (X \setminus A)^* = (X^* \cap A) \cup (X \setminus A)^*$.

4. If (X, τ, I) is Hayashi-Samuel space, $X^* = X$ and by (3), $A \cup \emptyset^* = A$.

For convergence of the associated filter to a point in $\nabla_2(A)$, we discuss the following example.

Example 3.3. *(i). Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ *and* $I = \{\emptyset, \{a\}\}$. *Then* X^* n= $\{b, c\}$ *and* $\mathcal{F}_I = \{X, \{b, c\}\}$. *Consider* $A = \{a\}$. *Then* $b \in \nabla_2(A) = (X^* \cap A) \cup (X \setminus A)^* = \{b, c\}$ *and* $\mathcal{F}_I$ *converges to b.*

(ii). Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{c, b\}\}$ *and* $I = \{\emptyset, \{a\}\}$. *Then* $\mathcal{F}_I = \{\{b, c\}, X\}$.*Consider* $B = \{a\}$, *then* $c \in \nabla_2(B) = (X^* \cap A) \cup (X \setminus A)^* = \{a\} \cup \{b, c\}^* = X$, *but* $\mathcal{F}_I \nrightarrow c$.

Theorem 8. *Let* (X, τ, I) *be an ideal topological space. Then for* $A \in \mathcal{P}(X)$,

1. $\Delta_3(X \setminus A) = \Delta_3(A)$.

2. $\Delta_3(X) = X \setminus X^*$

3. $\nabla_3(A) = X \setminus \Delta_3(X \setminus A) = X \setminus \Delta_3(A)$.

Proof. 1. $\Delta_3(X \setminus A) = ((X \setminus A) \setminus (X \setminus A)^*) \cup (\psi(X \setminus A) \setminus (X \setminus A)) = ((X \setminus A) \cap (X \setminus (X \setminus A)^*) \cup ((X \setminus A^*) \setminus (X \setminus A)) = ((X \setminus A) \cap \psi(A)) \cup ((X \setminus A^*) \cap A) = (\psi(A) \setminus A) \cup (A \setminus A^*) = \Delta_3(A)$.

2. $\Delta_3(X) = (X \setminus X^*) \cup (\psi(X) \setminus X) = (X \setminus X^*) \cup (X \setminus X) = X \setminus X^*$.

3. The proof is obvious by the definitions and 1.

Following examples will discuss the convergence of the associated filter to a point of $\Delta_3(A)$ and $\nabla_3(A)$.

Example 3.4. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Consider* $A = \{a\}$, *then* $\psi(A) = X \setminus \{b, c\}^* = X \setminus \{b, c\} = \{a\}$. *Therefore, for* $a \in \Delta_3(A) = \{a\}$, $\mathcal{F}_{\mathrm{I}} = \{X, \{b, c\}\}$. *Then* $\mathcal{F}_{\mathrm{I}} \nrightarrow a$. *Next suppose* $B = \{a, b\}$, *then* $B^* = \{b, c\}$ *and* $\psi(B) = X \setminus \{a\}^* = X$. *Thus for* $c \in \Delta_3(A) = \{a\} \cup \{c\} = \{a, c\}$, $\mathcal{F}_{\mathrm{I}} \longrightarrow c$.

Example 3.5. *(i) Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{c, b\}\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Then* $\mathcal{F}_{\mathrm{I}} = \{\{b, c\}, X\}$. *Now* $\nabla_3(X) = X \setminus \Delta_3(X \setminus X) = X \setminus \Delta_3(X) = X \setminus (X \setminus X^*) = X^* = \{a, b, c\}$. *Thus for* $c \in \nabla_3(X)$, $\mathcal{F}_{\mathrm{I}} \nrightarrow c$.

(ii) Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Then* $\mathcal{F}_{\mathrm{I}} = \{\{b, c\}, X\}$. *Consider* $B = \{a, b\}$, *then* $B^* = \{b, c\}$ *and* $\psi(B) = X \setminus \{a\}^* = X$. *Thus, for* $b \in X \setminus \Delta_3(A) = X \setminus \{a\} \cup \{c\} = \{b\}$, $\mathcal{F}_{\mathrm{I}} \longrightarrow b$.

4. Conclusions

Furthermore, in the conclusion part of this paper we consider following:

Lemma 2. [1] *Let* $f : X \to Y$ *be a bijective function. If* $\mathcal{I}$ *is a proper ideal on X, then*

$f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ *is a proper ideal on* Y.

Theorem 9. *[15] Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* (Y, σ) *be a topological space such that* $f: X \to Y$ *is a homeomorphism. Then* $f(A^*(\mathcal{I})) = (f(A))^*(f(\mathcal{I}))$ *for any* $A \in \mathcal{P}(X)$.

Proof. For $A \in \mathcal{P}(X)$ let $x \in X$ with $f(x) \notin f(A^*(\mathcal{I}, \tau))$. This implies that $x \notin A^*(\mathcal{I}, \tau)$. Thus there exists $U_x \in \tau(x)$ such that $U_x \cap A \in \mathcal{I}$. Therefore, $f(U_x \cap A) \in f(\mathcal{I})$, hence $f(U_x) \cap f(A) \in f(\mathcal{I})$. So, we have, $f(x) \notin [f(A)]^*(f(\mathcal{I}), \sigma)$, since $f(U_x) \in \sigma f(x)$. Therefore, $f[A^*(\mathcal{I}, \tau)] \supseteq [f(A)]^*(f(\mathcal{I}), \sigma)$.

Reverse inclusion:

Let $t \in X$ with $f(t) \notin (f(A)^*(f(\mathcal{I}), \sigma))$. Then there exists $U_{f(t)} \in \sigma(f(t))$ such that $U_{f(t)} \cap f(A) \in f(\mathcal{I})$. Thus, $f^{-1}[U_{f(t)} \cap f(A)] = f^{-1}[U_{f(t)} \cap A] \in f^{-1}(f(\mathcal{I})) = \mathcal{I}$, implies $t \notin A^*(\mathcal{I}, \tau)$, since $f^{-1}(U_{f(t)} \in \tau(t))$. Thus, $f(t) \notin A^*(\mathcal{I}, \tau)$. Hence we have, $[f(A)]^*(f(\mathcal{I}), \sigma) \supseteq f[A^*(\mathcal{I}, \tau)]$.

Corollary 3. [15] *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* (Y, σ) *be a topological space such that*

$f: X \to Y$ *is a homeomorphism. Then* $f(\psi(A)(\mathcal{I})) = \psi(f(A))(f(\mathcal{I}))$, *for any* $A \in \mathcal{P}(X)$.

Proof. Claim: $f(\psi(A)(\mathcal{I})) \subseteq \psi(f(A))(f(\mathcal{I}))$.

Let $y \in f(\psi(A)(\mathcal{I}))$. Then, $f^{-1}(y) \in \psi(A)(\mathcal{I}) = X \setminus (X \setminus A)^*$. This implies that

$f^{-1}(y) \notin (X \setminus A)^*$. Therefore, there exists an open set U *containing* $f^{-1}(y)$ such that $U \cap (X \setminus A) \in \mathcal{I}$. Therefore $f(U \cap (X \setminus A)) = f(U) \cap f(X \setminus A) = f(U) \cap (Y \setminus f(A)) \in f(\mathcal{I})$. This implies $y \notin (Y \setminus f(A))^*$ and $y \in Y \setminus (Y \setminus f(A))^*$. Hence we have $y \in \psi(f(A))(f(\mathcal{I}))$.

Claim: $\psi(f(A))(f(\mathcal{I})) \subseteq f(\psi(A)(\mathcal{I}))$.

Let $y \in \psi(f(A))(f(I))$. This implies that $y \not\in (Y \setminus f(A))^*$. Thus there exists an open set V in Y containing y such that $V \cap (Y \setminus f(A)) \in f(I)$. Then there exists $B \in I$ such that $V \cap (Y \setminus f(A)) = f(B)$. This implies that $f^{-1}(V \cap (Y \setminus f(A))) = f^{-1}(V) \cap f^{-1}(Y \setminus f(A))) = f^{-1}(V) \cap (X \setminus A) = B \in I$. Therefore, $f^{-1}(y) \not\in (X \setminus A)^*$ and hence $f^{-1}(y) \in X \setminus (X \setminus A)^* = \psi(A)(I)$. Consequently, we have $y \in f(\psi(A))(I))$.

Theorem 10. *Let (X, τ, I) be an ideal topological space and (Y, σ) be a topological space such that $f : X \to Y$ is a homeomorphism. Then for $A \subseteq X$, the following properties hold:*

1. $f(\wedge(A)(I)) = (\wedge(f(A)))(f(I))$,

2. $f(\Delta_i(A)(I)) = (\Delta_i(f(A))(f(I)))$ *for* $i = 1, 2, 3$,

3. $f(\nabla_i(A)(I)) = (\nabla_i(f(A))(f(I)))$ *for* $i = 1, 2, 3$.

Proof. 1. $f(\wedge(A)(I)) = f[\psi(A)(I) \setminus A] = f(\psi(A)(I)) \setminus f(A)$ (as f is injective) $= \psi(f(A))(f(I)$ (from Corollary 3) $= (\wedge(f(A)))(f(I))$.

2. For $i = 1$, $f(\Delta_1(A)(I)) = f[(\psi(A)(I) \setminus A^*(I)) \cup (\psi(A)(I) \setminus A)] = [f(\psi(A)(I) \setminus A^*(I))] \cup [f(\psi(A)(I) \setminus A)] = [f((\psi(A)(I)) \setminus f(A^*(I))] \cup [f(\psi(A)(I)) \setminus f(A)]$ (as f is injective) $= [(\psi(f(A))(f(I)) \setminus (f(A))^*(f(I))] \cup [(\psi(f(A))(f(I))\setminus f(A)]$ (by Theorem 9 and Corollary 3) $= (\Delta_1(f(A))(f(I)))$.

3. For $i = 1$, $f(\nabla_1(A)(I)) = f[X \setminus \Delta_1(X \setminus A)(I)] = f(X) \setminus f(\Delta_1(X \setminus A)(I))$ (as f is injective) $= Y \setminus \Delta_1(f(X) \setminus f(A))(f(I)) = Y \setminus \Delta_1(Y \setminus f(A))(f(I)) = (\nabla_1(f(A))(f(I)))$.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authorship contribution statement

Takashi Noiri: Writing, Reviewing and Editing, Conceptualization, Methodology.
Shyamapada Modak: Writing, Reviewing and Editing, Conceptualization, Methodology.
Sk Selim: Data Creation, Writing, Reviewing, Draft preparation, Investigation.

References

[1] Sk. Selim, T. Noiri, S. Modak, "Ideals and the associated filters on topological spaces," *Eurasian Bulletin of Mathematics*, vol. 2(3), pp. 80-85, 2019.

[2] D. Janković, T. R. Hamlett, "New topologies from old via ideals," *The American Mathematical Monthly*, vol. 97, pp. 295-310, 1990.

[3] T. Natkaniec, "On *I*-continuity and *I*-semicontinuity points," *Mathematica Slovaca*, vol. 36(3), pp. 297-312, 1986.

[4] K. Kuratowski, *Topology*, Vol. I, New York, Academic Press, 1966.

[5] E. Hayashi, "Topologies defined by local properties," *Mathematische Annalen*, vol. 156, pp. 205-215, 1964.

[6] H. Hashimoto, "On the $*$-topology and its applications," *Fundamenta Mathematicae*, vol. 91, pp. 5-10, 1976.

[7] T. R. Hamlett, D. Janković, "Ideals in topological spaces and the set operator Ψ," *Bollettino dell'Unione Matematica Italiana.*, vol. 7, no. (4-B), pp. 863-874, 1990.

[8] S. Modak, "Some new topologies on ideal topological spaces," *Proceedings of the National Academy of Sciences, India, Sect. A Phys. Sci.*, vol. 82(3), pp. 233-243, 2012.

[9] P. Samuel, "A topology formed from a given topology and ideal," *Journal of the London Mathematical Society*, vol. 10, pp. 409-416, 1975.

[10] A. Al-Omari, H. Al-Saadi, "A topology via ω-local functions in ideal spaces," Mathematica, vol. 60(83), pp. 103-110, 2018.

[11] C. Bandhopadhya, S. Modak, "A new topology via Ψ-operator," *Proceedings of the National Academy of Sciences,*

India, vol. 76(A), no. IV, pp. 317-320, 2006.
[12] S. Modak, C. Bandyopadhyay, “A note on Ψ-operator,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 30(1), 43-48, 2007.
[13] K. D. Joshi, *Introduction to General Topology*, Michigan: Wiley, 1983.
[14] N. Bourbaki, *General Topology*, Chapter 1-4, Verlag, Berlin, Heidelberg: Springer, 1989.
[15] S. Modak, Sk. Selim and Md. M. Islam, “Sets and functions in terms of local function,” Submitted.