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J-Trajectories in Locally Conformal Kähler Manifolds with Parallel Anti Lee Field

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ABSTRACT

We show that J-trajectories in a locally conformal Kähler manifold with parallel anti Lee field are of osculating order at most 3.

Keywords: *J-trajectory; LCK manifold; Kenmotsu manifold; magnetic curve.*

AMS Subject Classification (2020): *Primary: 53C15; Secondary: 53C25; 53C30; 53C55; 53C80*

1. Introduction

The classical theory of static electromagnetism is generalized to arbitrary dimensional Riemannian geometry. A *magnetic field* is a closed two-form on a Riemannian manifold. A trajectory of the *Lorentz equation*

$$\nabla_{\gamma'}\gamma' = q\phi\gamma'$$

is called a *magnetic trajectory*. Here q is a constant (called the *strength*), ϕ is the *Lorentz force* corresponding to the magnetic field and ∇ denotes the Levi-Civita connection.

More generally, let (M, g, F) be a Riemannian manifold with a g-skew symmetric endomorphism field F. Then a smooth curve γ in M is said to be a *F-trajectory* if it satisfies (*cf.* [6]):

$$\nabla_{\gamma'}\gamma' = qF\gamma'.$$

When $M = (M, J, g)$ is an almost Kähler manifold, then J-trajectories are nothing but Kähler magnetic trajectories with respect to the Kähler magnetic field $\Omega = g(\cdot, J)$. One of the fundamental results on Kähler magnetic field is that every (unit speed) Kähler magnetic trajectory in a Kähler manifold is a holomorphic circle, that is, a Frenet curve of osculating order 2 with constant curvature and contant complex torsions. Based on this fundamental result, global behaviors of circles in Kähler manifolds, especially complex space forms have been studied intensively.

As is well known, complex manifolds do not have Kähler metrics, in general. For instance, compact complex surfaces of odd first Betti number can not admit any Kähler metric compatible to the complex structure. As a generalization or alternative of Kähler metrics, locally conformal Kähler metrics (LCK metrics, in short) have been paid much attention of Differential Geometers. On an LCK manifold, there exist two characteristic vector fields, called the *Lee field* and *anti Lee field*, respectively.

Differential geometric studies on J-trajectories in LCK manifolds, *i.e.*, complex manifolds equipped with LCK metric was initiated by Ateş, Munteanu and Nistor [6]. They studied J-trajectories in the product manifold $\mathbb{S}^3 \times \mathbb{R}$ of the unit 3-sphere $\mathbb{S}^3$ and the real line $\mathbb{R}$. The product manifold $\mathbb{S}^3 \times \mathbb{R}$ has naturally defined LCK structure derived from the standard Sasakian structure of $\mathbb{S}^3$. In particular $\mathbb{S}^3 \times \mathbb{R}$ is a typical example of *Vaisman manifold*, that is, an LCK manifold with parallel Lee field. Vaisman showed that the universal covering

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of a complete Vaisman manifold is holomorphically isometric to a product manifold $N \times \mathbb{R}$, where N is a homothetic change of a Sasakian manifold [29]. Motivated by Vaisman's theorem and a work [6], we studied J-trajectories in Vaisman manifolds in our previous work [15].

On the other hand, products of the form $N \times \mathbb{R}$ where N is a Kenmotsu manifold are LCK manifolds with parallel *anti* Lee fields. Based on this fundamental fact, in this paper we study J-trajectories on LCK manifolds with parallel anti Lee fields. In particular we study J-trajectories in the product manifolds of Kenmotsu manifolds and the real line.

2. Preliminaries

2.1. LCK manifolds

Let $M = (M, g, J)$ be a Hermitian manifold with Kähler form Ω:

$$\Omega(X, Y) = g(X, JY).$$

Then M is said to be a *locally conformal Kähler manifold* (LCK manifold, in short) if there exits an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of M and a family of smooth functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ such that

$$d(e^{-\sigma_\alpha}\Omega) = 0 \ \text{ on } \ U_\alpha$$

for all α [28]. Namely the local conformal change $(U_\alpha, J|_{U_\alpha}, e^{-\sigma_\alpha} g)$ is Kähler.

In case $U_\alpha = M$, then M is said to be a *globally conformal Kähler manifold* (GCK, in short).

On an LCK manifold $\omega = d\sigma_\alpha$ is globally defined and satisfies

$$d\Omega = \omega \wedge \Omega. \tag{2.1}$$

The closed 1-form ω is called the *Lee form*.

Proposition 2.1. *Let M be a Hermitian manifold of complex dimension $n \geq 2$. Then*

- *In case $n \geq 3$, if there exits a 1-form ω satisfying (2.1), then M is an LCK manifold with Lee form ω.*
- *In case $n = 2$, there exits a 1-form ω satisfying (2.1). If ω is closed then M is an LCK manifold with Lee form ω.*

Remark 2.1. On an LCK manifold M, ω defines a de Rham cohomology class $[\omega] \in \mathrm{H}^1(M; \mathbb{R})$. We can see that M is GCK if and only if $[\omega] = 0$.

Let us denote by B the vector field metrically dual to ω and call it the *Lee field* or *Lee vector field*. The vector field $A = JB$ is called the *anti Lee field* (or anti Lee vector field). For the anti Lee field A, we use the sign convention of [23].

Even if the conformal change $e^{-\sigma_\alpha} g$ of g is locally defined, the Levi-Civita connection $\hat{\nabla}$ of $e^{-\sigma_\alpha} g$ is globally defiend on M. The connection $\hat{\nabla}$ is called the *Weyl connection* of (M, J, g) and given by

$$\hat{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\left(\omega(X)Y + \omega(Y)X - g(X, Y)B\right).$$

Since $\hat{\nabla} J = 0$, we get

$$(\nabla_X J)Y = \frac{1}{2}\left(\omega(JY)X - \omega(Y)JX + g(X, Y)A - \Omega(X, Y)B\right). \tag{2.2}$$

2.2. LCK manifolds with parallel Lee fields

Definition 2.1. An LCK manifold M is said to be a *Vaisman manifold* if its Lee form is parallel with respect to the Levi-Civita connection.

Vaisman proved the following fundamental theorem.

Theorem 2.1. *Let M be a complete Vaisman manifold. Then*

- *the Lee field B and anti Lee field A are infinitesimal automorphisms of (g, J) and*
- *the universal covering of M is a Riemannian product of $\mathbb{R}$ and an α-Sasakian manifold with $|\alpha| = |B|/2$.*

2.3. LCK manifolds with parallel anti Lee fields

Since the Lee field B is metrically equivalent to the Lee form ω, the Vaisman property is equivalent to the parallelism of B. Our interest is LCK manifolds with parallel anti Lee field. If $\nabla A = 0$, then by (2.2), we have [17]:

$$\nabla\omega = \frac{1}{2}\left(|B|^2 g - \omega\otimes\omega - \omega(J)\otimes\omega(J)\right). \tag{2.3}$$

LCK manifolds with parallel anti Lee field are characterized as follows [17]:

Proposition 2.2 (Kashiwada)**.** *On an LCK manifold M with parallel anti Lee field A, the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ locally generated by $\omega\circ J = 0$, respectively $\omega = 0$ are integrable. Their leaves are non-compact totally geodesic real hypersurfaces with an induced β-Kenmotsu structure and totally umbilical complex hypersurfaces on which the induced structure is Kähler, where $|B| = 2|\beta|$.*

Such LCK structures are naturally appear on the product of Kenmotsu manifolds and the real line.

2.4. Kenmotsu manifolds

For our use, here we recall basic materials on Kenmotsu manifolds [19].

Let N be a $(2n-1)$-dimensional *almost contact metric manifold* with structure tensor field $(\varphi,\xi,\eta,\bar{g})$. These structure tensor fields satisfy

$$\varphi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \eta\circ\varphi = 0, \quad \bar{g}(\varphi X,\varphi Y) = \bar{g}(X,Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(N)$.

The *fundamental* 2*-form* Φ of N is defined by

$$\Phi(X,Y) = \bar{g}(X,\varphi Y).$$

Let us consider a Riemannian product manifold $M = (N\times\mathbb{R}, \bar{g} + dt^2)$. We equip an almost complex structure J on M by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right), \quad X\in\mathfrak{X}(N), \quad f\in C^\infty(M). \tag{2.4}$$

Then (M,J) equipped with the product metric $g = \bar{g} + dt^2$ is an almost Hermitian manifold with Kähler form $\Omega = \Phi - 2\eta\wedge dt$.

An almost contact metric manifold N is said to be *normal* if J is integrable.

Definition 2.2. An almost contact metric manifold N is said to be a β-Kenmotsu manifold if

$$(\overline{\nabla}_X\varphi)Y = -\beta\left(\Phi(X,Y)\xi + \eta(Y)\varphi X\right), \quad \overline{\nabla}_X\xi = \beta(X - \eta(X)\xi). \tag{2.5}$$

Here β is a nozero constant. 1-Kenmotsu manifolds are referred as to Kenmotsu manifolds.

From this definition one can deduce that $\operatorname{div}\xi = 2\beta(n-1)$. Hence β-Kenmotsu manifolds can not be compact. In addition, β-Kenmotsu manifolds are normal and satisfy

$$d\eta = 0, \quad d\Phi = 2\beta\,\eta\wedge\Phi.$$

The action of local flows generated by ξ on the structure tensor fields is described as

$$\pounds_\xi g = 2\beta(g - \eta\otimes\eta), \quad \pounds_\xi\varphi = 0, \quad \pounds_\xi\eta = 0.$$

Thus g is not Killing vector field.

Example 2.1 (Warped product)**.** Let $\tilde{N} = (\tilde{N}, \tilde{J}, \tilde{g})$ be a Kähler manifold. We consider a warped product $N = \mathbb{R}\times_f \tilde{N}$ with base $\mathbb{R}$ and standard fiber $\tilde{N}$. Let z be a coordinate of $\mathbb{R}$. We choose the warping function $f(z)$ as $f(z) = ce^z$, where $c \neq 0$ is a constant. The warped product metric is denoted by $\bar{g} = dz^2 + f(z)^2\tilde{g}$.

We introduce an almost contact structure (φ,ξ,η) compatible to $\tilde{g}$ in the following manner:

$$\xi = \frac{\partial}{\partial z}, \quad \eta = dz,$$

On the tangent space $T_{(z,\tilde{p})}N = T_z\mathbb{R} \oplus T_{\tilde{p}}\tilde{N}$ of $N = \mathbb{R} \times_f \tilde{N}$ at $(z, \tilde{p})$, we introduce a linear endomorphism $\varphi_{(z,\tilde{p})}$ by

$$\varphi_{(z,\tilde{p})}X = \begin{cases} 0 \quad \text{on} \quad T_z\mathbb{R}, \\ \exp(z\xi)_{*\tilde{p}} \circ \tilde{J}_{*\tilde{p}} \circ \exp(-z\xi)_{*\tilde{p}}X, \quad \text{on} \quad T_{\tilde{p}}\tilde{N}. \end{cases}$$

Then the correspondence $p \longmapsto \varphi_p$ is a smooth endomorphism field on N and $(\varphi, \xi, \eta, \bar{g})$ is a Kenmotsu structure.

Kenmotsu [19] showed the following local structure theorem:

Theorem 2.2. *Let N be a Kenmotsu manifold. Then for any point $\bar{p} \in N$ there exists a neighborhood U of p and positive number ε such that U is represented as a warped product $U = (-\varepsilon, \varepsilon) \times_f \tilde{N}$ with warping function $f(w) = ce^w$ for some $c \in \mathbb{R}^\times$ and $\tilde{N}$ is a Kähler manifold.*

2.5.

Let us consider a Riemannian product $M = N \times \mathbb{R}$ of a β-Kenmotsu manifold N with the real line $\mathbb{R}$. We equip an almost complex structure J on M defined by (2.4). Then, since N is normal, M is a Hermitian manifold. The exterior derivative $d\Omega$ of the Kähler form Ω of $M = N \times \mathbb{R}$ is computed as

$$d\Omega = d(\Phi - 2\eta \wedge dt) = 2\beta\, \eta \wedge \Phi$$

On the other hand we notice that

$$\eta \wedge \Omega = \eta \wedge (\Phi - 2\eta \wedge dt) = \eta \wedge \Phi.$$

Hence Ω satisfies

$$d\Omega = (2\beta\eta) \wedge \Omega.$$

This formula shows that M is an LCK manifold with Lee form $\omega = 2\beta\eta$. The corresponding Lee field and anti Lee field are $B = 2\beta\xi$ and $A = 2\beta\partial_t$, respectively. Thus if we choose $\beta = \pm 1/2$, then $|B| = |A| = 1$.

Since the metric g is the product one, Levi-Civita connection ∇ of g satisfies

$$\nabla_A X = \nabla_X A = 0$$

for any $X \in \mathfrak{X}(N)$. In particular $\nabla A = 0$. Thus (M, J, g) is an LCK manifold with parallel anti Lee field. The covariant derivative ∇J is computed as follows:

Proposition 2.3. *Let $M = N \times \mathbb{R}$ be LCK manifold with β-Kenmotsu base manifold N. Then the covariant derivative ∇J is given by*

1. *For all $X, Y \in \mathfrak{X}(N)$,*
$$(\nabla_X J)Y = (\overline{\nabla}_X \varphi)Y + \beta g(\varphi X, \varphi Y)\partial_t = -\beta\left(\eta(Y)\varphi X + \Phi(X, Y)\xi - g(\varphi X, \varphi Y)\partial_t\right).$$
2. *In particular, for all $X, Y \in \mathfrak{X}(N)$,*
$$(\nabla_B J)Y = 0, \quad (\nabla_X J)B = -2\beta^2 \varphi X.$$
3. *For all $X, Y \in \mathfrak{X}(N)$,*
$$(\nabla_A J)Y = 0, \quad (\nabla_X J)A = -\frac{1}{2}|B|^2 X + 2\beta^2\eta(X)\xi.$$
4. $(\nabla_A J)A = 0$. *Hence* $\nabla_A J = 0$.

2.6.

Let us realize the hyperbolic 3-space $\mathbb{H}^3(-\beta^2)$ of constant curvature $-\beta^2$ as a homogeneous Kenmotsu manifold in the following manner. The hyperbolic 3-space $\mathbb{H}^3(-\beta^2)$ is identified with the solvable Lie group

$$\left\{ \left(\begin{array}{ccc} e^{-\beta z} & 0 & x \\ 0 & e^{-\beta z} & y \\ 0 & 0 & 1 \end{array} \right) \middle|\ x, y, z \in \mathbb{R} \right\} \subset \mathrm{GL}_3\mathbb{R} \tag{2.6}$$

equipped with the left invariant metric

$$g = e^{2\beta z}(dx^2 + dy^2) + dz^2.$$

As a Riemannian manifold $\mathbb{H}^3$ is the *warped product* $\mathbb{R} \times_{e^{\beta z}} \mathbb{R}^2$. The group operation of $\mathbb{H}^3(-\beta^2)$ is given explicitly by

$$(x, y, z) \cdot (\tilde{x}, \tilde{y}, \tilde{z}) = (x + e^{-\beta z}\tilde{x}, y + e^{-\beta z}\tilde{y}, z + \tilde{z}). \tag{2.7}$$

Let us define a representation ρ of $(\mathbb{R}(z), +)$ over $\mathbb{R}^2(x, y)$ by

$$\rho(z) = \begin{pmatrix} e^{-\beta z} & 0 \\ 0 & e^{-\beta z} \end{pmatrix}.$$

Then as a Lie group, $\mathbb{H}^3(-\beta^2)$ is a *semi-direct product* $\mathbb{R} \ltimes_\rho \mathbb{R}^2$.

The Lie algebra $\mathfrak{h}^3(-\beta^2)$ of $\mathbb{H}^3(-\beta^2)$ is

$$\left\{ \begin{pmatrix} -\beta w & 0 & u \\ 0 & -\beta w & v \\ 0 & 0 & 0 \end{pmatrix} \middle| \, u, v, w \in \mathbb{R} \right\}.$$

Take an orthonormal basis

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -\beta & 0 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We denote by e_i the left invariant vector field on $\mathbb{H}^3(-\beta^2)$ which is obtained by left translation of E_i. Then we have

$$e_1 = e^{-\beta z}\frac{\partial}{\partial x}, \quad e_2 = e^{-\beta z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \tag{2.8}$$

$$[e_1, e_2] = 0, \ \ [e_2, e_3] = \beta\, e_2, \ \ [e_3, e_1] = -\beta\, e_1. \tag{2.9}$$

The Levi-Civita connection $\overline{\nabla}$ of $\mathbb{H}^3(-\beta^2)$ is described as

$$\begin{aligned} \overline{\nabla}_{e_1} e_1 &= -\beta\, e_3, \quad \overline{\nabla}_{e_1} e_2 = 0, \quad \overline{\nabla}_{e_1} e_3 = \beta\, e_1, \\ \overline{\nabla}_{e_2} e_1 &= 0, \quad \overline{\nabla}_{e_2} e_2 = -\beta\, e_3, \quad \overline{\nabla}_{e_2} e_3 = \beta\, e_2, \\ \overline{\nabla}_{e_3} e_1 &= 0, \quad \overline{\nabla}_{e_3} e_2 = 0, \quad \overline{\nabla}_{e_3} e_3 = 0. \end{aligned} \tag{2.10}$$

These formulas show that the Lie algebra $\mathfrak{h}^3(-\beta^2)$ is *non-unimodular*.

Define an endomorphism field φ by

$$\varphi e_1 = e_2, \ \varphi e_2 = -e_1, \ \varphi e_3 = 0.$$

Next we put $\xi = e_3$ and $\eta = \mathrm{d}z$. Then $(\varphi, \xi, \eta, \bar{g})$ is a left invariant almost contact metric structure on $\mathbb{H}^3(-\beta^2)$. One can check that $(\mathbb{H}^3(-\beta^2), \varphi, \xi, \eta, \bar{g})$ is normal and satisfies

$$d\eta = 0, \quad d\Phi = 2\beta\eta \wedge \Phi.$$

One can check that the structure $(\varphi, \xi, \eta, \bar{g})$ is normal. Hence $\mathbb{H}^3(-\beta^2)$ is β-Kenmotsu manifold if $\beta \neq 0$. In particular, when $\beta = -1$, $\mathbb{H}^3(-1)$ is a Kenmotsu manifold. The homogeneous β-Kenmotsu manifold $\mathbb{H}^3(-\beta^2)$ is interpreted as the warped product $\mathbb{H}^3(-\beta^2) = \mathbb{R}(z) \times_{e^{\beta z}} \mathbb{E}^2(x, y)$.

3. J-trajectories on LCK manifolds

3.1. Frenet curves in almost Hermitian manifolds

Definition 3.1. If γ is a curve in a Riemannian manifold M, parametrized by arc length s, we say that γ is a *Frenet curve of osculating order* r when there exist orthonormal vector fields E_1, $E_2, \cdots, E_r$ along γ such that

$$\begin{aligned} &\gamma' = E_1, \ \nabla_{\gamma'} E_1 = \kappa_1 E_2, \ \nabla_{\gamma'} E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \cdots, \\ &\nabla_{\gamma'} E_{r-1} = -\kappa_{r-2} E_{r-2} + \kappa_{r-1} E_r, \ \nabla_{\gamma'} E_r = -\kappa_{r-1} E_{r-1}, \end{aligned} \tag{3.1}$$

where $\kappa_1, \kappa_2, \cdots, \kappa_{r-1}$ are positive C^∞ functions of s. The function κ_j is called the j-th *curvature* of γ.

A *geodesic* is regarded as a Frenet curve of osculating order 1. A *circle* is defined as a Frenet curve of osculating order 2 with *constant* κ_1. A *helix* of order r is a Frenet curve of osculating order r, such that all the curvatures $\kappa_1, \kappa_2, \cdots, \kappa_{r-1}$ are constant.

For Frenet curves in almost Hermitian manifolds, we recall the following notion:

Definition 3.2. Let $\gamma(s)$ be a Frenet curve of osculating order $r > 0$ in an almost Hermitian manifold (M, J, g). The *complex torsions* τ_{ij} $(1 \leq i < j \leq r)$ are smooth functions along γ defined by $\tau_{ij} = g(E_i, JE_j)$ (see [20]). A helix of order r in (M, J, g) is said to be a *holomorphic helix* of order r if all complex torsions are constant. In particular holomorphic helices of order 2 are called *holomorphic circles*.

For more informations on circles and helices in complex space forms, we refer to [1, 2, 3, 4, 20].

3.2. J-trajectory equation

Let $\gamma(u)$ be a smooth curve in an almost Hermitian manifold $M = (M, J, g)$. A curve $\gamma(u)$ is said to be a *J-trajectory* with strength q if it satisfies

$$\nabla_{\gamma'}\gamma' = qJ\gamma'$$

for some constant q. One can see that every J-trajectory has constant speed. Thus hereafter we parametrize J-trajectories by arc length parameter s.

Note that when M is an almost Kähler manifold, then its Kähler form Ω is referred as to the *Kähler magnetic field* on M. J-trajectories are called *Kähler magnetic trajectories* with respect to the Kähler magnetic field Ω.

Now we start our investigation on curvature properties of J-trajectories in an LCK manifold M with parallel anti Lee field.

First we observe that the first curvature κ_1 is constant $|q|$ by comparing the J-trajectory equation and the Frenet formula (3.1). The Frenet formula implies that the first normal vector field E_2 is given by $E_2 = \pm J\gamma'$. For simplicity of description, hereafter we choose $E_2 = J\gamma'$ and $q = \kappa_1 > 0$.

Remark 3.1. If a Frenet curve γ in an almost Hermitian manifold (M, J, g) is a J-trajectory, then

$$\tau_{12} = g(E_1, JE_2) = -1.$$

In case M is a Kähler manifold, then every J-trajectory is a holomorphic circle. For global behaviours of circles in complex projective space, we refer to an article [2] by Adachi, Maeda and Udagawa.

Now let $\gamma(s)$ be a non-geodesic J-trajectory in an LCK manifold with $E_2 = J\gamma'$. Then by using the formula (2.2) and Frenet equations, we have

$$2(\nabla_{\gamma'}J)\gamma' = 2\kappa_2 E_3 = \omega(E_2)E_1 - \omega(E_1)E_2 + A. \tag{3.2}$$

Note that (3.2) is rewritten as

$$2\kappa_2 E_3 = A - g(A, E_1)E_1 - g(A, E_2)E_2. \tag{3.3}$$

The equation (3.2) implies that when M is Kähler, $\kappa_2 = 0$. This conclusion is consistent with the fact "every J-trajectory of a Kähler manifold is a holomorphic circle" mentioned in Remark 3.1.

Moreover from (3.2) we notice that γ is of order 2 if and only if

$$A = g(A, E_1)E_1 + g(A, E_2)E_2.$$

Proposition 3.1. *Let γ be a non-geodesic J-trajectory with strength $q > 0$ parametrized by arc length in an LCK manifold M with parallel anti Lee field. Then the first and second curvatures of γ are given by*

$$\kappa_1 = q, \quad \kappa_2 = \frac{1}{2}\sqrt{|A|^2 - \omega(\gamma')^2 - \omega(J\gamma')^2} = \frac{1}{\sqrt{2}}\sqrt{(\nabla_{\gamma'}\omega)\gamma'}. \tag{3.4}$$

In particular κ_1 is constant.

- *The J-trajectory is a Frenet curve of osculating order 2 if and only if $(\nabla_{\gamma'}\omega)\gamma' = 0$.*
- *Assume that a J-trajectory is of order $r \geq 3$. If $\omega(\gamma') = 0$, then the second curvature κ_2 is constant.*

Proof. Under the parallelism of A, from (3.2) and (2.3), we have

$$4\kappa_2^2 = |A|^2 - \omega(E_1)^2 - \omega(E_2)^2 = |A|^2 - g(A, E_1)^2 - g(A, E_2)^2 = 2(\nabla_{\gamma'}\omega)\gamma'. \tag{3.5}$$

This formula shows that $\kappa_2 = 0$ if and only if $(\nabla_{\gamma'}\omega)\gamma' = 0$.

Next assume that γ is of order $r \geq 3$. By the parallelism of A, $|A|^2$ is constant. Assume that $\kappa_2 \neq 0$. Differentiating (3.5) by s, we get

$$\begin{aligned}
8\kappa_2\kappa_2' =& -\frac{d}{ds}\sum_{i=1}^{2} g(A, E_i)^2 = -2\sum_{i=1}^{2} g(A, E_i)g(A, \nabla_{\gamma'}E_i) \\
=& -2\left\{g(A, E_1)g(A, \kappa_1 E_2) + g(A, E_2)g(A, -\kappa_1 E_1 + \kappa_2 E_3)\right\} \\
=& -2g(A, E_2)g(A, \kappa_2 E_3) = -g(A, E_2)g(A, \omega(E_2)E_1 - \omega(E_1)E_2 + A) \\
=& -g(A, E_2)\left\{\omega(E_2)g(A, E_1) - \omega(E_1)g(A, E_2)\right\} \\
=& -g(A, E_2)\left\{|A|^2 - g(A, E_1)^2 - g(A, E_2)^2\right\} \\
=& -4\kappa_2^2\, g(A, E_2) = -4\omega(\gamma')\kappa_2^2.
\end{aligned}$$

Thus κ_2 is constant if and only if $\omega(\gamma') = 0$. □

Now we arrive at the main result of this paper.

Theorem 3.1. *Let γ be a non-geodesic J-trajectory with strength $q > 0$ parametrized by arc length in an LCK manifold M with parallel anti Lee field. Then the order of γ is at most 3. All the complex torsions of γ are constant.*

Proof. Assume that the order of γ is $r \geq 3$ and $\kappa_2 \neq 0$. Then differentiating the left hand side of the equation (3.3) along γ, we get

$$\nabla_{\gamma'}(2\kappa_2 E_3) = 2(\kappa_2' E_3 + \kappa_2 \nabla_{\gamma'} E_3) = 2\left(-\kappa_2^2 E_2 + \kappa_2' E_3 + \kappa_2\kappa_3 E_4\right).$$

Here we used the Frenet equations (3.1).

On the other hand, differentiating the right hand side of the equation (3.3) along γ, we get

$$\begin{aligned}
&\nabla_{\gamma'}(A - g(A, E_1)E_1 - g(A, E_2)E_2) \\
=&(g(A, E_2)\kappa_2 - g(A, E_1)')E_1 - (g(A, E_2)' + g(A, E_1)\kappa_1)E_2 - g(A, E_2)\kappa_2 E_3 \\
=&\kappa_2\omega(JE_3)E_2 + 2\kappa_2' E_3.
\end{aligned}$$

Here we used $\nabla A = 0$ and the formula $g(A, E_2) = \omega(\gamma')\kappa_2 = -2\kappa_2'$. Using (3.3) and (3.5), we get

$$\nabla_{\gamma'}(A - g(A, E_1)E_1 - g(A, E_2)E_2) = -2\kappa_2^2\, E_2 + 2\kappa_2' E_3.$$

Henceforth we obtain $\kappa_2\kappa_3 E_4 = 0$.

The complex torsions are computed as

$$\tau_{13} = g(E_1, JE_3) = -g(JE_1, E_3) = -g(E_2, E_3) = 0, \quad \tau_{23} = g(E_2, JE_3) = -g(JE_2, E_3) = g(E_1, E_3) = 0.$$

□

In our previous paper we proved that non-geodesic J-trajectories in a Vaisman manifold are Frenet curves of osculating order at most 4 [15].

4. J-trajectories in the product manifolds

In this section we study J-trajectories in the product manifold $M = N \times \mathbb{R}$ with β-Kenmotsu base manifold.

Let $\gamma(s) = (\overline{\gamma}(s), t(s))$ be a unit speed non-geodesic J-trajectory in $M = N \times \mathbb{R}$, where $\overline{\gamma}(s)$ is a curve in an almost contact metric manifold N. Then the unit tangent vector field $T(s) = \gamma'(s)$ is expressed as

$$T(s) = \gamma'(s) = \overline{\gamma}'(s) + t'(s)\frac{\partial}{\partial t}.$$

The arc length parametrization condition is

$$\bar{g}(\overline{\gamma}'(s), \overline{\gamma}'(s)) + t'(s)^2 = 1. \tag{4.1}$$

The acceleration vector field is computed as

$$\nabla_{\gamma'}\gamma' = \overline{\nabla}_{\overline{\gamma}'}\overline{\gamma}' + t''(s)\frac{\partial}{\partial t}.$$

On the other hand, we have

$$J\gamma' = \varphi\overline{\gamma}' - t'\xi + \eta(\overline{\gamma}')\frac{\partial}{\partial t}. \tag{4.2}$$

From these we deduce the following proposition [15]:

Proposition 4.1. *An arc length parametrized curve $\gamma(s)$ is a J-trajectory in $M = N \times \mathbb{R}$ with almost contact metric base manifold N if and only if it satisfies*

$$\overline{\nabla}_{\overline{\gamma}'}\overline{\gamma}' = q(\varphi\overline{\gamma}' - t'\xi), \quad t'' = q\eta(\overline{\gamma}'). \tag{4.3}$$

In particular, when $t(s) = \mathrm{constant}$, then the J-trajectory equation reduces to almost Legendre φ-trajectory equation for $\overline{\gamma}$ (see Appendix).

Now let us assume that the base manifold is a β-Kenmotsu manifold.

Proposition 4.2. *Let γ be a non-geodesic J-trajectory with strength $q \neq 0$ parametrized by arc length in $M = N \times \mathbb{R}$ with β-Kenmotsu base manifold N. Then γ has the curvatures*

$$\kappa_1 = |q|, \quad \kappa_2 = |\beta\varphi\overline{\gamma}'|.$$

Proof. From Proposition 3.1, we have $2\kappa_2^2 = (\nabla_{\gamma'}\omega)\gamma'$. Next since $\omega = 2\beta\eta$, we get

$$(\nabla_{\gamma'}\omega)\gamma' = 2\beta(\overline{\nabla}_{\overline{\gamma}'}\eta)\overline{\gamma}' = 2\beta^2(\bar{g}(\overline{\gamma}',\overline{\gamma}') - \eta(\overline{\gamma}')^2) = 2\beta^2\bar{g}(\varphi\overline{\gamma}'\varphi\overline{\gamma}').$$

□

Let $\theta(s)$ be the angle function between ξ and $\overline{\gamma}'$. Then we have $\eta(\overline{\gamma}') = |\overline{\gamma}'|\cos\theta$. The second curvature κ_2 is rewritten as

$$\kappa_2 = |\beta\sin\theta\,\overline{\gamma}'|$$

in terms of θ.

Here we compute the derivative of $\cos\theta$. We may consider the case $\cos\theta \neq 0$ and $\sin\theta \neq 0$. By using the formulas

$$\frac{d}{ds}\eta(\overline{\gamma}') = -qt', \quad \frac{d}{ds}|\overline{\gamma}'| = -\frac{qt'\eta(\overline{\gamma}')}{|\overline{\gamma}'|},$$

we get

$$\frac{d}{ds}\cos\theta = -\frac{qt'}{|\overline{\gamma}'|^3}(|\overline{\gamma}'|^2 - \eta(\overline{\gamma}')^2) = -\frac{qt'\sin^2\theta}{|\overline{\gamma}'|^2}.$$

Proposition 4.3. *Let $\gamma = (\overline{\gamma}, t)$ be a J-trajectory with strength $q > 0$ parametrized by arc length in $M = N \times \mathbb{R}$ with β-Kenmotsu base manifold N. Assume that $|\overline{\gamma}'| \neq 0$ and $\sin\theta \neq 0$. Then θ is constant if and only if t is constant. In such a case $\cos\theta = 0$.*

5. J-trajectories in $\mathbb{H}^3 \times \mathbb{R}$

5.1.

In this section we describe J-trajectories in the LCK manifold $M^4(\beta) = \mathbb{H}^3(-\beta^2) \times \mathbb{R}$. The LCK manifold $M^4(\beta)$ is realized as $\mathbb{R}^4(x, y, z, t)$ with metric

$$e^{2\beta z}(dx^2 + dy^2) + dz^2 + dt^2$$

and complex structure

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad Je_3 = e_4, \quad Je_4 = -e_3.$$

Here

$$e_1 = e^{-\beta z}\frac{\partial}{\partial x}, \quad e_2 = e^{-\beta z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}.$$

The left invariant orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ satisfies

$$\begin{gathered}\nabla_{e_1}e_1 = -\beta\, e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = \beta\, e_1, \quad \nabla_{e_1}e_4 = 0,\\ \nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -\beta\, e_3, \quad \nabla_{e_2}e_3 = \beta\, e_2, \quad \nabla_{e_2}e_4 = 0,\\ \nabla_{e_3}e_1 = \nabla_{e_3}e_2 = \nabla_{e_3}e_3 = \nabla_{e_3}e_4 = 0,\\ \nabla_{e_4}e_1 = \nabla_{e_4}e_2 = \nabla_{e_4}e_3 = \nabla_{e_4}e_4 = 0.\end{gathered} \tag{5.1}$$

Note that $M^4(\beta)$ is realized as a solvable Lie group

$$\left\{ \begin{pmatrix} e^{-\beta z} & 0 & x & 0 \\ 0 & e^{-\beta z} & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix} \;\middle|\; x, y, z, t \in \mathbb{R} \right\} \subset \mathrm{GL}_4\mathbb{R}$$

Moreover $M^4(\beta)$ is also regarded as a warped product

$$\mathbb{R}^2(z, t) \times_f \mathbb{R}^2(x, y)$$

with warping function $f(z, t) = \exp(\beta z)$. The Lee form is $\omega = 2\beta\, dt$. The corresponding Lee field and anti Lee fields are $B = 2\beta\xi$ and $A = 2\beta\partial_t$, respectively.

5.2.

Let $\gamma(s) = (\overline{\gamma}(s), t(s)) = (x(s), y(s), z(s), t(s))$ be a J-trajectory in $M^4(\beta)$. We express the unit tangent vector field $T(s) = \gamma'(s)$ as

$$T(s) = T_1(s)e_1 + T_2(s)e_2 + T_3(s)e_3 + T_4(s)e_4,$$

where

$$T_1(s) = e^{\beta z(s)}x'(s), \quad T_2(s) = e^{\beta z(s)}y'(s), \quad T_3(s) = z'(s), \quad T_4(s) = t'(s).$$

The unit speed condition is

$$T_1(s)^2 + T_2(s)^2 + T_3(s)^2 + T_4(s)^2 = e^{2\beta z(s)}(x'(s)^2 + y'(s)^2) + z'(s)^2 + t'(s)^2 = 1.$$

By using the table (5.1), J-trajectory equation is deduced as the system

$$T_1' + \beta T_3T_1 = -qT_2, \quad T_2' + \beta T_3T_2 = qT_1, \quad T_3' - \beta(T_1^2 + T_2^2) = -qT_4, \quad T_4' = qT_3. \tag{5.2}$$

The second curvature κ_2 is computed as

$$\kappa_2(s) = |\beta\varphi\overline{\gamma}'(s)| = |\beta|\sqrt{T_1(s)^2 + T_2(s)^2} = |\beta|\, e^{\beta z(s)}\sqrt{x'(s)^2 + y'(s)^2}. \tag{5.3}$$

Example 5.1 (Trajectories of order 2)**.** From (5.3) equation one can see that γ is of order 2 if and only if $x(s)$ and $y(s)$ are constant. Here we determine J-trajectories with $q > 0$ and $\kappa_2 = 0$. Such a J-trajectory is expressed as

$$\gamma(s) = (x_0, y_0, z(s), t(s))$$

for some constants x_0 and y_0. The system (5.2) is reduced to

$$T_3' = -qT_4, \quad T_4' = qT_3.$$

From this reduced system we get

$$(T_3(s), T_4(s)) = (R\cos(qs), -R\sin(qs))$$

for some nonzero constant R. Thus the J-trajectory $\gamma(s)$ satisfying the initial condition $\gamma(0) = (x_0, y_0, z_0, t_0)$ is

$$\left(x_0, y_0, z_0 + \frac{R}{q}\sin(qs), t_0 + \frac{R}{q}(\cos(qs) - 1)\right).$$

This is a circle in the totally geodesic flat plane $\mathbb{R}^2(z, t)$. Note that this circle is a holomorphic circle in $\mathbb{H}^3(-\beta^2) \times \mathbb{R}$. The contact angle θ of $\overline{\gamma}$ is a non-constant and given by $\theta(s) = qs$.

Example 5.2 (Helical J-trajectories)**.** Next we look for J-trajectories with $q > 0$ and constant $\kappa_2 > 0$. From Proposition 3.1, κ_2 is a positive constant if and only if $\omega(\gamma') = 0$. Note that $\overline{\gamma}$ is almost Legendre.

In the present case, we have $\omega(\gamma') = 2\beta T_3(s) = 2\beta z'(s)$. Thus $z(s) = z_0$. In this case the system (5.2) is reduced to

$$T_1' = -qT_2, \quad T_2' = qT_1, \quad \beta(T_1^2 + T_2^2) = qT_4, \quad T_4' = 0.$$

Thus the J-trajectory $\gamma(s)$ satisfying the initial condition $\gamma(0) = (x_0, y_0, z_0, t_0)$ is

$$\left(x_0 + \sqrt{\frac{qR}{\beta}}(\cos(qs) - 1), y_0 + \sqrt{\frac{qR}{\beta}}\sin(qs), z_0, t_0 + Rs\right),$$

where R is a non-zero constant such that $R/\beta > 0$. The J-trajectory is a holomorphic helix with $\kappa_1 = q$ and $\kappa_2 = \sqrt{q|R|}$.

5.3.

Let us return general situation. The case $T_1^2 + T_2^2 = 0$ was treated in Example 5.1, we may consider the case $T_1^2 + T_2^2 \neq 0$. From the first and second equations of (5.2), we get

$$T_1T_1' + T_2T_2' = -\beta T_3(T_1^2 + T_2^2).$$

The case $T_1^2 + T_2^2 = 0$ was treated in Example 5.1, we may consider the case $T_1^2 + T_2^2 \neq 0$. We get

$$\frac{(T_1^2 + T_2^2)'}{T_1^2 + T_2^2} = -2\beta T_3.$$

Integrating this equations, we have

$$\log(T_1^2 + T_2^2) = -2\beta z(s) + \text{constant}.$$

Thus we may write

$$T_1(s)^2 + T_2(s)^2 = r^2\, \exp\left(-2\beta z(s)\right)$$

for some positive constant r. This fact implies that

$$x'(s)^2 + y'(s)^2 = r^2\, \exp(-4\beta z(s)).$$

The functions T_1 and T_2 is expressed as

$$T_1(s) = r\exp(-\beta z(s))\cos\psi(s), \quad T_2(s) = r\exp(-\beta z(s))\sin\psi(s) \tag{5.4}$$

for some function $\psi(s)$.

If $\cos\psi = 0$, then $T_1 = 0$ and hence the J-trajectory equations (5.2) become

$$qT_2 = 0, \quad T_2' = -\beta T_3T_2, \quad T_3' - \beta(T_1^2 + T_2^2) = -qT_4, \quad T_4' = qT_3.$$

Thus q should be 0. Hence γ is a geodesic (see Appendix B).

Assume that $\cos\psi \neq 0$, then by using the first and second equations of (5.2), we have

$$\frac{1}{\cos^2\psi}\frac{d\psi}{ds} = \frac{d}{ds}\tan\psi = \frac{d}{ds}\left(\frac{T_2}{T_1}\right) = \frac{T_2'T_1 - T_2T_1'}{T_1^2} = \frac{q(T_1^2 + T_2^2)}{T_1^2} = q(1 + \tan^2\psi) = \frac{q}{\cos^2\psi}.$$

These computation imply that $\psi' = q$. Hence we obtain $\psi(s) = qs + \psi_0$ for some constant ψ_0.

Theorem 5.1. *Let $z(s)$ be a solution to the second order ordinary differential equation:*

$$\frac{d^2z}{ds^2} = -q^2z + \beta r^2 e^{-2\beta z} + C, \tag{5.5}$$

where C is a constant. Then the curves $(x(s), y(s), z(s), t(s))$ defined by

$$\begin{aligned} x(s) =& r\int \exp(-2\beta z(s))\, \cos(qs + \psi_0)\, ds, \\ y(s) =& r\int \exp(-2\beta z(s))\, \sin(qs + \psi_0)\, ds, \\ t(s) =& q\int z(s)\, ds + Cs + t_0 \end{aligned}$$

is a J-trajectory in $\mathbb{H}^3(-\beta^2) \times \mathbb{R}$.

Proof. The third equation
$$T_3' = \beta(T_1^2 + T_2^2) - qT_4,$$
of (5.2) is rewritten as
$$\frac{d^2z}{ds^2} = \beta(T_1^2 + T_2^2) - qT_4 = \beta r^2 e^{-2\beta z} - qT_4.$$
From the fourth equation of (5.2), we have
$$T_4(s) = qz(s) + C,$$
where C is a constant. Hence we obtain (5.5). □

Remark 5.1. In this section we investigate J-trajectories in the solvable Lie group $M^4(\beta) = \mathbb{H}^3 \times \mathbb{R}^3$. The solvable Lie group model (2.6) of the hyperbolic 3-space belong to the following two-parameter family of solvable Lie groups:
$$\mathrm{S}(\alpha,\beta) = \left\{ \left(\begin{array}{ccc} e^{-\alpha z} & 0 & x \\ 0 & e^{-\beta z} & y \\ 0 & 0 & 1 \end{array} \right) \middle| \; x, y, z \in \mathbb{R} \right\}.$$
In fact $\mathbb{H}^3(-\beta^2) = \mathrm{S}(\beta,\beta)$.

Solvmanifolds with LCK structures have been studied extensively. de Andrés, Cordero, Fernández, and Mencía [5] gave an interesting family of 4-dimensional solvable Lie groups equipped with LCK structure. Let $G(k,n)$ be the connected solvable Lie group consisting of matrices of the form
$$\left(\begin{array}{ccccc} e^{kz} & 0 & 0 & 0 & x \\ -nye^{kz} & 1 & 0 & 0 & t \\ 0 & 0 & e^{-kz} & 0 & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{array} \right),$$
where $k \in \mathbb{R}$ satisfies $\cosh k \in \mathbb{Z}^+ \setminus \{1\}$ and $n \in \mathbb{Z}$.

Take a left invariant one-forms
$$\vartheta^1 = dx - kxdz, \quad \vartheta^2 = dy + kydz, \quad \vartheta^3 = dz, \quad \vartheta^4 = dt + knxydz + nxdy.$$
Then $g = (\vartheta^1)^2 + (\vartheta^2)^2 + (\vartheta^3)^2 + (\vartheta^4)^2$ is a left invariant Riemannian metric on $G(k,n)$. Denote by $\{E_1, E_2, E_3, E_4\}$ the left invariant orthonormal frame field metrically dual to $\{\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4\}$. Then
$$JE_1 = \frac{n\lambda}{k}E_4, \quad JE_2 = E_3, \quad JE_3 = -\frac{k}{n\lambda}E_1, \quad JE_4 = -E_2.$$
is a left invariant g-orthogonal complex structure on $G(k,n)$. The resulting homogenous Hermitian manifold $G(k,n)$ is LCK if $n \neq 0$. Note that $G(k,n)$ is represented as a semi-direct product $H \ltimes_\rho \mathbb{R}^2$
$$H = \left\{ \left(\begin{array}{cc} e^{-kz} & y \\ 0 & 1 \end{array} \right) \middle| \, y, z \in \mathbb{R} \right\}$$
with abelian group $\mathbb{R}^2(x,t)$ via the representation
$$\rho \left(\begin{array}{cc} e^{-kz} & y \\ 0 & 1 \end{array} \right) = \left(\begin{array}{cc} e^{kz} & 0 \\ -nye^{kz} & 1 \end{array} \right).$$
The LCK solvable Lie group $G(k,n)$ admits a compact quotient $G(k,n)/\Gamma(k,n)$ [5] (see also [26, Theorem 2, Remark 3.1]). Moreover there exits a compact quotient $\mathrm{S}(-k,k))/\Gamma(k)$ so that $G(k,n)/\Gamma(k,n) \to \mathrm{S}(-k,k)/\Gamma(k)$ is a principal circle bundle (see [9, §2.4]).

Kamishima [16] proved that the compact quotient $M(k,1) = G(k,1)/\Gamma$ is holomorphically isometric to Inoue surface equipped with the LCK structure introduced by Tricerri [27]. Note that $M(k,1)$ is not Vaisman. Moreover $M(k,1)$ is holomorphically isometric to a compact quotient of Sol_1^4. Here Sol_1^4 is one of the model spaces of 4-dimensional geometries, see [10, 30]. Wall proved that Sol_1^4 admits no compatible Kähler structure [30, Theorem 1.2].

Kasuya proved that *Oeljeklaus-Toma manifolds* [21] (of type $(s,1)$) are solvmanifolds and have no Vaisman structures [18]. Note that Oeljeklaus-Toma manifolds of type $(1,1)$ are Inoue surfaces.

It would be interesting to study J-trajectories in LCK solvmanifolds, especially periodic J-trajectories in $G(k,n)/\Gamma(k,n)$.

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A. φ-trajectories on Kenmotsu manifolds

A.1.

Let N be an almost contact metric manifold with Levi-Civita connection $\overline{\nabla}$. A curve $\overline{\gamma}(u)$ in N is said to be a *φ-trajectory* if it satisfies

$$\overline{\nabla}_{\overline{\gamma}'}\overline{\gamma}' = \bar{q}\varphi\overline{\gamma}'$$

for some constant $\bar{q}$. In case the fundamental 2-form Φ is closed, then φ-trajectories are called *contact magnetic curves*. One can see that φ-trajectories have constant speed.

Let $\overline{\gamma}(s)$ be a constant speed φ-trajectory with strength q in a β-Kenmotsu manifold N. Denote by $\theta(s)$ the angle function of $\overline{\gamma}'(s)$ and ξ:

$$\cos\theta(s) = \frac{g(\overline{\gamma}'(s),\xi)}{|\overline{\gamma}'(s)|} = \frac{\eta(\overline{\gamma}'(s))}{|\overline{\gamma}'(s)|}$$

The angle function $\theta(s)$ is called the *contact angle* of $\overline{\gamma}$. A curve $\overline{\gamma}$ is said to be a *slant curve* if θ is constant [14]. In particular, curves with $\cos\theta = 0$ are called *almost Legendre curves* or *almost contact curves*.

Now let us investigate the contact angle of a constant speed φ-trajectory $\overline{\gamma}$. Denote the speed of $\overline{\gamma}$ by $a > 0$. Then we get

$$\begin{aligned} 0 =& \bar{g}(q\varphi\overline{\gamma}',\xi) = \bar{g}(\overline{\nabla}_{\overline{\gamma}'}\overline{\gamma}',\xi) = \eta(\overline{\gamma}')' - \bar{g}(\overline{\gamma}',\overline{\nabla}_{\overline{\gamma}'}\xi) = \eta(\overline{\gamma}')' - \beta\eta(\overline{\gamma}')^2 \\ =& (a\cos\theta)' - \beta\, a^2\, \cos^2\theta = -a\theta'(\sin\theta) - \beta a^2\cos^2\theta. \end{aligned}$$

Hence we obtain

$$(\sin\theta)\,\theta' = -a\beta\cos^2\theta.$$

From this equation we deduce the following fact.

Proposition A.1. *Let N be a β-Kenmotsu manifold. A constant speed curve φ-trajectory $\gamma(s)$ is a slant curve, then $\sin\theta = 0$ or $\cos\theta = 0$. In the former case, $q = 0$ and γ is a geodesic.*

Remark A.1. Pandey and Mohammad claimed that every φ-trajectory in a Kenmotsu manifold is slant. In addition they claimed the existence of helical φ-trajectories with constant contact angle so that $\sin\theta \neq 0$ and $\cos\theta \neq 0$ in [24]. In the proof of [24, Theorem 1], they used K-contact property. However Kenmotsu manifolds can not satisfy the *contact metric condition* $\Phi = d\eta$ since $d\eta = 0$.

A.2.

Now let us investigate φ-trajectories in the hyperbolic 3-space $\mathbb{H}^3(-\beta^2)$.

Let $\overline{\gamma}(s) = (x(s), y(s), z(s))$ be a *constant speed* curve in $\mathbb{H}^3(-\beta^2)$. Then we have

$$T(s) = \overline{\gamma}'(s) = x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} + z'(s)\frac{\partial}{\partial z} = e^{\beta z(s)}x'(s)e_1 + e^{\beta z(s)}y'(s)e_2 + z'(s)e_3.$$

We put

$$T_1(s) = e^{\beta z(s)}x'(s), \quad T_2(s) = e^{\beta z(s)}y'(s), \quad T_3(s) = z'(s).$$

By the constant speed condition, we may put

$$T_1(s)^2 + T_2(s)^2 + T_3(s)^2 = a^2$$

for some positive constant a.

Let $\theta(s)$ be a angle function of $\overline{\gamma}'(s)$ and ξ:

$$\cos\theta(s) = \frac{g(\overline{\gamma}'(s),\xi)}{|\overline{\gamma}'(s)|} = \frac{T_3(s)}{\sqrt{T_1(s)^2 + T_2(s)^2 + T_3(s)^2}} = \frac{z'(s)}{\sqrt{e^{2\beta z(s)}(x'(s)^2 + y'(s)^2) + z'(s)^2}}.$$

In case θ is constant, then $\overline{\gamma}$ is said to be a *slant curve* [14]. If $\overline{\gamma}$ is slant, then $z'(s) = a\cos\theta$. From this we get $z(s) = (a\cos\theta)s + z_0$. Next, by the constant speed condition, $T_1(s)^2 + T_2(s)^2 = a^2\sin^2\theta$. Thus $(T_1(s), T_2(s))$ is represented as $(a\sin\theta\cos\psi(s), a\sin\theta\sin\psi(s))$ for some function $\psi(s)$.

Proposition A.2 ([8]). *Any slant curve in $\mathbb{H}^3(-\beta^2)$ is represented as*

$$\begin{pmatrix} a\sin\theta\int_0^s \exp(-\beta((a\cos\theta)u+z_0)\cos\theta)\cos\psi(u)du+x_0 \\ a\sin\theta\int_0^s \exp(-\beta((a\cos\theta)u+z_0)\cos\theta)\sin\psi(u)du+y_0 \\ (a\cos\theta)s+z_0 \end{pmatrix}.$$

In particular, Legendre curves are parametrized as

$$\left(\pm a\int_0^s e^{-\beta z_0}\cos\psi(u)du+x_0,\ \pm a\int_0^s e^{-\beta z_0}\sin\psi(u)du+y_0,\ z_0\right).$$

Now let $\overline{\gamma}$ be a φ-trajectory. Note that $\overline{\gamma}$ has constant speed $a>0$.
The acceleration vector field is

$$\bar{\nabla}_{\overline{\gamma}'}\overline{\gamma}' = (T_1'+\beta T_3T_1)e_1+(T_2'+\beta T_3T_2)e_2+(T_3'-\beta T_1^2-\beta T_2^2)e_3.$$

The φ-trajectory equation is the system

$$T_1'+\beta T_3T_1=-qT_2,\quad T_2'+\beta T_3T_2=qT_1,\quad T_3'-\beta(T_1^2+T_2^2)=0.$$

If $\overline{\gamma}$ is a slant φ-trajectory, then $T_1^2+T_2^2=0$. Hence both $x(s)$ and $y(s)$ are constant. The velocity is $T=T_3e_3$. On the other hand we notice that $|T|=a=|T_3|=a|\cos\theta|$. Hence $\cos\theta=\pm1$.
Thus $\overline{\gamma}$ is a vertical geodesic parametrized as

$$(x_0, y_0, \pm as+z_0)$$

with velocity $\overline{\gamma}'=\pm a\xi$.

Proposition A.3. *The only slant φ-trajectories in $\mathbb{H}^3(-\beta^2)$ are vertical geodesics. In particular there are no almost Legendre φ-trajectories.*

B. Geodesics in $M^4(\beta)$

In this section we study geodesics in the LCK manifold $M^4(\beta)=\mathbb{H}^3(-\beta^2)\times\mathbb{R}$.
From (5.2), $\gamma(s)=(x(s),y(s),z(s),t(s))$ is a geodesic if and only if

$$T_1'+\beta T_3T_1=0,\quad T_2'+\beta T_3T_2=0,\quad T_3'-\beta(T_1^2+T_2^2)=0,\quad T_4'=0. \tag{B.1}$$

Remark B.1. Since $M^4(\beta)$ is represented by $M^4(\beta)=\mathbb{R}^2(z,t)\times_{\exp(\beta z)}\mathbb{R}^2(x,y)$ as a warped product, one can deduce the geodesic equations (B.1) by applying the following proposition [22, p. 208]:

Proposition B.1. *Let (B,g_B) and (F,g_F) be Riemannian manifolds. Take a warped product $M=B\times_f F$. Then a unit speed curve $\gamma(s)=(\gamma_B(s),\gamma_F(s))$ in M is a geodesic if and only if it satisfies*

$$\nabla^B_{\gamma_B'}\gamma_B' = f\,g_F(\gamma_F',\gamma_F')\,\mathrm{grad}_B\, f,\quad \nabla^F_{\gamma_F'}\gamma_F' = -\frac{2}{f(\gamma)}\frac{d}{ds}f(\gamma)\,\gamma_F'. \tag{B.2}$$

Here ∇^B and ∇^F are Levi-Civita connections of B and F, respectively. The function

$$f(\gamma)^4\,g_F(\gamma_F',\gamma_F') \tag{B.3}$$

is a conserved quantity of a geodesic γ.

First of all we notice that $t(s)=t_0+cs$ for some constant c because of the fourth equation of (B.1). Next, the conserved quantity (B.3) is $e^{4\beta z}(T_1^2+T_2^2)$. Put $r=e^{2\beta z}\sqrt{T_1^2+T_2^2}\geq 0$. Then T_1 and T_2 are expressed as

$$T_1(s)=r\exp(-2\beta z(s))\cos\psi(s),\quad T_2(s)=r\exp(-2\beta z(s))\sin\psi(s)$$

for some function $\psi(s)$. In other words, we retrieve (5.4).

Example B.1 (Horizontal line)**.** In case $T_1^2 + T_2^2 = 0$, *i.e.*, $r = 0$, we get

$$T_3' = 0, \quad T_3^2 + T_4^2 = 1.$$

Hence T_3 and T_4 are expressed as

$$T_3(s) = \cos\mu, \quad T_4 = \sin\mu$$

for some constant μ. Thus we obtain

$$\gamma(s) = (x_0, y_0, z_0 + (\cos\mu)s - 1, t_0 + (\sin\mu)s).$$

This is a line in the totally geodesic plane $\{(x_0, y_0, z, t) \mid z, t \in \mathbb{R}\}$. In other words, $\gamma(s)$ is a line horizontal with respect to the Riemannian submersion $\mathbb{R}^2(z,t) \times_{e^{\beta z}} \mathbb{R}^2(x,y) \to \mathbb{R}^2(z,t)$.

Next we study geodesics with $T_1^2 + T_2^2 \neq 0$. Then from the third equation of (B.1), we get

$$T_3'(s) = \beta r^2 \exp(-2\beta z(s)).$$

This equation is rewritten as

$$\frac{d^2 z}{ds^2} = \beta r^2 \, e^{-2\beta z}. \tag{B.4}$$

This ODE is a special one of the *Toda lattice* [13, 25]. The ODE (B.4) is also called 1*-dimensional Liouville equation* or *Bratu equation* [7, 11, 12].

Assume that $\cos\psi \neq 0$, then by using the first and second equations of (B.1), we have

$$\frac{1}{\cos^2\psi}\frac{d\psi}{ds} = \frac{d}{ds}\tan\psi = \frac{d}{ds}\left(\frac{T_2}{T_1}\right) = \frac{T_2' T_1 - T_2 T_1'}{T_1^2} = 0.$$

Hence $\psi(s)$ is a constant. In case $\cos\psi(s)$ is identically zero, then $\psi(s)$ is a constant $\pm\pi/2$. Hence we conclude that $\psi(s)$ is a constant and denote it by μ.

Proposition B.2. *Let $z(s)$ be a solution to the Toda lattice* (B.4) *then the curve $\gamma(s) = (x(s), y(s), z(s), t_0)$ defined by*

$$x(s) = r\cos\mu \int \exp(-2\beta z(s))\, ds, \quad y(s) = r\sin\mu \int \exp(-2\beta z(s))\, ds,$$

is a geodesic in $\mathbb{H}^3(-\beta^2) \times \mathbb{R}$.

For simplicity we choose $\beta = 4$ and $r = \sqrt{2}$, then the general solution to the Toda lattice (B.4) satisfying the initial condition $z(0) = z_0$ is given by [13, 25]:

$$z(s) = \frac{1}{2}\log\cosh(2e^{-2z_0 s}) + z_0.$$

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