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On Crossed Squares of Commutative Algebras

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Abstract
In this work, we show a categorical property for crossed squares of commutative algebras by defining a specific object in this category and then we give the construction of the pullback with this object.

Keywords: *Crossed module; Crossed square; Pullback.*

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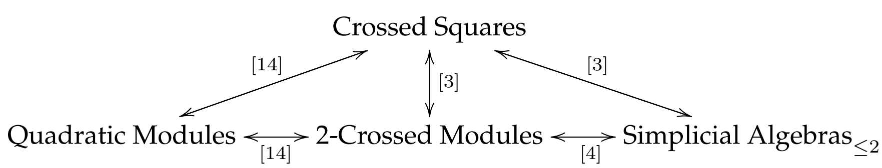
1. Introduction

Whitehead introduced crossed modules [1] as a homotopy 2-type connected space. In his work Whitehead shows that for a CW-complex

$$\partial : \pi_2(X, P, *) \rightarrow \pi_1(P, *)$$

is a crossed module. Later Porter adapts this notion for crossed modules of algebras [8]. The commutative algebra case for crossed modules can be found in the works of Gerstenhaber [9] and Lichtenbaum-Schlessinger [10].

Guin- Waléry and Loday defined crossed squares in [11] as an algebraic model for homotopy 3-type connected spaces. Thus crossed squares model homotopy types in dimensions bigger than 3. Later Ellis defined the commutative algebra version of crossed squares in [2]. 2- crossed modules and quadratic modules are also models for homotopy 3- types. Quadratic modules defined by Baues and 2-crossed modules defined by Conduché. The commutative algebra versions of quadratic modules and 2-crossed modules defined in [13], [14] respectively. Also Conduché shows that a mapping cone of a crossed square is a 2-crossed module [12]. The relations between the category of crossed squares and related categories such as simplicial groups, 2-crossed modules are given in [3], [4] and [13] as illustrated in the following diagram.



Categorical properties of these related categories such as pullback, limit and colimit are investigated in [5] [6], [7].The pullback construction is highly related with fibration. The fibration of 2-crossed modules is given in [16]. The category of pairs of crossed modules is mentioned in [7] to investigate the bifibration of crossed squares of groups. In section 3 we adapt this category for commutative algebras and construct a crossed squares from any pair of crossed modules. In last section, we construct the pullback of crossed squares over commutative algebras.

2. Preliminaries

In this section, we will recall the definitions of crossed modules and crossed squares. Now we give the definition of crossed modules of algebras from [15].

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Definition 2.1. Let C and R be two k-algebras and R acts on C. The morphism

$$\delta : C \to R$$

of k-algebras is called pre crossed module if

$$\delta(r \cdot c) = r\delta(c)$$

for all c in C and r in R. In addition to this condition if $\partial : C \to R$ satisfy

$$c \cdot \delta(c') = cc'$$

then $\partial : C \to R$ is called a crossed module and denoted with (C, R, δ).

Example 2.2. Let R be an algebra and I be an ideal of G. (I, R, inc) with $inc : I \hookrightarrow R$ is a crossed module. Where R acts on I by conjugation.

Example 2.3. Let M be a R- module. $(M, R, 0_M)$ is a crossed module with zero morphism $0_M : M \longrightarrow R$, $0_M(m) = e_R$.

Definition 2.4. A commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\beta} & C \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\sigma} \\ E & \xrightarrow[\delta]{} & R \end{array}$$

of commutative algebras with actions of the algebra R on C, D and E and h-map

$$h : C \times E \to D$$

is called *crossed square* [2] if for all $c, c' \in C$, $d \in D$ and $e, e' \in E$, $r \in R$, $k \in \mathbf{k}$:

1. The $\beta, \alpha, \sigma, \delta$ and $\sigma\beta = \delta\alpha$ are crossed modules;
2. β, α preserve the action of R;
3. $kh(c, e) = h(kc, e) = h(c, ke)$;
4. $h(c, e + e') = h(c, e) + h(c, e')$;
5. $h(c + c', e) = h(c, e) + h(c', e)$;
6. $r \cdot h(c, e) = h(r \cdot c, e) = h(c, r \cdot e)$;
7. $\beta h(c, e) = c \cdot e$;
8. $\alpha h(c, e) = e \cdot c$;
9. $h(c, \alpha(d)) = c \cdot d$;
10. $h(\beta(d), e) = e \cdot d$.

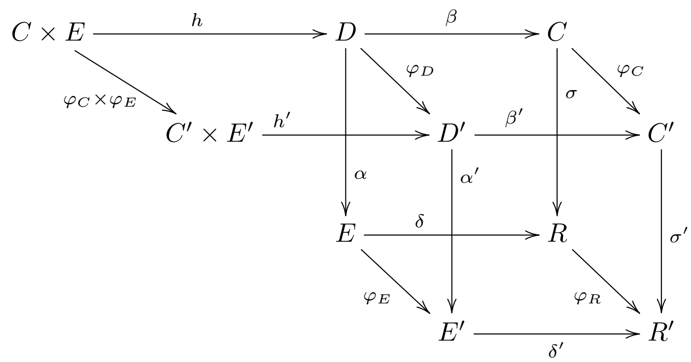
We will denote such a crossed square with $\begin{pmatrix} & C & \\ D & & R \\ & E & \end{pmatrix}$. Let

$$\varphi : \begin{pmatrix} & C & \\ D & & R \\ & E & \end{pmatrix} \to \begin{pmatrix} & C' & \\ D' & & R' \\ & E' & \end{pmatrix}$$

be a morphism of crossed squares. Then the morphisms

$$\begin{array}{llll} \varphi_D : & D \longrightarrow D' & \varphi_C : & C \longrightarrow C' \\ \varphi_E : & E \longrightarrow E' & \varphi_R : & R \longrightarrow R' \end{array}$$

are crossed module morphisms making the diagram



is commutative and the homomorphisms $\varphi_D, \varphi_C, \varphi_E$ are φ_R-equivariant. We will denote this category by $\mathbf{Crs}^2$.

Example 2.5. Let (C, R, ∂) be a crossed modules over commutative algebras and (C', R', ∂') be any ideal of (C, R, ∂). The diagram

$$\begin{array}{ccc} C' & \xrightarrow{\mu} & C \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\partial} \\ R' & \xrightarrow[v]{} & R \end{array}$$

with $h : C \times R' \to C'$ h-map given by $h(cr') = cr'$ for c in C and r' in R' is a crossed square.

Example 2.6. Let I and J be two ideals of a commutative ring R. The diagram

$$\begin{array}{ccc} J \cap I & \hookrightarrow & I \\ \downarrow & & \downarrow \\ J & \xrightarrow[v]{} & R \end{array}$$

is a crossed square with $h : J \times I \to J \cap I$ and h-map is given by $h(ji) = ji$.

3. Crossed Squares from Pairs of Crossed Modules

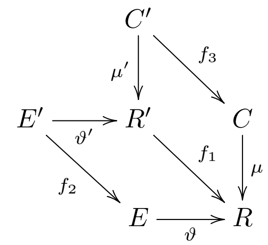
In [7] Brown and Sivera mentioned bifibration of crossed squares over the category of pairs of crossed modules in groups. In this section, we will give the definition of the category, pairs of crossed modules, for commutative algebras.

Definition 3.1. Let $\mu : C \to P$ and $\vartheta : E \to P$ be crossed modules of commutative algebras. The category, pairs of crossed modules, $XMod^2$ consists of objects

$$\begin{array}{ccc} & & S \\ & & \downarrow{\scriptstyle\mu} \\ E & \xrightarrow[\vartheta]{} & R \end{array}$$

and with the morphisms preserving the action of R on C and E. Shortly we will write $(C, E, R, \mu, \vartheta)$ for a pair of crossed modules.

Let $\begin{pmatrix} & C & \\ D & & R \\ & E & \end{pmatrix}$ be a crossed square and $f = (f_1, f_2, f_3) : (C', E', R', \mu', \vartheta') \to (C, E, R, \mu, \vartheta)$ be a morphism in $XMod^2$ as given by the following diagram.

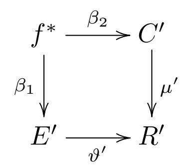


We define

$$f^* = \{(e', c', d) \in E' \times C' \times D : \vartheta'(e') = \mu'(c'), f_2(e') = \alpha(d), f_3(c') = \beta(d)\}$$

and $\beta_1(e', c', d) = c'$, $\beta_2(e', c', d) = e'$ to give the next proposition where $\sigma : D \to C$ and $\delta : D \to E$.

Proposition 3.2. *The diagram*



constructed with the data above is an object in Crs^2.

Proof. 1. From the definition μ' and ϑ' are crossed modules. First, let us show that β_1 is a crossed module.

$$\begin{aligned} \beta_1(c'' \cdot (e', c', d)) &= \beta_1(\mu'(c'') \cdot e', c'' \cdot c', f_3(c'') \cdot d) \\ &= c'' \cdot c' \\ &= c'' \cdot \beta_1(e', c', d) \end{aligned}$$

for $c'' \in C', (e', c', d) \in f^*$ and

$$\begin{aligned} (\beta_1(e', c', d)) \cdot (e', c'', d') &= c' \cdot (e'', c'', d') \\ &= (\mu'(c') \cdot e'', c' \cdot c'', f_3(c'') \cdot d) \\ &= (\vartheta'(e') \cdot e'', c'c', \sigma(d) \cdot d') \\ &= (e'e'', c'c'', dd') \\ &= (e', c', d)(e'', c'', d') \end{aligned}$$

for $(e', c', d), (e'', c'', d') \in f^*$. Similiar way β_2 is a crossed module. Since compositions of two crossed modules $\mu'\beta_1, \vartheta'\beta_2$ are crossed modules and from the definition of f^* it is clear that $\mu'\beta_1 = \vartheta'\beta_2$.

2.β_1 and β_2 preserves the action for $r' \in R'$ and $(e', c', d) \in f^*$

$$\begin{aligned} \beta_2(r' \cdot (e', c', d)) &= \beta_2(r' \cdot e', r' \cdot c', f_1(r') \cdot d) \\ &= r' \cdot e' \\ &= r' \cdot \beta_2(e', c', d) \end{aligned}$$

3. Define

$$\begin{aligned} h' : E' \times C' &\to E' \times C' \times D \\ (e', c') &\mapsto (\mu'(c') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c''))) \end{aligned}$$

where $h : E \times C \to D$ is the h-map of $\begin{pmatrix} & C & \\ D & & R \\ & E & \end{pmatrix}$. For $c' \in C'$, $e' \in E'$ and $k \in k$ we have

$$\begin{aligned} k \cdot h'(e', c') &= k(\mu'(c') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c''))) \\ &= (k \cdot \mu'(c') \cdot e', k.\vartheta'(e') \cdot c'', k \cdot h(f_2(e'), f_3'(c''))) \\ &= (\mu'(c') \cdot ke', \vartheta'(ke') \cdot c'', h(kf_2(e'), f_3'(c''))) \\ &= (\mu'(c') \cdot ke', \vartheta'(ke')c'', h(f_2(ke'), f_3'(c''))) \\ &= h'(ke', c') \end{aligned}$$

$$\begin{aligned} k \cdot h'(e', c') &= k \cdot (\mu'(c') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c''))) \\ &= (k \cdot \mu'(c') \cdot e', k \cdot \vartheta'(e') \cdot c'', k \cdot h(f_2(e'), f_3'(c''))) \\ &= (\mu'(kc') \cdot e', \vartheta'(e') \cdot kc'', h(f_2(e'), kf_3'(c''))) \\ &= (\mu'(kc') \cdot e', \vartheta'(e') \cdot kc'', h(f_2(e'), f_3'(kc''))) \\ &= h'(e', kc') \end{aligned}$$

4. For $c', c'' \in C'$ and $e' \in E'$;

$$\begin{aligned} h'(e', c' + c'') &= (\mu'(c' + c'') \cdot e', \vartheta'(e') \cdot (c' + c''), h(f_2(e'), f_3'(c' + c''))) \\ &= ((\mu'(c') \cdot e' + \mu'(c'') \cdot e', \vartheta'(e') \cdot c' + \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c') + f_3'(c''))) \\ &= (\mu'(c') \cdot e' + \mu'(c'') \cdot e', \mu'(c') \cdot e')) + \vartheta'(e''', h((f_2(e'), f_3'(c')), h(f_2)(e') f_3'(c'')) \\ &= (\mu'(c') \cdot e', \mu'(c') \cdot e', h(f_2(e'), f_3'(c'))) + (\mu'(c'') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e') f_3'(c''))) \\ &= h'(e', c') + h'(e', c'') \end{aligned}$$

5.For $c' \in C'$ and $e', e'' \in E'$;

$$\begin{aligned} h'(e' + e'', c') &= (\mu'(c') \cdot (e' + e''), \vartheta'(e' + e'') \cdot c', h(f_2(e' + e''), f_3'(c'))) \\ &= (\mu'(c') \cdot e' + \mu'(c') \cdot e'', (\vartheta'(e') + \vartheta'(e'')) \cdot c', h(f_2(e') + f_2(e''), , f_3'(c'))) \\ &= (\mu'(c') \cdot e' + \mu'(c') \cdot e'', \vartheta'(e') \cdot c' + \vartheta'(e'') \cdot c', h((f_2(e'), f_3'(c')) + h((f_2(e''), f_3'(c'))) \\ &= (\mu'(c') \cdot e', \vartheta'(e') \cdot c', h((f_2(e'), f_3'(c'))) + (\mu'(c') \cdot e'', \vartheta'(e'') \cdot c', h((f_2(e''), f_3'(c'))) \\ &= h'(e', c') + h'(e'', c') \end{aligned}$$

6. For $c' \in C', e', \in E'$ and $r' \in R'$;

$$\begin{aligned} r' \cdot h'(e', c') &= r' \cdot (\mu'(c') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c'))) \\ &= r' \cdot (\mu'(c') \cdot e'), r' \cdot (\vartheta'(e') \cdot c''), r' \cdot h(f_2(e'), f_3'(c')) \\ &= (r' \cdot \mu'(c')) \cdot e', (r' \cdot \vartheta'(e')) \cdot c'', h(r' \cdot f_2(e'), f_3'(c')) \\ &= (\mu'(c') \cdot r' \cdot e', \vartheta'(r'e') \cdot c'', h(f_2(r' \cdot e'), f_3'(c'))) \\ &= (\mu'(c') \cdot (r' \cdot e'), \vartheta'(r' \cdot e') \cdot c'', h(f_2(r' \cdot e'), f_3'(c'))) \\ &= h'(r' \cdot e', c') \end{aligned}$$

$$\begin{aligned} r' \cdot h'(e', c') &= r' \cdot (\mu'(c') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c'))) \\ &= r' \cdot (\mu'(c') \cdot e'), r' \cdot (\vartheta'(e') \cdot c''), r' \cdot h(f_2(e'), f_3'(c')) \\ &= (r' \cdot \mu'(c')) \cdot e', (r' \cdot \vartheta'(e')) \cdot c'', h(r' \cdot f_2(e'), f_3'(c')) \\ &= (\mu'(r'c') \cdot e', (\vartheta'(e') \cdot r') \cdot c', h(f_2(e'), f_3'(r'c'))) \\ &= (\mu'(r' \cdot c') \cdot e', \vartheta'(e') \cdot (r' \cdot c'), h(f_2(e'), f_3'(r' \cdot c'))) \\ &= h'(e', r' \cdot c') \end{aligned}$$

7. For $c' \in C'$, $e' \in E'$ and $r' \in R'$;

$$\begin{aligned} r' \cdot h'(e', c') &= r' \cdot (\mu'(c') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c'))) \\ &= r' \cdot (\mu'(c') \cdot e'), r' \cdot (\vartheta'(e') \cdot c''), r' \cdot h(f_2(e'), f_3'(c')) \\ &= (r' \cdot \mu'(c')) \cdot e', (r' \cdot \vartheta'(e')) \cdot c'', h(r' \cdot f_2(e'), f_3'(c')) \\ &= (\mu'(c') \cdot r' \cdot e', \vartheta'(r'e') \cdot c'', h(f_2(r' \cdot e'), f_3'(c'))) \\ &= (\mu'(c') \cdot (r' \cdot e'), \vartheta'(r' \cdot e') \cdot c'', h(f_2(r' \cdot e'), f_3'(c'))) \\ &= h'(r' \cdot e', c') \end{aligned}$$

$$\begin{aligned} r' \cdot h'(e', c') &= r' \cdot (\mu'(c') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c'))) \\ &= r' \cdot (\mu'(c') \cdot e'), r' \cdot (\vartheta'(e') \cdot c''), r' \cdot h(f_2(e'), f_3'(c')) \\ &= (r' \cdot \mu'(c')) \cdot e', (r' \cdot \vartheta'(e')) \cdot c'', h(r' \cdot f_2(e'), f_3'(c')) \\ &= (\mu'(r'c') \cdot e', (\vartheta'(e') \cdot r') \cdot c', h(f_2(e'), f_3'(r'c'))) \\ &= (\mu'(r' \cdot c') \cdot e', \vartheta'(e') \cdot (r' \cdot c'), h(f_2(e'), f_3'(r' \cdot c'))) \\ &= h'(e', r' \cdot c') \end{aligned}$$

8. For $c' \in C'$, $e' \in E'$ and $r' \in R'$;

$$\begin{aligned} r' \cdot h'(e', c') &= r' \cdot (\mu'(c') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c'))) \\ &= r' \cdot (\mu'(c') \cdot e'), r' \cdot (\vartheta'(e') \cdot c''), r' \cdot h(f_2(e'), f_3'(c')) \\ &= (r' \cdot \mu'(c')) \cdot e', (r' \cdot \vartheta'(e')) \cdot c'', h(r' \cdot f_2(e'), f_3'(c')) \\ &= (\mu'(c') \cdot r' \cdot e', \vartheta'(r'e') \cdot c'', h(f_2(r' \cdot e'), f_3'(c'))) \\ &= (\mu'(c') \cdot (r' \cdot e'), \vartheta'(r' \cdot e') \cdot c'', h(f_2(r' \cdot e'), f_3'(c'))) \\ &= h'(r' \cdot e', c') \end{aligned}$$

$$\begin{aligned}
r' \cdot h'(e', c') &= r' \cdot (\mu'(c') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c'))) \\
&= r' \cdot (\mu'(c') \cdot e'), r' \cdot (\vartheta'(e') \cdot c''), r' \cdot h(f_2(e'), f_3'(c')) \\
&= (r' \cdot \mu'(c')) \cdot e', (r' \cdot \vartheta'(e')) \cdot c'', h(r' \cdot f_2(e'), f_3'(c')) \\
&= (\mu'(r'c') \cdot e', (\vartheta'(e') \cdot r') \cdot c', h(f_2(e'), f_3'(r'c'))) \\
&= (\mu'(r' \cdot c') \cdot e', \vartheta'(e') \cdot (r' \cdot c'), h(f_2(e'), f_3'(r' \cdot c'))) \\
&= h'(e', r' \cdot c')
\end{aligned}$$

9. For $(e', c', d) \in f^*$ and $e'' \in E'$;

$$\begin{aligned}
h'(e', \beta_1(e', c', d)) &= h'(e'', c') \\
&= (\mu'(c') \cdot e'', \vartheta'(e'') \cdot c', h(f_2(e''), f'(c'))) \\
&= \vartheta'(e') \cdot e'', \vartheta'(e'') \cdot c', h(f_2(e''), \sigma(d)) \\
&= (e' \cdot e'', e'' \cdot c', f_2(e'') \cdot d) \\
&= e'' \cdot (e', c', d)
\end{aligned}$$

10. For $(e', c', d) \in f^*$ and $c'' \in C'$;

$$\begin{aligned}
h'(\beta_2(e', c', d), c'') &= h'(e', c') \\
&= (\mu'(c'') \cdot e', \vartheta'(e') \cdot c'', h(f_2(e'), f_3'(c''))) \\
&= (c'' \cdot e', \mu'(c') \cdot c'', h(\delta(d), f_3'(c''))) \\
&= (c'' \cdot e', c' \cdot c'', f_3(c'') \cdot d) \\
&= c'' \cdot (e', c', d)
\end{aligned}$$

□

Conclusion 3.3. There exist a functor

$$F : XMod^2 \to Crs^2$$

from the category of pairs of crossed modules to that of crossed squares.

4. Pullback in Crs^2

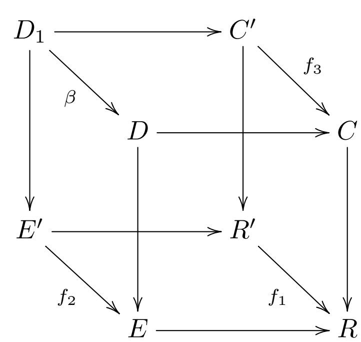
In this section, we will give the construction of a pullback object in Crs^2 using pairs of crossed modules.

Definition 4.1. Given a crossed square $\begin{pmatrix} & C & \\ D & & R \\ & E & \end{pmatrix}$ and a morphism of pairs of crossed modules $f = (f_1, f_2, f_3) : (C', E', R', \mu', \vartheta') \to (C, E, R, \mu, \vartheta)$ the pullback crossed square can be given by

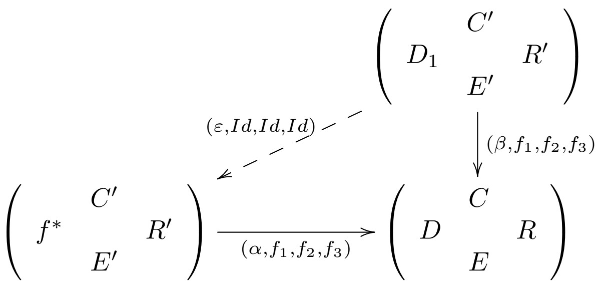
i) A crossed square $\begin{pmatrix} & C' & \\ f^* & & R' \\ & E' & \end{pmatrix}$ such that

$$f^* = \{(e', c', d) \in E' \times C' \times D : \vartheta'(e') = \mu'(c'), f_2(e') = \alpha(d), f_3(c') = \beta(d)\}$$

ii) Given any morphism of crossed squares



there is a unique crossed square morphism making the diagram



commutative.

Next we define a morphism

$$\begin{array}{rcl} \alpha : f^* & \to & D \\ (e', c', d) & \mapsto & d \end{array}$$

to give the following proposition.

Proposition 4.2. *(f^*, β_1, β_2) is the pullback object for the pair (μ', ϑ').*

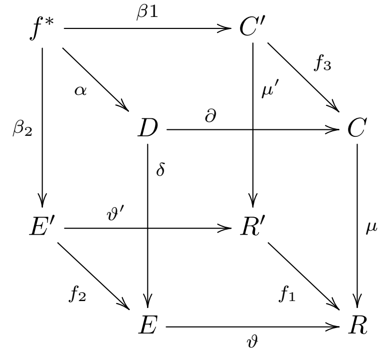
Proof. For $(e', c', d) \in E' \times C' \times D$;

$$\begin{array}{rcl} f_2\beta_2(e', c', d) & = & f_2(e') \\ & = & \delta(d) \\ & = & \delta\alpha(e', c', d) \\ & \Rightarrow & f_2\beta_2 = \delta\alpha \end{array}$$

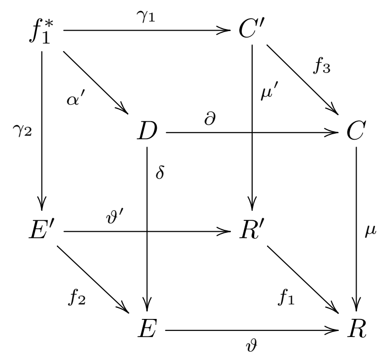
$$\begin{array}{rcl} \mu'\beta_1(e', c', d) & = & \mu'(c') \\ & = & \vartheta'(e') \\ & = & \vartheta'\beta_2(e', c', d) \\ & \Rightarrow & \mu'\beta_1 = \vartheta'\beta_2 \end{array}$$

$$\begin{array}{rcl} f_3\beta_1(e', c', d) & = & f_3(c') \\ & = & \partial(d) \\ & = & \partial\alpha(e', c', d) \\ & \Rightarrow & f_3\beta_1 = \partial\alpha \end{array}$$

Furthermore since $(f_1, f_2, f_3) \in Mor(XMod^2)$ we have $f_1\mu' = \mu f_3$, $\vartheta f_2 = f_1\vartheta'$ and $\left(\begin{array}{ccc} & C & \\ D & & R \\ & E & \end{array} \right)$ is a crossed square $\mu\partial = \vartheta\delta$ making the following diagram



commutative. Thus (α, f_1, f_2, f_3) is a morphism in Crs^2. Let $g = (\alpha', f_1, f_2, f_3)$ be another morphism in Crs^2



and an h-map

$$\begin{array}{rcl} h_1 : E' \times C' & \to & f_1^* \\ (e', c') & \mapsto & x \end{array}$$

be given. Define

$$\begin{array}{rcl} \xi : f_1^* & \to & f^* \\ x & \mapsto & (\gamma_1(x), \gamma_2(x), \alpha'(x)) \end{array}$$

Next we will show that $\xi : f_1^* \to f^*$ is well-defined. We know that for

$$(e', c', d) \in f^* \Leftrightarrow \vartheta'(e') = \mu'(c'), f_3(c') = \partial(d), f_2(e') = \delta(d)$$

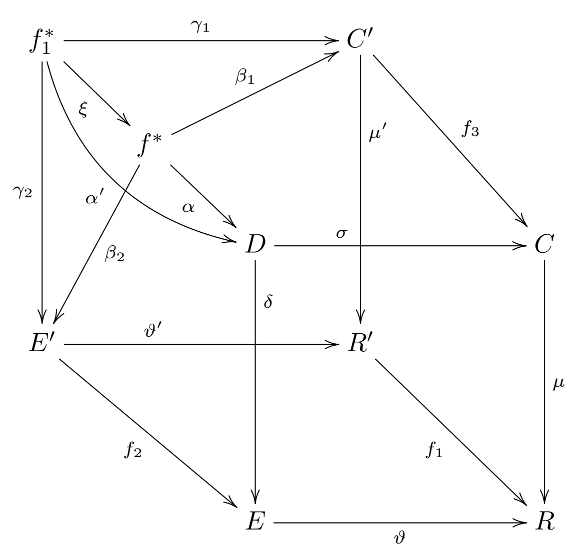
For $x \in f_1^*$

$$\begin{array}{rcccccl} f_3(\gamma_2(x)) & = & f_3\gamma_2(x) & = & \partial\alpha'(x) & = & \partial(\alpha'(x)) \\ f_2(\gamma_1(x)) & = & f_2\gamma_1(x) & = & \delta\alpha'(x) & = & \delta(\alpha'(x)) \end{array}$$

This shows that $(\gamma_1(x), \gamma_2(x), \alpha'(x)) \in f^*$. Then we get

$$\begin{array}{rcccl} \alpha\xi(x) & = & \alpha(\gamma_1(x), \gamma_2(x), \alpha'(x)) & = & \alpha'(x) \\ \beta_1\xi(x) & = & \beta_1(\gamma_1(x), \gamma_2(x), \alpha'(x)) & = & \gamma_1(x) \\ \beta_2\xi(x) & = & \beta_2(\gamma_1(x), \gamma_2(x), \alpha'(x)) & = & \gamma_2(x) \end{array}$$

That is we have the following diagram



Let

$$\begin{array}{rcl} \xi' : f_1^* & \to & f^* \\ x & \mapsto & (e', c', d) \end{array}$$

be another morphism satisfying

$$\beta_1 \xi' (x) = \gamma_1 (x) , \beta_2 \xi' (x) = \gamma_2 (x) , \alpha \xi' (x) = \alpha' (x) .$$

For $x \in f_1^*$ we have;

$$\begin{array}{lllll} \beta_1 \xi' (x) & = & \beta_1 (e', c', d) & = & c' & = \gamma_1 (x) \\ \beta_2 \xi' (x) & = & \beta_2 (e', c', d) & = & e' & = \gamma_2 (x) \\ \alpha \xi' (x) & = & \alpha (e', c', d) & = & d & = \alpha' (x) \end{array}$$

this implies $(\gamma_1 (x) , \gamma_2 (x) , \alpha' (x)) = (e', c', d)$. Since

$$\begin{array}{lll} \xi' (x) & = & (e', c', d) \\ & = & (\gamma_1 (x) , \gamma_2 (x) , \alpha' (x)) \\ & = & \xi (x) \\ & \Rightarrow & \xi' (x) = \xi (x) \\ & \Rightarrow & \xi' = \xi \end{array}$$

ξ is unique. As a result (f^*, β_1, β_2) is the pullback object for the pair (μ', ϑ'). □

Conclusion 4.3. Pullback object exists in Crs^2.

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