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İstatistikçiler Dergisi 4 (2011) 16-30

İstatistikçiler Dergisi

Fuzzy pricing of high excess loss layer when modeling the tail with generalized Pareto distribution

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Abstract

Good estimates for the tails of loss severity distributions are essentials for pricing or positioning high-excess loss layers (in reinsurance). Extreme value theory (EVT) provides a framework to formalize the study of behaviour in the tails of loss severity distributions. In EVT, the excess losses over a high threshold are modelled using generalized Pareto distribution (GPD). In any data analysis, there are various layers of uncertainty such as parameter and/or model uncertainty. These uncertainities are magnified in extreme value analysis. The aim of this study is to obtain fuzzy price for high excess loss layer when GPD provides good fitting to the tail of claim data. For this purpose, parameters of GPD are estimated using Buckley's approach.

Keywords: *Buckley' s approach; Extreme value theory; Fuzzy parameter estimation; Fuzzy pricing for high excess of loss layer; Generalized Pareto distribution.*

Özet

Kuyruk bölgesi genelleştirilmiş Pareto dağılımı ile modellendiğinde hasar fazlasının bulanık fiyatlandırması

Reasüransta, hasar şiddeti dağılımlarının kuyruk bölgesinin doğru tahmini fiyatlandırma ve hasar fazlasının belirlenmesinde önemlidir. Hasar şiddeti dağılımlarının kuyruk bölgesinin modellenmesinde uç değer teorisinden yararlanılmaktadır. Uç değer teorisinde, oldukça yüksek bir eşik değerini aşan hasarlar genelleştirilmiş Pareto dağılımı (GPD) kullanılarak modellenmektedir. Herhangi bir veri analizinde, parametre ve/veya model belirsizliği gibi çeşitli belirsizlikler sözkonusudur. Uç değer analizinde, bu belirsizlikler daha da artmaktadır. Bu çalışmanın amacı, hasar verisinin kuyruk bölgesine GPD'nın iyi uyum sağlaması durumunda hasar fazlasının bulanık fiyatlarının elde edilmesidir. Bu amaçla, Buckley' in yaklaşımı kullanarak GPD' nın parametreleri tahmin edilmiştir.

Anahtar sözcükler: *Buckley yaklaşımı; Bulanık parametre tahmini; Genelleştirilmiş Pareto dağılımı; Hasar fazlasının bulanık fiyatlandırması; Uç değer teorisi.*

1. Introduction

Determining the fair value of price is one of the most difficult and most important aspects of the insurance business. Price directly affects the demand for an insurance product. If premiums are set too low, it can quickly deprive of insurance company' s capital; or they are set too high, the insurers will lose their competitiveness and as a result their insureds.

In non-life insurance, although the probabilities of large claims are relatively low, they usually represent the greatest part of indemnities paid by the company and may put the company under financial difficulties. Therefore, modeling large losses has great importance for both insurer and reinsurer. Although a large number of small losses that do not result in liabilities for the reinsurer are not reported by the insurance company, there are a small number of large losses that are unlikely to occur but once they occur they might cause substantial losses to the reinsurer.

In the case of pricing high excess of loss layers, the reinsurer's concern lies in those rare events that might cause very large losses to the primary insurer and that therefore are likely to affect the reinsurer. Also, it is of interest to have good explanatory models for those large losses in order to calculate premiums that are neither too low nor too high.

Hence, a real understanding of statistical modeling for extreme events became of great interest among pricing staff and underwriters in many insurance and reinsurance companies.

If a reinsurer is using historical information to find statistical models for pricing high excess of loss layers, then he/she would be more interested in good models for the largest losses [23]. Extreme value theory (EVT) provides solid fundamentals for the statistical modeling of such events.

Beirlants and Teugels [4] and Embrecht and Klüppelberg[14] have argued that EVT motivates a number of sensible approaches to this problem. In particular, the peak over thresholds has been advocated in this context (Rootzen and Tajvidi [27], McNeil [24]). For many different underlying distributions, excess losses over high thresholds are modeled using generalized Pareto distribution (GPD). Embrechts et al. [15] give detailed summary and analytical proofs of all the properties of extreme value distributions including GPD. In the meantime, Hosking and Wallis [19] study approximate confidence intervals of the parameters and quantile estimation for GPD.

There is uncertainty while pricing insurance contracts. The insurer wants to be sure that adequate premiums are charged for any line of business. However, when it comes to extreme events or observations, the insurer/reinsurer needs to make inferences about possible occurrences outside the observations where there is very little information, therefore the uncertainty of any estimator is even higher.

While determining prices, one must take into account different levels of uncertainty such as parameter, model and data uncertainties.

Fuzzy logic proposed by Zadeh [33] enables us to deal with more general sources of uncertainty in both empirical data and/or models for data analysis.

Fuzzy set theory which introduces sets of objects whose boundaries are not sharply defined aims at modeling imprecise information which exists in real world situations. Fuzzy set theory could provide decision procedures that are much more flexible than those originating from conventional set theory [22].

Derrig and Ostaszewski [11] pointed out that deregulation and global competition of the last two decades have opened the door for new methodologies among them being fuzzy methods and more progress in this area would be seen in the view of increasing competitiveness and globalization of the insurance industry.

In the actuarial field, Fuzzy Set Theory is used to model problems that require a great deal of actuarial subjective judgment and problems for which the information available is scarce or vague.

The earliest work directly applying fuzzy set theory to actuarial science was by DeWit [12] where he pointed out that the process of insurance underwriting is indeed fraught with uncertainty which may not be properly described by probability. It was subsequently applied to insurance claim cost forecasting by Cummins and Derrig [7], to ratemaking and risk classification by Verrall and Yakoubov [31], Derrig and Ostaszewski [9], Young [32], Ebanks et al. [13], to underwriting and reinsurance decisions by DeWit [12], Lemaire [22] and Horgby et al. [18] and to pricing by Lemaire [22], Karkowski and Ostaszewski [20], Berliner and Buehlmann [5], Babad and Berliner [2], Terceno et al. [30], Cummins and Derrig [8], Derrig and Ostaszewski [10] and Simonelli [29], Young [32]. A potential range of actuarial applications is discussed in Ostaszewski [25].

GPD providing good estimates for the tails of loss severity distributions is very important for pricing in reinsurance. Using traditional pricing method, actuary calculates single premium. However, this premium usually can not be charged due to competitiveness and globalization in insurance market. For the management it is important to know how much more or less than this single premium can be charged. Fuzzy price may be very valuable tool in assessing the range of price to be considered as well as in informing the management about uncertainty of the price calculation process.

As the aim of this study is to obtain fuzzy price for high excess loss layer when GPD provides good fitting to the tail of loss severity data, parameters of GPD are estimated using Buckley' s approach which is based on a set of confidence interval to produce a triangular shaped fuzzy number.

The structure of the paper is as follows. In the next section, we briefly give information about fuzzy sets and numbers. Section 3 includes the estimation of the parameters using Buckley's approach. In Section 4, we present information about EVT. Fuzzy parameter estimation of GPD is given in Section 5. Section 6 includes pricing high excess loss layer using traditional and fuzzy respectively. Numerical example is provided in Section 7. Finally, Section 8 concludes the paper.

2. Fuzzy sets and fuzzy numbers

Zadeh [33] introduced for the first time sets of objects whose boundaries are not sharply defined in the paper titled as "Fuzzy Sets" which gave rise to enormous interest among researchers and initiated fuzzy set theory. Fuzzy set theory aims at modeling imprecise information which exists in real world situations. As many practical problems are extremely complex and ill-designed, it is difficult to model with precision.

There is a clear distinction between fuzzy sets and probability theory. Although probability concepts are derived from considerations about the uncertainty of propositions about the real world, fuzzy concepts are closely related to the multi-valued logic treatments of issues of imprecision in the definition of entities. Therefore, if there is a problem to be modeled which has some inherent imprecision, fuzzy set theory provides a better framework than probability theory. Dealing with problems in which some of the principal sources of uncertainty are non-statistical in nature, the effectiveness of probability theory is limited.

A class of objects whose boundaries are not sharply defined is called as a fuzzy set. If $X = \{x\}$ denote a collection of objects, a fuzzy set A in X is a set of ordered pairs

$$A = \{x, \mu_A(x)\}, \quad x \in X$$

where $\mu_A(x)$ is the grade of membership of x in A, $\mu_A(x): X \to M$ is a function from X to membership space M and produces values in [0, 1] for all x. Hence the degree of membership of x in A is represented by $\mu_A(x)$ which is a function having values between 0 and 1 representing respectively the lowest and highest grade of membership [22].

A fuzzy number which is the main instrument used in Fuzzy Set Theory for quantifying uncertain quantities is a fuzzy subset of the real line whose highest membership values are clustered around a given real number. The membership function is monotonic on both sides of this real number. Depending on the situation, there are different classes of membership functions such as triangular, trapezoidal, Gaussian and generalized bell shape as given in Figure 1 [28].

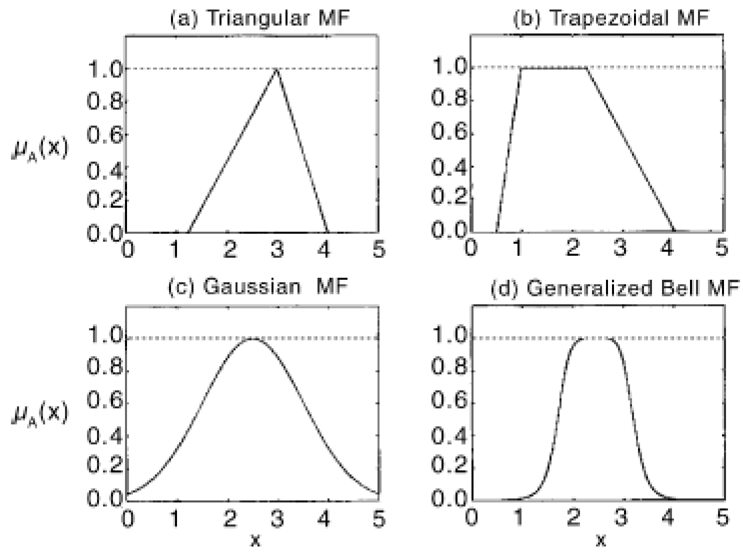

Figure 1. Examples of membership functions.

In practice, triangular fuzzy numbers are preferred as they are easy to handle arithmetically and can be interpreted intuitively. A triangular fuzzy number $\tilde{A}$ can be symbolized as $\tilde{A}=(a,l_a,r_a)$ where a is the centre and l_a and r_a are the left and right spreads, respectively as seen in Figure 2.

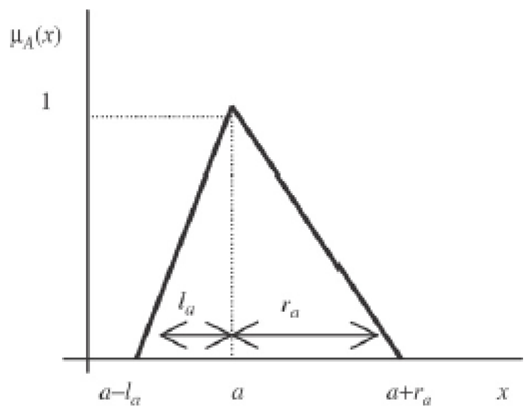

Figure 2. A triangular fuzzy number $\tilde{A}=(a,l_a,r_a)$.

The membership function $\mu_{\tilde{A}}(x)$ which characterizes a triangular fuzzy number is:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a+l_a}{l_a} & a-l_a < x \le a \\ \dfrac{a+r_a-x}{r_a} & a < x \le a+r_a \\ 0 & otherwise \end{cases}.$$

In addition to membership function, α -cuts (A_α), slices through a fuzzy set producing regular (non-fuzzy) sets, also characterize a triangular fuzzy number:

$$A_\alpha = [\underline{A}(\alpha), \overline{A}(\alpha)] = [a - l_a(1-\alpha), a - r_a(1-\alpha)].$$

As seen in Figure 2, from a statistical point of view, it can be said that, if $\tilde{A}$ is a random variable, a is its mode and $\mu_{\tilde{A}}(x)$ plays a similar role to the density function. Also, A_α may have a similar meaning to a confidence interval [1].

3. Parameter estimation using Buckley' s approach

Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with probability density (or probability mass) function $f(x;\theta)$ for single parameter θ, with observed value $x_1, x_2, ..., x_n$. Assume that θ is unknown parameter and it must be estimated. Let $Y = u(X_1, ..., X_n)$ be a statistics used to estimate θ. Given the values of these random variables $X_i = x_i, 1 \le i \le n$, we obtain a point estimate $\hat{\theta} = y = \mathrm{u}(\mathrm{x}_1, \ \ldots, \ \mathrm{x}_n)$ for θ. Since we never expect this point estimate to exactly equal to θ, we often also compute a $(1-\beta)100\%$ confidence interval for θ.

Denote a $(1-\beta)100\%$ confidence interval for θ by $[\theta_1(\beta), \theta_2(\beta)]$ for $0.01 \le \beta \le 1$. Add to this interval $[\hat{\theta}, \hat{\theta}]$ for the 0% confidence interval for θ. As seen in Figure 3, placing these confidence intervals one on top of the other, one can obtain a triangular shaped fuzzy number θ whose α -cuts are the following confidence intervals:

$$\tilde{\theta}[\alpha] = [\theta_1(\alpha), \theta_2(\alpha)] \quad \text{for } 0,01 < \alpha < 1.$$

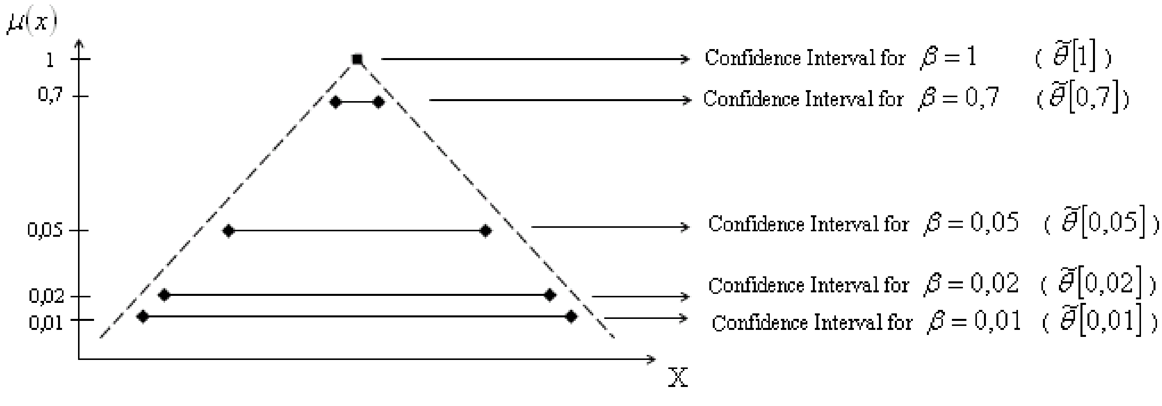

Figure 3. Fuzzy parameter estimation for θ.

Hence, more information about θ is used rather than just a point estimate, or just a single interval estimate. It is easy to generalize Buckley's method in the case where θ is a vector of parameters [6].

4. Extreme value theory

EVT provides a framework to formalize the study of behavior in the tails of a distribution. When modeling the maxima of a random variable, EVT plays the same fundamental role as the Central Limit theorem plays when modeling sums of random variables. In both cases, the theory tells us what the limiting distributions are.

To identify the extremes in real data, there are generally two approaches: block maxima method and peaks over threshold method. Although the block maxima method, a traditional method, is used to analyze data with seasonality, the threshold method is widely used in recent application as it uses data more efficiently [17].

In the following subsections, the fundamental theoretical results of these two methods are given.

4.1. Distribution of maxima

Suppose $X_1, X_2, ..., X_n$ are independent and identically distributed (i.i.d.) random variables from an unknown distribution F. Let us denote the maximum of the first n observations by $M_n = max(X_1, ..., X_n)$ and suppose further that the sequences of real numbers $a_n > 0$ and b_n can be found. The sequence of normalized maxima $(M_n - b_n)/a_n$ converges in distribution:

$$P\{(M_n - b_n)/a_n \le x\} = F^n(a_n x + b_n) \to H(x), \text{ as } n \to \infty \quad (1)$$

for some non-degenerate distribution function *H(x)*. If this condition holds it can be said that *F* is in the maximum domain of attraction of *H* and be written as $F \in MDA(H)$.

Fisher and Tippett [16] showed that

$F \in MDA(H) \Rightarrow H$ is of the type H_ξ for some ξ.

Thus, if suitably normalized maxima converge in distribution, then the limit distribution must be an extreme value distribution for some value of the shape parameter ξ, location parameter μ and scale parameter σ [24].

In the mean time, if the condition given in Equation (1) holds, then H belongs to one of the three standard extreme value distributions:

Frechet: $\Phi_\alpha(x) = \begin{cases} 0 & , x \le 0 \\ e^{-x^{-\alpha}} & , x > 0 \end{cases}$ where $\alpha > 0$

Weibull: $\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & , x \le 0 \\ 1 & , x > 0 \end{cases}$ where $\alpha > 0$.

Gumbel: $\Lambda(x) = e^{-e^{-x}}$ where $x \in R$

As Frechet distribution has a polynomially decaying tail, it fits well to heavy tail distributions. The class of distributions where the tail decays like a power function includes the Pareto, Burr, Loggamma, Cauchy and t-distributions as well as various mixture models. Gumbel distribution which has exponentially decaying tail suits well to thin tail distribution. Normal, Exponential, Gamma and Lognormal distributions are distributions in the maximum domain of attraction of the Gumbel. Weibull distribution is the asymptotic distribution of the finite endpoint distributions. Distributions in the maximum domain of attraction of Weibull are Uniform and Beta distributions.

Figure 4 involves the probability density functions for the standard Frechet,Weibull and Gumbel distributions.

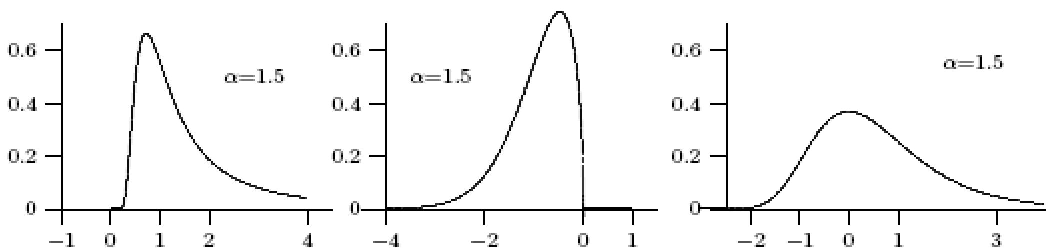

Figure 4. Densities for the Frechet, Weibull and Gumbel functions.

One-parameter representation of these standard extreme value distributions known as the Generalized Extreme Value (GEV) distribution is as follows:

$$H_\xi(x) = \begin{cases} e^{-(1+\xi x)^{-1/\xi}} & if\ \xi \neq 0 \\ e^{-e^{-x}} & if\ \xi = 0 \end{cases} \qquad (2)$$

where x is such that $1+\xi x > 0$ and ξ is shape parameter or tail index.

GEV distribution is obtained by setting $\xi = \alpha^{-1}$ for the Frechet distribution ($\xi > 0$), $\xi = -\alpha^{-1}$ for the Weibull distribution ($\xi < 0$) and by interpreting the Gumbel distribution as the limit case for $\xi = 0$ [17].

4.2. Distribution of exceedances

Suppose we have a sequence of i.i.d. observations $X_1, X_2, ..., X_n$ from an unknown distribution function F an example of which is given in Figure 5 and are interested in estimating the distribution function F_u of values of x above a certain threshold u.

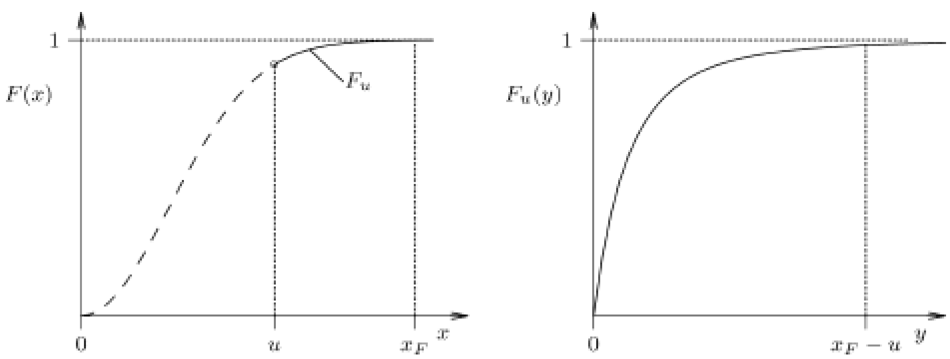

Figure 5. Distribution function *F* and conditional distribution function F_u .

The distribution function F_u is called the *conditional excess distribution function* and is defined as

$$F_u(y) = P(X - u \le y \mid X > u), \quad 0 \le y \le x_F - u \tag{3}$$

where $y = x - u$ are the excesses and $x_F \le \infty$ is the right endpoint of *F*. F_u can also be written in terms of *F* as follows:

$$F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)} = \frac{F(x) - F(u)}{1 - F(u)}. \tag{4}$$

Balkema and de Haan [3] and Pickands [26] show that conditional excess distribution function for a large u is GPD, the limiting distribution of the excesses, under MDA condition. That is, when $u \to \infty$

$$F_u(y) \approx G_{\xi,\sigma}(y)$$

where $G_{\xi,\sigma}(y)$ is GPD whose distribution function is:

$$G_{\xi,\sigma}(y) = \begin{cases} 1 - \left(1 + \dfrac{\xi}{\sigma} y\right)^{-1/\xi} & \text{if } \xi \ne 0 \\ 1 - e^{-y/\sigma} & \text{if } \xi = 0 \end{cases}$$

for $y \in [0, (x_F - u)]$ if $\xi \ge 0$ and $y \in [0, -\sigma/\xi]$ if $\xi < 0$.

GPD can also be expressed as a function of *x* as follows:

$$G_{\xi,\sigma} = 1 - (1 + \xi(x - u)/\sigma)^{-1/\xi}$$

where $x = u + y$.

Figure 6 illustrates the shape of $G_{\xi,\sigma}$ when ξ takes a negative, a positive and a zero value when the scaling parameter σ is equal to one [17].

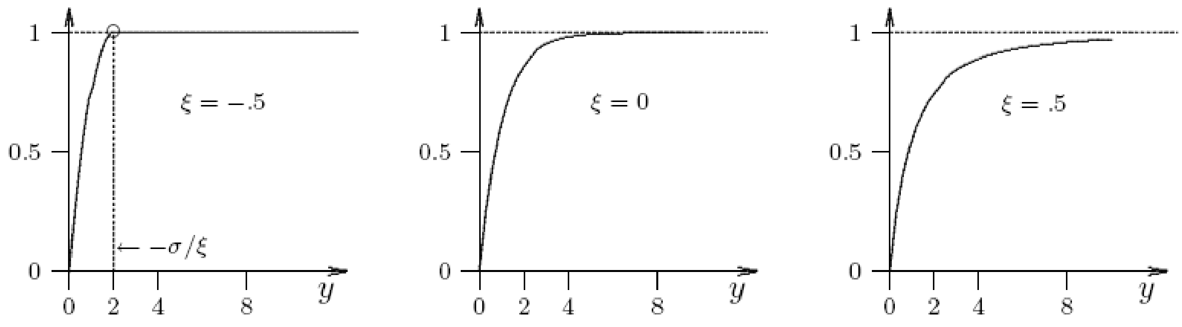

Figure 6. Shape of the generalized Pareto distribution $G_{\xi,\sigma}(x)$ for σ = 1.

5. Fuzzy parameter estimation of GPD

As mentioned in Section 3, Buckley introduced a method of estimation of a parameter which uses a set of confidence intervals to produce a triangular shaped fuzzy number for the estimator. In other words, Buckley estimates the parameters of a statistical model using set of confidence intervals. Therefore, the confidence intervals of the parameters of GPD should be constructed in order to obtain parameter estimation of GPD using Buckley's approach.

There are varieties of methods including the maximum likelihood method, the method of moment and the method of probability weighted moments to fit GPD to data on the excesses of high threshold. Parameter estimators of GPD obtained using these methods are asymptotically normally distributed for some values of the shape parameter.

If $h(\hat{\theta})$ is asymptotically normally distributed with $n \operatorname{var} h(\hat{\theta}) \sim v(\theta)$ as $n \to \infty$, then the confidence interval for $h(\theta)$ with asymptotic confidence level β is

$$P\,[h(\hat{\theta}) - \{n^{-1}v(\hat{\theta})\}^{1/2} z_{(1-\beta)/2} \le h(\theta) \le h(\hat{\theta}) + \{n^{-1}v(\hat{\theta})\}^{1/2} z_{(1-\beta)/2}] = 1-\beta \tag{5}$$

where z_β is the β^{th} quantile of the standard normal distribution [19].

As maximum likelihood estimation of the parameters of GPD is used in our application, asymptotic behavior of maximum likelihood estimators and their confidence intervals are given here.

For $\xi > -0.5$, the maximum likelihood estimates $(\hat{\xi}_{N_u}, \hat{\sigma}_{N_u})$ based on sample of N_u excesses of high threshold u are asymptotically normally distributed [24]:

$$N_u^{1/2}\begin{pmatrix}\hat{\xi}_{N_u}\\ \hat{\sigma}_{N_u}\end{pmatrix} \xrightarrow{d} N\left[\begin{pmatrix}\xi\\ \sigma\end{pmatrix}, \begin{pmatrix}(1+\xi)^2 & \sigma(1+\xi)\\ \sigma(1+\xi) & 2\sigma^2(1+\xi)\end{pmatrix}\right], \text{ as } N_u \to \infty \tag{6}$$

From Equations (5) and (6), the confidence interval for the shape parameter ξ with asymptotic confidence level β is:

$$P\left\{\hat{\xi} - \left[n^{-1}\left(\frac{(1+\xi)^2}{n}\right)\right]^{1/2} z_{(1-\beta)/2} \le \xi \le \hat{\xi} + \left[n^{-1}\left(\frac{(1+\xi)^2}{n}\right)\right]^{1/2} z_{(1-\beta)/2}\right\} = 1-\beta \tag{7}$$

and the confidence interval for the scale parameter σ with asymptotic confidence level β is:

$$P\left\{\hat{\sigma} - \left[n^{-1}\left(\frac{2\sigma^2(1+\xi)}{n}\right)\right]^{1/2} z_{(1-\beta)/2} \le \sigma \le \hat{\sigma} - \left[n^{-1}\left(\frac{2\sigma^2(1+\xi)}{n}\right)\right]^{1/2} z_{(1-\beta)/2}\right\} = 1-\beta \tag{8}$$

The estimators of the parameters of GPD as a triangular shaped fuzzy numbers can be obtained by placing the confidence intervals one on the top of the other for $0.01 \le \beta \le 1$ in Equations (7) and (8).

6. Pricing high excess loss layer

Pricing an insurance product is one of the most difficult and most important aspect of the insurance business. In many real-world situations, the single (crisp) premium calculated by the actuary is not the price that is eventually charged. The buyer may respond with a counteroffer; the insurer's management may decide to charge more or less than the actuary's recommendation; and so on. The traditional crisp premium conveys no information that is helpful in deciding how much more or less than the crisp premium it is ‘‘reasonable’’ to charge. Fuzzy premiums, on the other hand, provide an analogue to the crisp premium in the form of the fuzzy mean value but also provide information on the strength of membership in the fuzzy set of ‘‘acceptable’’ or nearby premiums [8].

Fuzzy price may be a very valuable tool in assessing the range of price to be considered as well as in informing the management about uncertainty of the price calculation process.

GPD providing good estimates for the tails of loss severity distributions is very important for pricing in reinsurance. As traditional pricing methods have not been accounted the degree of uncertainty and also uncertainty is always present in pricing, in this study a reasonable interval for price to be charged from the insureds is obtained by estimating the parameters of GPD using fuzzy approach.

In the following subsections, traditional pricing for high excess loss layer is given and this method is converted into fuzzy pricing by estimating the parameters of GPD using Buckley's approach.

6.1. Traditional pricing

Traditional price for a general layer (r,R) is:

$$P = \int_r^R (x-r) f_{X^\delta}(x)dx + (R-r)(1-F_{X^\delta}(R)) \tag{9}$$

where $f_{X^\delta}(x) = dF_{X^\delta}(x)/dx$ denotes the density function for the losses truncated at δ. We can estimate $F_{X^\delta}(x)$ for x > u using the tail estimation procedure and fitting GPD to the excesses by choosing a high threshold u<r. The estimate of $F_{X^\delta}(x)$ is:

$$\hat{F}_{X^\delta}(x) = (1-F_n(u))G_{\hat{\xi},\mu,\hat{\sigma}}(x) + F_n(u) \tag{10}$$

where $\hat{\xi}$ and $\hat{\sigma}$ are maximum-likelihood estimators and $F_n(u)$ is an estimator of $P\{X^\delta \le u\}$ based on the empirical distribution function of the data [24].

6.2. Fuzzy pricing

As the parameters of GPD are estimated as triangular fuzzy numbers, fuzzy prices for high excess loss layer can be obtained using integration of fuzzifying function over a non fuzzy interval.

In non-fuzzy interval $[a,b] \in R$, let the fuzzifying function have fuzzy value $\overline{f}(x)$ for x $\in$[a,b]. Integration $\overline{I}(a,b)$ of the fuzzifying function in [a,b] is defined as follows:

$$\overline{I}(a,b) = \left\{ \left(\int_a^b f_\alpha^-(x)dx + \int_a^b f_\alpha^+(x)dx, \alpha \right) \mid \alpha \in [0,1] \right\}$$

Here f_α^+ and f_α^- are α -cuts functions of $\overline{f}(x)$. Plus sign (+) in the formulation explains the enumeration in fuzzy set but not addition. Hence the total integration is obtained by aggregating integrations of each α -cut function. If a-cut operation is applied to the fuzzifying function f_α^+ or f_α^- can be calculated. Therefore, integration can be written as below:

$$\overline{I}_a^- = \int_a^b f_\alpha^-(x)dx \text{ and } \tilde{I}_a^+ = \int_a^b f_\alpha^+(x)dx$$

As a result, the possibility of $\overline{I}_a^-$ or $\overline{I}_a^+$ to be a member of total integration $\overline{I}(a,b)$ is α [21].

As the parameters ξ and σ of GPD are obtained using fuzzy approach, $\hat{F}_{X^{\delta}}(x)$ given in Equation (10) is a fuzzifying function of the distribution function for the losses truncated at δ. Therefore, Equation (9) can be expressed as:

$$\begin{aligned}\bar{P} &= \int_r^R (x-r) f_{X^{\bar{\delta}}}(x)\,dx + (R-r)\left(1-F_{X^{\bar{\delta}}}(R)\right) \mid \alpha \in [0,1], \xi \in \bar{\xi}[\alpha], \sigma^2 \in \bar{\sigma}^2[\alpha] \\ &= \int_r^R x f_{X^{\bar{\delta}}}(x)\,d_x - r\int_r^R f_{X^{\bar{\delta}}}(x)\,d_x + (R-r)\left(1-F_{X^{\bar{\delta}}}(R)\right) \mid \alpha \in [0,1], \xi \in \bar{\xi}[\alpha], \sigma^2 \in \bar{\sigma}^2[\alpha]\end{aligned} \quad (11)$$

where $\bar{P}$ is fuzzy price for high excess loss layer.

7. Applications

As mentioned before, the aim of this study is to obtain fuzzy price for high excess loss layer when GPD provides good fitting to the tail of claim data. As the point estimates of the parameters of GPD are sufficient to calculate fuzzy price for the high excess loss layer, in this study the maximum likelihood estimates of the GPD calculated by McNeill [24] for Danish fire losses over 1 Million Danish Krone comprising 2156 losses from 1980 to 1990 are used. McNeill [24] carried out a number of exploratory graphical methods providing useful preliminary information about the data and especially in tail and concluded that GPD is best fitting distribution among the truncated Lognormal, the ordinary Pareto and the GPD and optimal threshold value is 10.

The maximum likelihood estimates of the GPD for optimal threshold (10) calculated by McNeill [24] are given in Table 1.

Table 1. Maximum likelihood estimates for Danish Fire Losses.

Model	*u*	*Excesses*	$\hat{\xi}$	$\hat{\sigma}$
GPD	10	109	0.5	6.9

In this study, traditional and fuzzy prices are calculated for high excess loss layers from 100 to 150. Using maximum likelihood estimates given in Table 1 and Equation (9), traditional price for high excess loss layer from 100 to 150 is calculated as P=0.0258.

While calculating the traditional price, the degree of uncertainty has not been accounted and this price has no information about how much more or less is reasonable to charge.

Estimating the parameters of GPD using fuzzy approach, a reasonable interval for price can be obtained. In order to obtain fuzzy parameter estimation we need estimation of the parameters using classical approach such as maximum likelihood. Therefore, the maximum likelihood estimates in Table 1 are used to obtain fuzzy estimation.

As fuzzy parameter estimations of GPD are obtained by Buckley's approach which is based on a set of confidence intervals to produce a triangular shaped fuzzy number, firstly we construct the confidence intervals for shape parameter ξ and scale parameter σ.

The confidence interval for shape parameter ξ with asymptotic confidence level $\beta = 0.05$ is:

$P\{0,473028 \leq \xi \leq 0,526972\} = 0.95$

So, fuzzy parameter estimate for shape parameter ξ obtained using confidence intervals for $0.01 \leq \beta \leq 1$ are given in Figure 7.

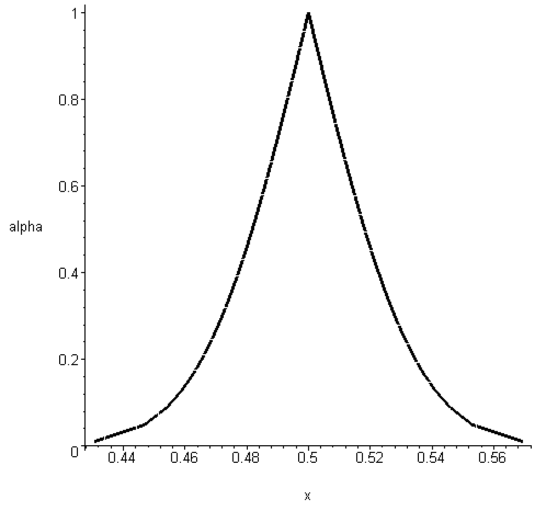

Figure 7. Fuzzy estimate of ξ.

The confidence interval for scale parameter σ with asymptotic confidence level $\beta = 0.05$ is:

$P\{6.685099 \leq \sigma \leq 7,114901\} = 0.95$

So, fuzzy parameter estimate for scale parameter σ obtained using confidence intervals for $0.01 \leq \beta \leq 1$ are given in Figure 8.

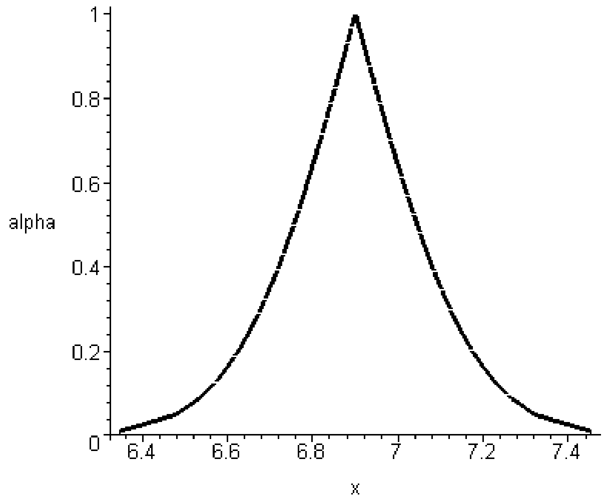

Figure 8. Fuzzy Estimate of σ **.**

After estimating the parameters of GPD using fuzzy approach, fuzzy prices for high-excess loss layer from 100 to 150 are calculated using Equation (11) for different α -cuts. The results are given in Table 2. The graph of fuzzy high excess loss layer prices is also given in Figure 9.

Table 2. Fuzzy high-excess loss layer prices for different α -cuts.

α	0	0,2	0,4	0,6	0,8	1
$\tilde{P}[\alpha]$	0,0208 ; 0,0314	0,0218 ; 0,03028	0,0228 ; 0,02916	0,0238 ; 0,02804	0,0248 ; 0,02692	0,0258 ; 0,0258

As seen in Table 2, fuzzy price for α = 1 is equal to traditional price calculated using Equation (9).

The main advantage of estimating the price by means of triangular fuzzy numbers is to obtain the variability of the price. Fuzzy price involves traditional price which is the center of the triangular fuzzy number and the spreads of the triangular fuzzy numbers give the variability of traditional price in other words give information about how much more or less than the traditional price can be charged. As seen in Table 2, the fuzzy price spreads from 0.0208 to 0.0314 for α =0 denoting maximum uncertainty about the crisp result 0.0258. For α =0.8, minimum and maximum possible values of price are 0.0248 and 0.02692. When uncertainty is minimum, α is taken as 1. So, it can be said that the higher value of α , the higher the confidence in the price.

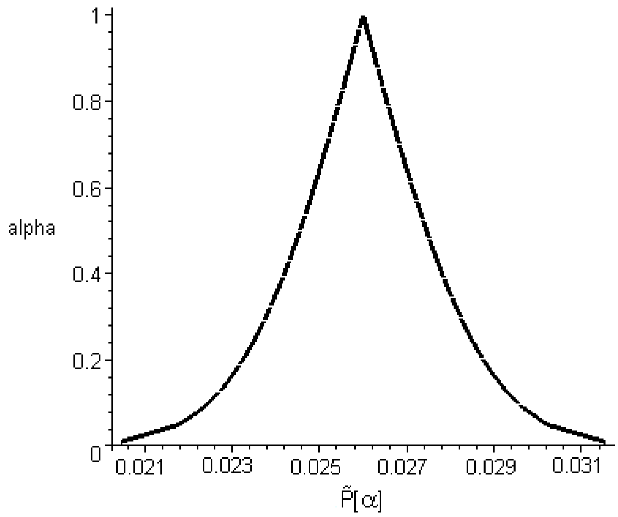

Figure 9. Fuzzy high-excess loss layer prices

The choice of α is subjective. Managements decide the value of α by taking into account the real-world situation. As buyer makes a price offer, fuzzy price is a valuable tool for the management to decide whether the buyer' s offer is acceptable or not.

Fuzzy price involves not only minimum and maximum possible values (the range of price) for different α -cuts but also traditional price. The prices outside of this range are not feasible. Charging a price lower than minimum of this range may put the insurance company under financial difficulties and also setting a price higher than maximum value of this range may result in losing competitiveness.

8. Conclusion

Due to competitiveness and globalization in insurance market, the point estimation of price calculated by an actuary may not be charged from the insured which convey no information that is helpful in deciding how much more or less it is reasonable to charge.

Fuzzy price, on the other hand, may be a very valuable tool in assessing the range of price to be considered as well as in informing the management about uncertainty of the price calculation process. Uncertainty is always present when fitting a loss distribution to historical data especially in the case of fitting the tail of the loss distribution as there would be very few observations which fall in the tail of the distribution. It is known that fitting GPD to insurance losses which exceed high threshold is a useful method for estimating the tails of the loss severity distribution. As the confidence interval for the parameters of GPD allows us to judge the uncertainty of the estimation, in this study parameter estimators of GPD are obtained using Buckley's approach based on a set of confidence intervals which is a way of dealing with uncertainty. Estimating the parameters of GPD using fuzzy enables us to calculate a reasonable interval for high excess loss layer prices, a reference quantity not only for the fair value of prices but also for its variability, to be charged from the insured.

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