Konuralp Journal of Mathematics, 8 (1) (2020) 21-29

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Some Results of Certain Class of Multivalently Bavilevič Functions

Tamer M. Seoudy[1,2*]

[1]*Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt*
[2]*Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah, Saudi Arabia*

Abstract

By making use of the principle of subordination, we introduce a certain class of multivalent analytic Bavilevič functions. Also, we obtain subordination and superordination properties, distortion theorems and inequality properties. The results presented here would provide extensions of those given in earlier works.

Keywords: *Analytic functions, subordination, superordination, Bavilevič functions.*
2010 Mathematics Subject Classification: *30C45.*

1. Introduction

Let $\mathscr{H}(\mathbb{U})$ be the class of analytic functions in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and $\mathscr{H}[a,n]$ be subclass of $\mathscr{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots\ (z\in\mathbb{U}).$$

Also, let $\mathscr{A}(p,n)$ denote the subclass of $\mathscr{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z)=z^p+\sum_{k=p+n}^{\infty}a_kz^k \quad (p,n\in\mathbb{N}=\{1,2,3,\ldots\}). \tag{1.1}$$

We write $\mathscr{A}(p,1)=\mathscr{A}(p)$ and $\mathscr{A}(1,1)=\mathscr{A}$. If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, or $g(z)$ is superordinate to $f(z)$, written symbolically, $f\prec g$ in $\mathbb{U}$ or $f(z)\prec g(z)\ (z\in\mathbb{U})$, if there exists a Schwarz function $\omega(z)$, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1(z\in\mathbb{U})$ such that $f(z)=g(\omega(z))\ (z\in\mathbb{U})$. Further more, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [7] and [2]):

$$f(z)\prec g(z) \quad (z\in\mathbb{U})\Longleftrightarrow f(0)=g(0) \quad \text{and} \quad f(\mathbb{U})\subset g(\mathbb{U}).$$

Let $\phi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the first order differential subordination:

$$\phi\left(p(z),zp'(z);z\right)\prec h(z), \tag{1.2}$$

then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z)\prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all dominants of (1.2) is called the best dominant.
If $p(z)$ and $\phi(p(z),zp'(z);z)$ are univalent in $\mathbb{U}$ and if $p(z)$ satisfies first order differential superordination:

$$h(z)\prec\phi\left(p(z),zp'(z);z\right), \tag{1.3}$$

then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z)\prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all subordinants of (1.3) is called the best subordinant.

Email addresses: **tms00@fayoum.edu.eg, tmsaman@uqu.edu.sa (T. M. Seoudy)**

Definition 1.1. *A function $f_i \in \mathscr{A}(p,n)\,(i=1,\dots,m)$ is said to be the class $\mathscr{B}_p^n(\lambda,\alpha_i,m;A,B)$ if it satisfies the following subordination condition:*

$$(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum_{i=1}^{m}\alpha_i}\prec\frac{1+Az}{1+Bz}, \tag{1.4}$$

where all the powers are principal values and throughout the paper unless otherwise mentioned the parameters λ, α_i, p, n, m, A, B are constrained as follows: $\lambda\in\mathbb{C}$, $\alpha_i>0$, $i=1,\dots,m$, $p,n\in\mathbb{N}$, $-1\le B<A\le 1$ and $z\in\mathbb{U}$.

Furthermore, the function $f_i\in\mathscr{B}_p^n(\lambda,\alpha_i,m;\beta)$ if and only if $f_i\in\mathscr{A}(p,n)$ satisfies the following condition:

$$\Re\left\{(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum_{i=1}^{m}\alpha_i}\right\}>\beta$$

where $0\le\beta<1$ and we write $\mathscr{B}_p^n(0,\alpha,m;\beta)=\mathscr{B}_p^n(\alpha,m;\beta)$. We note that

(*i*) $\mathscr{B}_p^n(\lambda,\alpha,1;A,B)=\mathscr{B}_p^n(\lambda,\alpha;A,B)$ (see Liu [5]);
(*ii*) $\mathscr{B}_p^n(1,\alpha,1;A,B)=\mathscr{B}_p^n(\alpha;A,B)$ (see Yang [13]);
(*iii*) $\mathscr{B}_1^1(1,\alpha,1;A,B)=\mathscr{B}(\alpha;A,B)$, where $\mathscr{B}(\alpha;A,B)$ is the class studied by Singh [12] and Owa and Obradovic [8];
(*iv*) $\mathscr{B}_1^n(\lambda,\alpha,1;A,0)=\mathscr{B}^n(\lambda,\alpha;A)$, where $\mathscr{B}^n(\lambda,\alpha;A)$ is the class introduced by Ponnusamy and Rajasekaran [10].
(*v*) $\mathscr{B}_1^n(1,\alpha,1;1-2\beta,-1)=\mathscr{B}^n(\alpha;\beta)$ $(0\le\beta<1)$, where $\mathscr{B}^n(\alpha;\beta)$ is the class considered by Owa [9].

In order to establish our main results, we need the following definition and lemmas.

Definition 1.2. *[6] Denote by Q the set of all functions f that are analytic and injective on $\bar{\mathbb{U}}\backslash E(f)$, where*

$$E(f)=\left\{\zeta\in\partial\mathbb{U}:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and such that $f'(\zeta)\neq 0$ for $\zeta\in\bar{\mathbb{U}}\backslash E(f)$.

Lemma 1.3. *([3] and [7]) Let the function h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0)=1$. Suppose also that $g(z)$ given by*

$$g(z)=1+c_nz^n+c_{n+1}z^{n+1}+\dots \tag{1.5}$$

is analytic in $\mathbb{U}$. If

$$g(z)+\frac{zg'(z)}{\gamma}\prec h(z)\quad(\Re(\gamma)\ge 0;\gamma\neq 0;z\in\mathbb{U}), \tag{1.6}$$

then

$$p(z)\prec q(z)=\frac{\gamma}{n}z^{-\frac{\gamma}{n}}\int_0^z h(t)\,t^{\frac{\gamma}{n}-1}\,dt\prec h(z),$$

and $q(z)$ is the best dominant.

Lemma 1.4. *[11] Let q be a convex univalent function in $\mathbb{U}$ and let $\sigma\in\mathbb{C},\eta\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ with*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function p is analytic in $\mathbb{U}$ and

$$\sigma p(z)+\eta zp'(z)\prec\sigma q(z)+\eta zq'(z),$$

then $p(z)\prec q(z)$ and q is the best dominant.

Lemma 1.5. *[6] Let q be convex univalent in $\mathbb{U}$ and $\kappa\in\mathbb{C}$. Further assume that $\Re(\kappa)>0$. If*

$$p(z)\in\mathscr{H}[q(0),1]\cap Q,$$

and $p(z)+\kappa zp'(z)$ is univalent in $\mathbb{U}$, then

$$q(z)+\kappa zq'(z)\prec p(z)+\kappa zp'(z),$$

implies $q(z)\prec p(z)$ and q is the best subordinant.

Lemma 1.6. *[4] Let F be analytic and convex in $\mathbb{U}$. If $f,g\in\mathscr{A}$ and $f,g\prec\mathrm{F}$, then*

$$\lambda f+(1-\lambda)g\prec\mathrm{F}\ (0\le\lambda\le 1).$$

Lemma 1.7. *[14] For real or complex numbers $a,b,c\,(c\neq 0,-1,-2,\dots)$ and $z\in\mathbb{U}$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z)\ \ (\Re(c)>\Re(b)>0); \tag{1.7}$$

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\ {}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right); \tag{1.8}$$

In the present paper, we obtain subordination and superordination properties, convolution properties and distortion theorem of the class $\mathscr{B}_p^n(\lambda,\alpha_i,m;A,B)$.

2. Main results

We begin by presenting our first subordination property given by Theorem 2.1 below.

Theorem 2.1. *Let $f_i \in \mathscr{B}_p^n(\lambda, \alpha_i, m; A, B)$ with $\Re(\lambda) > 0$. Then*

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} \prec Q(z) \prec \frac{1+Az}{1+Bz}, \tag{2.1}$$

where the function $Q(z)$ given by

$$Q(z) = \begin{cases} \frac{A}{B} + \frac{(B-A)}{B(1+Bz)}\ {}_2F_1\left(1,1;\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}+1;\frac{Bz}{1+Bz}\right) & (B \neq 0) \\ 1+\frac{p\sum_{i=1}^{m}\alpha_i}{p\sum_{i=1}^{m}\alpha_i+\lambda n}Az & (B=0) \end{cases} \tag{2.2}$$

is the best dominant. Furthermore

$$\Re\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} > \rho \quad (z \in \mathbb{U}), \tag{2.3}$$

where

$$\rho = \begin{cases} \frac{A}{B} + \frac{(B-A)}{B(1-B)}\ {}_2F_1\left(1,1;\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}+1;\frac{B}{B-1}\right) & (B \neq 0) \\ 1-\frac{p\sum_{i=1}^{m}\alpha_i}{p\sum_{i=1}^{m}\alpha_i+\lambda n}A & (B=0). \end{cases} \tag{2.4}$$

The estimate (2.3) is the best possible.

Proof. Let $f_i \in \mathscr{B}_p^n(\lambda, \alpha_i, m; A, B)$ and suppose that

$$g(z) = \prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} \quad (z \in \mathbb{U}). \tag{2.5}$$

Then the function $g(z)$ is of the form (1.5), analytic in $\mathbb{U}$ and $g(0) = 1$. By taking the derivatives in the both sides of (2.5), we get

$$(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} + \frac{\lambda\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum_{i=1}^{m}\alpha_i} = g(z) + \frac{\lambda\, zg'(z)}{p\sum_{i=1}^{m}\alpha_i}. \tag{2.6}$$

Since $f_i \in \mathscr{N}_p^n(\lambda, \alpha_i, m; A, B)$, we have

$$g(z) + \frac{\lambda\, zg'(z)}{p\sum_{i=1}^{m}\alpha_i} \prec \frac{1+Az}{1+Bz}.$$

Now, by using Lemma 1.3 for $\gamma = \frac{p\sum_{i=1}^{m}\alpha_i}{\lambda}$, we deduce that

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} \prec Q(z) = \frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n} z^{-\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}}\int_0^z t^{\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}-1}\left(\frac{1+At}{1+Bt}\right)dt = \frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}-1}du \tag{2.7}$$

$$= \begin{cases} \frac{A}{B} + \frac{(B-A)}{B(1+Bz)}\ {}_2F_1\left(1,1;\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}+1;\frac{Bz}{1+Bz}\right) & (B \neq 0) \\ 1+\frac{p\sum_{i=1}^{m}\alpha_i}{p\sum_{i=1}^{m}\alpha_i+\lambda n}Az & (B=0), \end{cases} \tag{2.8}$$

where we have made a change of variables followed by the use of identities in Lemma 1.7 with $a = 1$, $b = \frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}$ and $c = b+1$. This proves the assertion (2.1).
Next, in order to prove the assertion (2.3) of Theorem 2.1, it suffices to show that

$$\inf_{|z|<1}\{\Re(Q(z))\} = Q(-1). \tag{2.9}$$

Indeed, we have for $|z| \leq r < 1$,

$$\Re\left(\frac{1+Az}{1+Bz}\right) \geq \frac{1-Ar}{1-Br}.$$

Setting

$$G(z,s) = \frac{1+Asz}{1+Bsz}$$

and

$$dv(s) = \frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n} s^{\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}-1} ds \quad (0 \leq s \leq 1),$$

which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z) = \int_0^1 G(z,s)\ dv(s),$$

so that

$$\Re\{Q(z)\} \geq \int_0^1 \frac{1-Asr}{1-Bsr}\, dv(s) = Q(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (2.3). Finally, the estimate (2.3) is the best possible because the function $Q(z)$ is the best dominant of (2.1). This completes the proof of Theorem 2.1. □

Corollary 2.2. *Let $f \in \mathscr{B}_p^n(\lambda, \alpha_i, m; A^*, B)$ with $\Re(\lambda) > 0$ and $B \neq 0$, then*

$$\Re\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} > 0 \quad (z \in \mathbb{U}),$$

where A^* is given by

$$A^* = \frac{B\ {}_2F_1\left(1,1;\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}+1;\frac{B}{B-1}\right)}{{}_2F_1\left(1,1;\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}+1;\frac{B}{B-1}\right)+B-1}. \tag{2.10}$$

The result is sharp.

Proof. In view of Theorem 2.1, if

$$(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} + \frac{\lambda\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum_{i=1}^{m}\alpha_i} \prec \frac{1+A^*z}{1+Bz},$$

then

$$\Re\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} > \left[\frac{A^*}{B} + \frac{(B-A^*)}{B(1-B)}\ {}_2F_1\left(1,1;\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}+1;\frac{B}{B-1}\right)\right]^{\frac{1}{\alpha}}. \tag{2.11}$$

On substituting the value of A^* given by (2.10) in the right hand side of the inequality (2.11), we have

$$\Re\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} > 0 \quad (z \in \mathbb{U}),$$

which proves Corollary 2.2. □

Putting $A = 1-2\sigma\,(0 \leq \sigma < 1)$ and $B = -1$ in Theorem 2.1, we have

Corollary 2.3. *Let $f \in \mathscr{B}_p^n(\lambda, \alpha_i, m; \sigma)$ with $\Re(\lambda) > 0$, then*

$$\Re\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} > \sigma + (1-\sigma)\left[{}_2F_1\left(1,1;\frac{p\sum_{i=1}^{m}\alpha_i}{\lambda n}+1;\frac{1}{2}\right)-1\right].$$

The result is sharp.

Putting $m=1$ in Corollary 2.3, we have

Corollary 2.4. *Let $f\in\mathscr{B}_p^n(\lambda,\alpha;\sigma)$ with $\Re(\lambda)>0$, then*

$$\Re\left(\frac{f(z)}{z^p}\right)^{\alpha}>\sigma+(1-\sigma)\left[{}_2F_1\left(1,1;\frac{p\alpha}{\lambda n}+1;\frac{1}{2}\right)-1\right].$$

The result is sharp.

Theorem 2.5. *Let $q(z)$ be univalent in $\mathbb{U}$, $\lambda\in\mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-\Re\left(\frac{p\sum\limits_{i=1}^{m}\alpha_i}{\lambda}\right)\right\}. \tag{2.12}$$

If $f\in\mathscr{A}(p,n)$ satisfies the following subordination condition:

$$(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}\prec q(z)+\frac{\lambda\, zq'(z)}{p\sum\limits_{i=1}^{m}\alpha_i}, \tag{2.13}$$

then

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Let the function $g(z)$ be defined by (2.5). We know that (2.6) holds true. Combining (2.6) and (2.13), we find that

$$g(z)+\frac{\lambda\, zg'(z)}{p\sum\limits_{i=1}^{m}\alpha_i}\prec q(z)+\frac{\lambda\, zq'(z)}{p\sum\limits_{i=1}^{m}\alpha_i}. \tag{2.14}$$

By using Lemma 1.4 and (2.14), we easily get the assertion of Theorem 2.5. □

Taking $q(z)=\frac{1+Az}{1+Bz}\ (-1\le B<A\le 1)$ in Theorem 2.5, we get the following result.

Corollary 2.6. *Let $\lambda\in\mathbb{C}^*$ and $-1\le B<A\le 1$. Suppose also that*

$$\Re\left(\frac{1-Bz}{1+Bz}\right)>\max\left\{0,-\Re\left(\frac{p\sum\limits_{i=1}^{m}\alpha_i}{\lambda}\right)\right\}.$$

If $f\in\mathscr{A}(p,n)$ satisfies the following subordination condition:

$$(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}\prec\frac{1+Az}{1+Bz}+\frac{\lambda(A-B)z}{p\sum\limits_{i=1}^{m}\alpha_i\,(1+Bz)^2},$$

then

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\prec\frac{1+Az}{1+Bz},$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

We now derive the following superordination result.

Theorem 2.7. *Let q be convex univalent in $\mathbb{U}$, $\lambda\in\mathbb{C}$ with $\Re(\lambda)>0$. Also let*

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\in\mathscr{H}[q(0),1]\cap Q$$

and

$$(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}$$

be univalent in $\mathbb{U}$. If $f \in \mathscr{A}(p)$ satisfies the following superordination condition:

$$q(z)+\frac{\lambda\ zq^{'}(z)}{p\sum\limits_{i=1}^{m}\alpha_i}\prec(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i},$$

then

$$q(z)\prec\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i},$$

and the function $q(z)$ is the best subordinant.

Proof. Let the function $g(z)$ be defined by (2.5). Then

$$q(z)+\frac{\lambda\ zq'(z)}{p\sum\limits_{i=1}^{m}\alpha_i}\prec(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}=g(z)+\frac{\lambda\ zg'(z)}{p\sum\limits_{i=1}^{m}\alpha_i}.$$

An application of Lemma 1.5 yields the assertion of Theorem 2.7. □

Taking $q(z)=\frac{1+Az}{1+Bz}\,(-1\le B<A\le 1)$ in Theorem 2.7, we get the following corollary.

Corollary 2.8. *Let $-1\le B<A\le 1$, $\lambda\in\mathbb{C}$ with $\Re(\lambda)>0$. Also let*

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\in\mathscr{H}[1,1]\cap Q,$$

and

$$(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}$$

be univalent in $\mathbb{U}$. If $f\in\mathscr{A}(p)$ satisfies the following superordination condition:

$$\frac{1+Az}{1+Bz}+\frac{\lambda\ (A-B)z}{p\sum\limits_{i=1}^{m}\alpha_i\ (1+Bz)^2}\prec(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i},$$

then

$$\frac{1+Az}{1+Bz}\prec\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant.

Combining the above results of subordination and superordination. we easily get the following “Sandwich-type result”.

Theorem 2.9. *Let q_1 be convex univalent and let q_2 be univalent in $\mathbb{U}$, $\lambda\in\mathbb{C}$ with $\Re(\lambda)>0$. Let q_2 satisfies (2.12). If*

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\in\mathscr{H}[q_1(0),1]\cap Q,$$

and

$$(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}$$

be univalent in $\mathbb{U}$, also

$$q_1(z)+\frac{\lambda\ zq_1'(z)}{p\sum\limits_{i=1}^{m}\alpha_i}\prec(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}\prec q_2(z)+\frac{\lambda\ zq_2'(z)}{p\sum\limits_{i=1}^{m}\alpha_i},$$

then

$$q_1(z)\prec\prod_{i=1}^{m}\left(\frac{z^p}{f_i(z)}\right)^{\alpha_i}\prec q_2(z),$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Theorem 2.10. *If $f_i \in \mathscr{B}_p^n(\alpha_i,m;\beta)$ with $\lambda>0$ and $0\leq\beta<1$. Then $f_i\in\mathscr{B}_p^n(\lambda,\alpha_i,m;\beta)$ for $|z|<R$, where*

$$R=\left(\sqrt{\left(\frac{n\lambda}{p\sum\limits_{i=1}^{m}\alpha_i}\right)^2+1}-\frac{n\lambda}{p\sum\limits_{i=1}^{m}\alpha_i}\right)^{\frac{1}{n}}. \tag{2.15}$$

The bound R is the best possible.

Proof. We begin by writing

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}=\beta+(1-\beta)\,g(z)\quad(z\in\mathbb{U}). \tag{2.16}$$

Then clearly, the function $g(z)$ is of the form (1.5), is analytic and has a positive real part in $\mathbb{U}$. By taking the derivatives in the both sides in equality (2.16), we get

$$\frac{1}{1-\beta}\left\{(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}-\beta\right\}=g(z)+\frac{\lambda\,zg'(z)}{p\sum\limits_{i=1}^{m}\alpha_i}. \tag{2.17}$$

By making use of the following well-known estimate (see [1, Theorem 1]):

$$\frac{|zg'(z)|}{\Re\{g(z)\}}\leq\frac{2nr^n}{1-r^{2n}}\quad(|z|=r<1)$$

in (2.17), we obtain that

$$\frac{1}{1-\beta}\Re\left\{(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}-\beta\right\}\geq\Re\{g(z)\}\left(1-\frac{2\lambda nr^n}{p\sum\limits_{i=1}^{m}\alpha_i\,\left(1-r^{2n}\right)}\right). \tag{2.18}$$

It is seen that the right-hand side of (2.18) is positive, provided that $r<R$, where R is given by (2.15). In order to show that the bound R is the best possible, we consider the function $f\in\mathscr{A}(p,n)$ defined by

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}=\beta+(1-\beta)\frac{1+z^n}{1-z^n}\quad(z\in\mathbb{U}).$$

By noting that

$$\frac{1}{1-\beta}\left\{(1-\lambda)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}-\beta\right\}=\frac{1+z^n}{1-z^n}+\frac{2\lambda\;nz^n}{p\sum\limits_{i=1}^{m}\alpha_i\;(1-z^n)^2}=0, \tag{2.19}$$

for $z=R\,\exp\left(\frac{\pi i}{n}\right)$, we conclude that the bound is the best possible. Theorem 2.10 is thus proved. □

Theorem 2.11. *Let $\lambda_2\geq\lambda_1\geq0$ and $-1\leq B_1\leq B_2<A_2\leq A_1\leq1$. Then*

$$\mathscr{B}_p^n(\lambda_2,\alpha_i,m;A_2,B_2)\subset\mathscr{B}_p^n(\lambda_1,\alpha_i,m;A_1,B_1). \tag{2.20}$$

Proof. Suppose that $f_i\in\mathscr{B}_p^n(\lambda_2,\alpha_i,m;A_2,B_2)$. We know that

$$(1-\lambda_2)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}\prec\frac{1+A_2z}{1+B_2z}.$$

Since $-1\leq B_1\leq B_2<A_2\leq A_1\leq1$, we easily find that

$$(1-\lambda_2)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}\prec\frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}, \tag{2.21}$$

that is $f_i\in\mathscr{B}_p^n(\lambda_2,\alpha_i,m;A_1,B_1)$. Thus the assertion of Theorem 2.11 holds for $\lambda_2=\lambda_1\geq0$. If $\lambda_2>\lambda_1\geq0$, by Theorem 2.1 and (2.21), we know that $f_i\in\mathscr{B}_p^n(0,\alpha_i,m;A_1,B_1)$, that is,

$$\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\prec\frac{1+A_1z}{1+B_1z}. \tag{2.22}$$

At the same time, we have

$$(1-\lambda_1)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}$$

$$=\left(1-\frac{\lambda_1}{\lambda_2}\right)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda_1}{\lambda_2}\left[(1-\lambda_2)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}\right]. \tag{2.23}$$

Moreover, since $0\leq\frac{\lambda_1}{\lambda_2}<1$,and the function $\frac{1+A_1z}{1+B_1z}\,(-1\leq B_1<A_1\leq 1; z\in\mathbb{U})$ is analytic and convex in $\mathbb{U}$. Combining (2.21)-(2.23) and Lemma 1.6, we find that

$$(1-\lambda_1)\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}+\frac{\lambda\prod\limits_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\sum\limits_{i=1}^{m}\left(\alpha_i\frac{zf_i'(z)}{pf_i(z)}\right)}{\sum\limits_{i=1}^{m}\alpha_i}\prec\frac{1+A_1z}{1+B_1z},$$

that is $f_i\in\mathscr{B}_p^n(\lambda_1,\alpha_i.m;A_1,B_1)$, which implies that the assertion (2.20) of Theorem 2.11 holds. □

Theorem 2.12. *Let $f\in\mathscr{B}_p^n(\lambda,\alpha_i,m;A,B)$ with $\lambda>0$ and $-1\leq B<A\leq 1$. Then*

$$\frac{p\sum\limits_{i=1}^{m}\alpha_i}{n\lambda}\int_0^1\frac{1-Au}{1-Bu}\,u^{\frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}-1}du<\Re\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}<\frac{p\sum\limits_{i=1}^{m}\alpha_i}{n\lambda}\int_0^1\frac{1+Au}{1+Bu}\,u^{\frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}-1}du. \tag{2.24}$$

Proof. Let $f\in\mathscr{B}_p^n(\lambda,\alpha_i,m;A,B)$ with $\lambda>0$. From Theorem 2.1, we know that (2.7) holds, which implies that

$$\Re\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}<\sup_{z\in\mathbb{U}}\Re\left\{\frac{p\sum\limits_{i=1}^{m}\alpha_i}{n\lambda}\int_0^1\frac{1+Azu}{1+Bzu}\,u^{\frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}-1}\,du\right\}$$

$$\leq\frac{p\sum\limits_{i=1}^{m}\alpha_i}{n\lambda}\int_0^1\sup_{z\in\mathbb{U}}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}-1}\,du<\frac{p\sum\limits_{i=1}^{m}\alpha_i}{n\lambda}\int_0^1\frac{1+Au}{1+Bu}\,u^{\frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}-1}\,du, \tag{2.25}$$

and

$$\Re\prod_{i=1}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}>\inf_{z\in\mathbb{U}}\Re\left\{\frac{p\sum\limits_{i=1}^{m}\alpha_i}{n\lambda}\int_0^1\frac{1+Azu}{1+Bzu}\,u^{\frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}-1}du\right\}$$

$$\geq\frac{p\sum\limits_{i=1}^{m}\alpha_i}{n\lambda}\int_0^1\inf_{z\in\mathbb{U}}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}-1}\,du>\frac{p\sum\limits_{i=1}^{m}\alpha_i}{n\lambda}\int_0^1\frac{1-Au}{1-Bu}\,u^{\frac{p\sum_{i=1}^{m}\alpha_i}{n\lambda}-1}du. \tag{2.26}$$

Combining (2.25) and (2.26), we get (2.24). The proof of Theorem 2.12 is evidently completed. □

Remark 2.13. *Putting $m=1$ in Theorems 2.1, 2.11 and 2.12,respectively, we obtain the results of Liu [5, Theorems 1, 2 and 3, respectively].*

Acknowledgement

We are grateful to the referees for their valuable suggestions.

References

[1] Bernardi, S. D., *New distortion theorems for functions of positive real part and applications to the partial sums of univalent convex functions*, Proc. Amer. Math. Soc., 45(1)(1974), 113-118.
[2] Bulboaca, T., *Differential Subordinations and Superordinations, Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.
[3] Hallenbeck, D. J. and Ruschewyh, S., *Subordination by convex functions*, Proc. Amer. Math. Soc., 52(1975), 191-195.
[4] Liu, M.-S., *On certain subclass of analytic functions*, J. South China Normal Univ., 4(2002),15-20 (in Chinese).
[5] Liu, M., *On certain subclass of $p-$valent functions*, Soochow J. Math., 26(2)(2000), 163-171.
[6] Miller, S. S. and Mocanu, P.T., *Subordinats of differential superordinations*, Complex Var., 48(2003), 815-826.
[7] Miller, S. S. and Mocanu, P.T., *Differential Subordination: Theory and Applications, in: Series in Pure and Applied Mathematics*, vol. 225, Marcel Dekker, New York, 2000.
[8] Owa, S. and Obradovic, M., *Certain subclasses of Bazilevic functions of type α*, Internat. J. Math. and Math. Sci., 9(2)(1986), 97-105.
[9] Owa, S., *On certain Bazilevic functions of order β*, Internat. J. Math. and Math. Sci., 15(3)(1992), 613-616.

[10] Ponnusamy S. and Rajasekaran, S., *New sufficient conditions for starlike and univalent functions*, Soochow J. Math., 21(1)(1995), 193-201.
[11] Shanmugam, T. N., Ravichandran V. and Sivasubramanian, S., *Differential sandwich theorems for subclasses of analytic functions*, Australian J. Math. Anal. Appl., 3(2006), Art. 8, 1-11.
[12] Singh, R., *On Bazilevic functions*, Proc. Amer. Math. Sci., 38(1973), 261-267.
[13] Yang, D. G., *Properties of certain classes of multivalent functions*, Bull. Inst. Math. Acad. Sinica, 22(4)(1994), 361-367.
[14] Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions*, Cambridge University Press, Cambridge, 1927.