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Some Global Optimality Results using the Contractive Conditions of Integral Type

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Abstract

In this paper, we establish new best proximity point theorems considering a classical global optimization problem of finding the minimum distance between pairs of closed sets using the contractive conditions of integral type on a complete metric space. These results can be used to find optimal approximate solutions by means of some contractive conditions of integral type. Also an illustrative example is given.

Keywords: *Global optimality, best proximity point, integral type condition, metric space, completeness.*
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1. Introduction

Fixed and proximity point problems are very famous since there exist some applications of these problems in various areas such as engineering, physical theories, computer science etc. Mainly, the proximity point problem is used to find the minimum distance between two sets with a function. Hence the proximity point problem can be studied on a metric space since a metric (or a distance function) is a function which defines a distance between each pair of elements of any set.
Fixed point theory has been studied on various metric spaces using some contractive conditions. For example, Mustafa and Sims introduced the notion of a G-metric space as a new approach to generalized metric spaces in [15]. Thereafter, some authors obtained new fixed-point results on a G-metric space (see [6], [13] and [14] for more details). Recently, Sedghi et al. defined the concept of an S-metric space and obtained some fixed-point theorems in [26]. Then using various contractive conditions, new existence and uniqueness fixed-point theorems have been proved in some papers. For some examples, one can see [16], [17], [18], [26] and [27]. Also some mathematicians investigated new contractive conditions of integral type to obtain interesting fixed-point results (see [19], [20] and [25] for more details).
There are several studies on the best proximity point problem in the literature. Some authors obtained new best proximity results using certain types of contractive inequalities and also investigated optimal approximate solutions of some fixed-point equations (see [1], [2], [3], [7], [8], [9], [10], [11], [12], [21], [22], [23] and [24] for more details).
Now we recall the following concepts:
Let (X,ρ) be a metric space and $E,F \subset X$.
A pair $(u,v) \in E \times F$ is called a best proximity pair if

$$\rho(u,v) = \rho(E,F) = \inf\{\rho(e,f) : e \in E \text{ and } f \in F\}.$$

If $E \neq \emptyset$, $F \neq \emptyset \subset X$ and $T : E \to F$ then

$$\rho(u,Tu) \geq \rho(E,F),$$

for all $u \in E$.
A point $e \in E$ is called a best proximity point with respect to T if

$$\rho(e,Te) = \rho(E,F),$$

that is, the function $\rho(e,Te)$ attains its global minimum at the point e and this value is equal to $\rho(E,F)$. Hence it is a problem of global minimization.
In the next section, we use the following class of functions given in [4] and [5], respectively, to obtain new proximity point results:
Let $\mathscr{S}$ be the class of functions $\theta : [0,\infty) \to [0,1)$ satisfy the condition $\theta(t_n) \to 1$ implies $t_n \to 0$.

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Let Ψ be the class of functions $\psi : [0,\infty) \to [0,\infty)$ such that ψ is continuous, strictly increasing and $\psi(0) = 0$.
On the other hand, the existence of optimal approximate solutions which are called best proximity points is investigated by best proximity point theorems to the fixed-point equation $Tu = u$ when there is no exact solution. As the distance between any element $u \in E$ and its image $Tu \in F$ is at least the distance between the sets E and F, a best proximity point theorem gets global minimum of $\rho(u,Tu)$ by stipulating an approximate solution u of the fixed point equation $Tu = u$ to satisfy the condition

$$\rho(u,Tu) = \rho(E,F).$$

The aim of this article is to establish new best proximity point theorems considering a classical global optimization problem of finding the minimum distance between pairs of closed sets using the contractive conditions of integral type on a complete metric space. Also an illustrative example is given.

2. Main Results

Now, we prove some proximity point results on a complete metric space. Throughout the whole paper we assume that $\kappa : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,\infty)$, nonnegative and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \kappa(t)dt > 0. \tag{2.1}$$

Let (X,ρ) be a complete metric space and $E \neq \emptyset$, $F \neq \emptyset$ any closed subsets of X in the next theorems, unless otherwise stated.

Theorem 2.1. *Let the function $\kappa : [0,\infty) \to [0,\infty)$ be defined as in (2.1) and $T : E \cup F \to E \cup F$ a self-mapping satisfying:*

1. *$T(E) \subseteq F$ and $T(F) \subseteq E$,*
2. $$\int_0^{\rho(Tu,Tv)} \kappa(t)dt \le \theta\left(\int_0^{M_1(u,v)} \kappa(t)dt\right)\int_0^{M_1(u,v)} \kappa(t)dt + \left(1-\theta\left(\int_0^{M_1(u,v)} \kappa(t)dt\right)\right)\int_0^{\rho(E,F)} \kappa(t)dt, \tag{2.2}$$

   where $u \in E$, $v \in F$, $\theta \in \mathscr{S}$ and $M_1(u,v) = \max\{\rho(u,v), \rho(u,Tu), \rho(v,Tv)\}$.

Let $u_0 \in E$ be any element and the sequence $\{u_n\}$ defined as $u_{n+1} = Tu_n$ for all $n \ge 0$. Then $\rho(u_n,Tu_n) \to \rho(E,F)$. If $\{u_{2n}\}$ has a convergent subsequence in E, the subsequence converges to a proximity point.

Proof. Using the hypothesis, we have $u_{n-1} \in E$, $u_n \in F$ or $u_n \in E$, $u_{n-1} \in F$, for all $n \ge 1$ according to n is odd or even. Using the inequality (2.2) and the definition of the sequence $\{u_n\}$ for all $n \ge 1$, we obtain

$$\int_0^{\rho(u_n,u_{n+1})} \kappa(t)dt = \int_0^{\rho(Tu_{n-1},Tu_n)} \kappa(t)dt \le \theta\left(\int_0^{M_1(u_{n-1},u_n)} \kappa(t)dt\right)\int_0^{M_1(u_{n-1},u_n)} \kappa(t)dt + \left(1-\theta\left(\int_0^{M_1(u_{n-1},u_n)} \kappa(t)dt\right)\right)\int_0^{\rho(E,F)} \kappa(t)dt, \tag{2.3}$$

where

$$\begin{aligned} M_1(u_{n-1},u_n) &= \max\{\rho(u_{n-1},u_n), \rho(u_{n-1},Tu_{n-1}), \rho(u_n,Tu_n)\} \\ &= \max\{\rho(u_{n-1},u_n), \rho(u_n,u_{n+1})\}. \end{aligned}$$

Let $\rho(u_n,u_{n+1}) > \rho(u_{n-1},u_n)$. Then we get

$$M_1(u_{n-1},u_n) = \rho(u_n,u_{n+1})$$

and so

$$\int_0^{\rho(u_n,u_{n+1})} \kappa(t)dt - \int_0^{\rho(E,F)} \kappa(t)dt \le \theta\left(\int_0^{\rho(u_n,u_{n+1})} \kappa(t)dt\right)\left(\int_0^{\rho(u_n,u_{n+1})} \kappa(t)dt - \int_0^{\rho(E,F)} \kappa(t)dt\right).$$

Hence we get $\theta\left(\int_0^{\rho(u_n,u_{n+1})} \kappa(t)dt\right) \ge 1$, which is a contradiction since $\theta \in \mathscr{S}$. So we have

$$\rho(u_n,u_{n+1}) \le \rho(u_{n-1},u_n)$$

and

$$M_1(u_{n-1},u_n) = \rho(u_{n-1},u_n). \tag{2.4}$$

Then there exists p with $0 < \rho(E,F) \le p$ such that

$$\lim_{n\to\infty} M_1(u_{n-1},u_n) = \lim_{n\to\infty} \rho(u_{n-1},u_n) = p. \tag{2.5}$$

Let $\rho(E,F) < p$. Using the conditions (2.3) and (2.4), we have

$$\frac{\int_0^{\rho(u_n,u_{n+1})} \kappa(t)dt - \int_0^{\rho(E,F)} \kappa(t)dt}{\int_0^{\rho(u_{n-1},u_n)} \kappa(t)dt - \int_0^{\rho(E,F)} \kappa(t)dt} \leq \theta\left(\int_0^{\rho(u_{n-1},u_n)} \kappa(t)dt\right) < 1,$$

for all $n \geq 1$ since $\theta \in \mathscr{S}$. If we take limit for $n \to \infty$ then using the equality (2.5), we get

$$\lim_{n\to\infty} \theta\left(\int_0^{\rho(u_{n-1},u_n)} \kappa(t)dt\right) = 1.$$

This implies that

$$\lim_{n\to\infty} \int_0^{\rho(u_{n-1},u_n)} \kappa(t)dt = 0,$$

which is a contradiction with $\rho(E,F) < p$. Hence $p = \rho(E,F)$ and

$$\rho(u_n, Tu_n) \to \rho(E,F), \tag{2.6}$$

as $n \to \infty$. Let $\{u_{2n_k}\}$ be a subsequence of $\{u_{2n}\}$ and

$$\lim_{k\to\infty} u_{2n_k} = y_0. \tag{2.7}$$

Since $\{u_{2n_k-1}\}$ is a sequence in F then we obtain

$$\begin{aligned}\int_0^{\rho(E,F)} \kappa(t)dt &\leq \int_0^{\rho(y_0,u_{2n_k-1})} \kappa(t)dt \leq \int_0^{\rho(y_0,u_{2n_k})+\rho(u_{2n_k},u_{2n_k-1})} \kappa(t)dt \leq \int_0^{\rho(y_0,u_{2n_k})} \kappa(t)dt + \int_0^{\rho(u_{2n_k},u_{2n_k-1})} \kappa(t)dt \\ &= \int_0^{\rho(y_0,u_{2n_k})} \kappa(t)dt + \int_0^{\rho(u_{2n_k-1},Tu_{2n_k-1})} \kappa(t)dt,\end{aligned}$$

for all $k > 0$. If we take limit for $k \to \infty$ then using the condition (2.6), we have

$$\lim_{k\to\infty} \int_0^{\rho(y_0,u_{2n_k-1})} \kappa(t)dt = \int_0^{\rho(E,F)} \kappa(t)dt. \tag{2.8}$$

Now we get

$$\begin{aligned}\int_0^{\rho(y_0,Ty_0)} \kappa(t)dt &\leq \int_0^{\rho(y_0,u_{2n_k})+\rho(u_{2n_k},Ty_0)} \kappa(t)dt \leq \int_0^{\rho(y_0,u_{2n_k})} \kappa(t)dt + \int_0^{\rho(Tu_{2n_k-1},Ty_0)} \kappa(t)dt \\ &\leq \int_0^{\rho(y_0,u_{2n_k})} \kappa(t)dt + \theta\left(\int_0^{M_1(u_{2n_k-1},y_0)} \kappa(t)dt\right) \int_0^{M_1(u_{2n_k-1},y_0)} \kappa(t)dt + \left(1-\theta\left(\int_0^{M_1(u_{2n_k-1},y_0)} \kappa(t)dt\right)\right) \int_0^{\rho(E,F)} \kappa(t)dt \\ &= \theta\left(\int_0^{M_1(u_{2n_k-1},y_0)} \kappa(t)dt\right)\left(\int_0^{M_1(u_{2n_k-1},y_0)} \kappa(t)dt - \int_0^{\rho(E,F)} \kappa(t)dt\right) + \int_0^{\rho(E,F)} \kappa(t)dt + \int_0^{\rho(y_0,u_{2n_k})} \kappa(t)dt.\end{aligned} \tag{2.9}$$

Using the conditions (2.6) and (2.8), we obtain

$$\begin{aligned}&\lim_{k\to\infty} \int_0^{M_1(x_{2n_k-1},y_0)} \kappa(t)dt \\ = &\lim_{k\to\infty} \int_0^{\max\{\rho(x_{2n_k-1},y_0),\rho(x_{2n_k-1},x_{2n_k}),\rho(y_0,Ty_0)\}} \kappa(t)dt = \int_0^{\rho(y_0,Ty_0)} \kappa(t)dt.\end{aligned} \tag{2.10}$$

We assume that $\rho(y_0,Ty_0) > \rho(E,F)$. If we take limit for $k \to \infty$ then using the conditions (2.9) and (2.10), we have

$$\int_0^{\rho(y_0,Ty_0)} \kappa(t)dt - \int_0^{\rho(E,F)} \kappa(t)dt \leq \lim_{k\to\infty} \theta\left(\int_0^{M_1(u_{2n_k-1},y_0)} \kappa(t)dt\right)\left(\int_0^{\rho(y_0,Ty_0)} \kappa(t)dt - \int_0^{\rho(E,F)} \kappa(t)dt\right) \leq \int_0^{\rho(y_0,Ty_0)} \kappa(t)dt - \int_0^{\rho(E,F)} \kappa(t)dt$$

and so

$$\lim_{k\to\infty} \theta\left(\int_0^{M_1(u_{2n_k-1},y_0)} \kappa(t)dt\right) = 1.$$

Since $\theta \in \mathscr{S}$, we get

$$\lim_{k\to\infty} \int_0^{M_1(u_{2n_k-1},y_0)} \kappa(t)dt = 0,$$

which implies that

$$\lim_{k\to\infty} M_1(u_{2n_k-1},y_0) = 0,$$

that is, $y_0 = Ty_0$, which is a contradiction with our assumption. Hence

$$\rho(y_0,Ty_0) = \rho(E,F).$$

Consequently, $\{u_{2n_k}\}$ converges to a proximity point. □

Considering $\kappa(t) = 1$ in Theorem 2.1 then we get Theorem 2 given in [8] on page 100.

Theorem 2.2. *Let the function $\kappa : [0,\infty) \to [0,\infty)$ be defined as in (2.1) and $T : E\cup F \to E\cup F$ a self-mapping satisfying*:

1. $T(E) \subseteq F$ *and* $T(F) \subseteq E$,
2. $$\int_0^{\rho(Tu,Tv)} \kappa(t)dt \le \theta\left(\int_0^{M_2(u,v)} \kappa(t)dt\right)\int_0^{M_2(u,v)} \kappa(t)dt + \left(1-\theta\left(\int_0^{M_2(u,v)} \kappa(t)dt\right)\right)\int_0^{\rho(E,F)} \kappa(t)dt,$$
   where $u \in E$, $v \in F$, $\theta \in \mathscr{S}$ and
   $$M_2(u,v) = \max\left\{ \rho(u,v), \rho(u,Tu), \rho(v,Tv), \frac{\rho(u,Tu)\rho(v,Tv)}{1+\rho(u,v)}, \frac{\rho(u,Tu)\rho(v,Tv)}{1+\rho(Tu,Tv)} \right\}.$$

Let $u_0 \in E$ be any element and the sequence $\{u_n\}$ defined as $u_{n+1} = Tu_n$ for all $n \ge 0$. Then $\rho(u_n,Tu_n) \to \rho(E,F)$. If $\{u_{2n}\}$ has a convergent subsequence in E, the subsequence converges to a proximity point.

Proof. The proof is similar to the proof of Theorem 2.1 since

$$M_2(u_{n-1},u_n) = \max\{\rho(u_{n-1},u_n), \rho(u_{n+1},u_n)\}.$$

□

In the following theorem, we obtain a new best proximity result using the class Ψ and a function $\alpha : X\times X \to [1,\infty)$.

Theorem 2.3. *Let the function $\kappa : [0,\infty) \to [0,\infty)$ be defined as in (2.1) and $T : E\cup F \to E\cup F$ a self-mapping satisfying*:

1. $T(E) \subseteq F$ *and* $T(F) \subseteq E$,
2. $$\alpha(u,v)\psi\left(\int_0^{\rho(Tu,Tv)} \kappa(t)dt\right) \le \theta\left(\psi\left(\int_0^{M_1(u,v)} \kappa(t)dt\right)\right)\psi\left(\int_0^{M_1(u,v)} \kappa(t)dt\right) + \left(1-\theta\left(\psi\left(\int_0^{M_1(u,v)} \kappa(t)dt\right)\right)\right)\int_0^{\rho(E,F)} \kappa(t)dt, \tag{2.11}$$
   where $u \in E$, $v \in F$, $\theta \in \mathscr{S}$, $\alpha : X\times X \to [1,\infty)$ and $\psi \in \Psi$.

Let $u_0 \in E$ be any element and the sequence $\{u_n\}$ defined as $u_{n+1} = Tu_n$ for all $n \ge 0$. Then $\rho(u_n,Tu_n) \to \rho(E,F)$. If $\{u_{2n}\}$ has a convergent subsequence in E, the subsequence converges to a proximity point.

Proof. Using the hypothesis, we have

$$u_{n-1} \in E,\ u_n \in F \text{ or } u_n \in E,\ u_{n-1} \in F,$$

for all $n \ge 1$ according to n is odd or even. Using the inequality (2.11), for all $n \ge 1$, we obtain

$$\begin{aligned}
\alpha(u_{n-1},u_n)\psi\left(\int_0^{\rho(u_n,u_{n+1})} \kappa(t)dt\right) &= \alpha(u_{n-1},u_n)\psi\left(\int_0^{\rho(Tu_{n-1},Tu_n)} \kappa(t)dt\right) \\
&\le \theta\left(\psi\left(\int_0^{M_1(u_{n-1},u_n)} \kappa(t)dt\right)\right)\psi\left(\int_0^{M_1(u_{n-1},u_n)} \kappa(t)dt\right) + \left(1-\theta\left(\psi\left(\int_0^{M_1(u_{n-1},u_n)} \kappa(t)dt\right)\right)\right)\int_0^{\rho(E,F)} \kappa(t)dt,
\end{aligned} \tag{2.12}$$

where

$$M_1(u_{n-1},u_n) = \max\{\rho(u_{n-1},u_n), \rho(u_n,u_{n+1})\}.$$

Let $\rho(u_n,u_{n+1}) > \rho(u_{n-1},u_n)$. Then we get

$$M_1(u_{n-1},u_n) = \rho(u_n,u_{n+1})$$

and so

$$\alpha(u_{n-1},u_n)\,\psi\left(\int_0^{\rho(u_n,u_{n+1})}\kappa(t)dt\right)-\int_0^{\rho(E,F)}\kappa(t)dt\le\theta\left(\psi\left(\int_0^{\rho(u_n,u_{n+1})}\kappa(t)dt\right)\right)\left[\psi\left(\int_0^{\rho(u_n,u_{n+1})}\kappa(t)dt\right)-\int_0^{\rho(E,F)}\kappa(t)dt\right].$$

Hence we get $\theta\left(\psi\left(\int_0^{\rho(u_n,u_{n+1})}\kappa(t)dt\right)\right)\ge 1$. Therefore we have a contradiction with $\theta\in\mathscr{S}$ and

$$M_1(u_{n-1},u_n)=\rho(u_{n-1},u_n). \tag{2.13}$$

Then there exists p with $0<\rho(E,F)\le p$ such that

$$\lim_{n\to\infty}M_1(u_{n-1},u_n)=\lim_{n\to\infty}\rho(u_{n-1},u_n)=p. \tag{2.14}$$

Let $\rho(E,F)<p$. Using the conditions (2.12) and (2.13), we have

$$\frac{\alpha(u,v)\,\psi\left(\int_0^{\rho(u_n,u_{n+1})}\kappa(t)dt\right)-\int_0^{\rho(E,F)}\kappa(t)dt}{\psi\left(\int_0^{\rho(u_{n-1},u_n)}\kappa(t)dt\right)-\int_0^{\rho(E,F)}\kappa(t)dt}\le\theta\left(\psi\left(\int_0^{\rho(u_{n-1},u_n)}\kappa(t)dt\right)\right)<1,$$

for all $n\ge 1$ since $\theta\in\mathscr{S}$. If we take limit for $n\to\infty$ then using the equality (2.14), we get

$$\lim_{n\to\infty}\theta\left(\psi\left(\int_0^{\rho(u_{n-1},u_n)}\kappa(t)dt\right)\right)=1.$$

This implies that

$$\lim_{n\to\infty}\psi\left(\int_0^{\rho(u_{n-1},u_n)}\kappa(t)dt\right)=0$$

and

$$\lim_{n\to\infty}\int_0^{\rho(u_{n-1},u_n)}\kappa(t)dt=0,$$

which is a contradiction with $\rho(E,F)<p$. Hence $p=\rho(E,F)$ and

$$\rho(u_n,Tu_n)\to\rho(E,F), \tag{2.15}$$

as $n\to\infty$. Let $\{u_{2n_k}\}$ be a subsequence of $\{u_{2n}\}$ and

$$\lim_{k\to\infty}u_{2n_k}=y_0. \tag{2.16}$$

Since $\{u_{2n_k-1}\}$ is a sequence in F then we obtain

$$\int_0^{\rho(E,F)}\kappa(t)dt\le\int_0^{\rho(y_0,u_{2n_k-1})}\kappa(t)dt\le\int_0^{\rho(y_0,u_{2n_k})+\rho(u_{2n_k},u_{2n_k-1})}\kappa(t)dt\le\int_0^{\rho(y_0,u_{2n_k})}\kappa(t)dt+\int_0^{\rho(u_{2n_k-1},Tu_{2n_k-1})}\kappa(t)dt,$$

for all $k>0$. If we take limit for $k\to\infty$ then using the condition (2.15), we have

$$\lim_{k\to\infty}\int_0^{\rho(y_0,u_{2n_k-1})}\kappa(t)dt=\int_0^{\rho(E,F)}\kappa(t)dt. \tag{2.17}$$

Now we get

$$\begin{aligned}
&\psi\left(\int_0^{\rho(y_0,Ty_0)}\kappa(t)dt\right)\le\psi\left(\int_0^{\rho(y_0,u_{2n_k})}\kappa(t)dt\right)+\psi\left(\int_0^{\rho(Tu_{2n_k-1},Ty_0)}\kappa(t)dt\right)\\
&\le\psi\left(\int_0^{\rho(y_0,u_{2n_k})}\kappa(t)dt\right)+\alpha(u_{2n_k-1},y_0)\,\psi\left(\int_0^{\rho(Tu_{2n_k-1},Ty_0)}\kappa(t)dt\right)\\
&\le\theta\left(\psi\left(\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt\right)\right)\psi\left(\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt\right)+\left(1-\theta\left(\psi\left(\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt\right)\right)\right)\int_0^{\rho(E,F)}\kappa(t)dt+\psi\left(\int_0^{\rho(y_0,u_{2n_k})}\kappa(t)dt\right)\\
&=\theta\left(\psi\left(\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt\right)\right)\left(\psi\left(\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt\right)-\int_0^{\rho(E,F)}\kappa(t)dt\right)+\int_0^{\rho(E,F)}\kappa(t)dt+\psi\left(\int_0^{\rho(y_0,u_{2n_k})}\kappa(t)dt\right).
\end{aligned} \tag{2.18}$$

Using the conditions (2.15) and (2.18) we obtain

$$\lim_{k\to\infty}\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt=\lim_{k\to\infty}\int_0^{\max\{\rho(u_{2n_k-1},y_0),\rho(u_{2n_k-1},u_{2n_k}),\rho(y_0,Ty_0)\}}\kappa(t)dt=\int_0^{\rho(y_0,Ty_0)}\kappa(t)dt. \tag{2.19}$$

We assume that $\rho(y_0,Ty_0)>\rho(E,F)$. If we take limit for $k\to\infty$ then using the conditions (2.18) and (2.19) we have

$$\begin{aligned}&\psi\left(\int_0^{\rho(y_0,Ty_0)}\kappa(t)dt\right)-\int_0^{\rho(E,F)}\kappa(t)dt\\ &\le\lim_{k\to\infty}\theta\left(\psi\left(\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt\right)\right)\left(\psi\left(\int_0^{\rho(y_0,Ty_0)}\kappa(t)dt\right)-\int_0^{\rho(E,F)}\kappa(t)dt\right)\le\psi\left(\int_0^{\rho(y_0,Ty_0)}\kappa(t)dt\right)-\int_0^{\rho(E,F)}\kappa(t)dt\end{aligned}$$

and so

$$\lim_{k\to\infty}\theta\left(\psi\left(\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt\right)\right)=1.$$

Since $\theta\in\mathscr{S}$, we get

$$\lim_{k\to\infty}\psi\left(\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt\right)=0$$

and

$$\lim_{k\to\infty}\int_0^{M_1(u_{2n_k-1},y_0)}\kappa(t)dt=0,$$

which implies that

$$\lim_{k\to\infty}M_1(u_{2n_k-1},y_0)=0,$$

that is, $y_0=Ty_0$, which is a contradiction with our assumption. Therefore we get

$$\rho(y_0,Ty_0)=\rho(E,F).$$

Consequently, $\{u_{2n_k}\}$ converges to a proximity point. □

Theorem 2.4. *Let the function $\kappa:[0,\infty)\to[0,\infty)$ be defined as in (2.1) and $T:E\cup F\to E\cup F$ a self-mapping satisfying:*

1. *$T(E)\subseteq F$ and $T(F)\subseteq E$,*
2. $$\alpha(u,v)\psi\left(\int_0^{\rho(Tu,Tv)}\kappa(t)dt\right)\le\theta\left(\psi\left(\int_0^{M_2(u,v)}\kappa(t)dt\right)\right)\psi\left(\int_0^{M_2(u,v)}\kappa(t)dt\right)+\left(1-\theta\left(\psi\left(\int_0^{M_2(u,v)}\kappa(t)dt\right)\right)\right)\int_0^{\rho(E,F)}\kappa(t)dt,$$

   where $u\in E$, $v\in F$, $\theta\in\mathscr{S}$, $\alpha:X\times X\to[1,\infty)$ and $\psi\in\Psi$.

Let $u_0\in E$ be any element and the sequence $\{u_n\}$ defined as $u_{n+1}=Tu_n$ for all $n\ge 0$. Then $\rho(u_n,Tu_n)\to\rho(E,F)$. If $\{u_{2n}\}$ has a convergent subsequence in E, the subsequence converges to a proximity point.

Proof. The proof is similar to the proof of Theorem 2.3 since

$$M_2(u_{n-1},u_n)=\max\{\rho(u_{n-1},u_n),\rho(u_{n+1},u_n)\}.$$

□

In the following theorem, we obtain another best proximity result using the class Ψ and two functions $\alpha,\beta:X\times X\to[1,\infty)$.

Theorem 2.5. *Let the function $\kappa:[0,\infty)\to[0,\infty)$ be defined as in (2.1) and $T:E\cup F\to E\cup F$ a self-mapping satisfying:*

1. *$T(E)\subseteq F$ and $T(F)\subseteq E$,*
2. $$\alpha(u,Tu)\beta(v,Tv)\psi\left(\int_0^{\rho(Tu,Tv)}\kappa(t)dt\right)\le\theta\left(\psi\left(\int_0^{M_1(u,v)}\kappa(t)dt\right)\right)\psi\left(\int_0^{M_1(u,v)}\kappa(t)dt\right)+\left(1-\theta\left(\psi\left(\int_0^{M_1(u,v)}\kappa(t)dt\right)\right)\right)\int_0^{\rho(E,F)}\kappa(t)dt,$$

   where $u\in E$, $v\in F$, $\theta\in\mathscr{S}$, $\alpha,\beta:X\times X\to[1,\infty)$ and $\psi\in\Psi$.

Let $u_0\in E$ be any element and the sequence $\{u_n\}$ defined as $u_{n+1}=Tu_n$ for all $n\ge 0$. Then $\rho(u_n,Tu_n)\to\rho(E,F)$. If $\{u_{2n}\}$ has a convergent subsequence in E, the subsequence converges to a proximity point.

Proof. The proof is obtained by the similar arguments used in the proof of Theorem 2.3. □

If we consider the function M_2 instead of the function M_1 in Theorem 2.5 then we get the following theorem.

Theorem 2.6. *Let the function $\kappa:[0,\infty)\to[0,\infty)$ be defined as in (2.1) and $T:E\cup F\to E\cup F$ a self-mapping satisfying*:

1. $T(E)\subseteq F$ *and* $T(F)\subseteq E$,
2. $$\alpha(u,Tu)\beta(v,Tv)\psi\left(\int_0^{\rho(Tu,Tv)}\kappa(t)dt\right)\le\theta\left(\psi\left(\int_0^{M_2(u,v)}\kappa(t)dt\right)\right)\psi\left(\int_0^{M_2(u,v)}\kappa(t)dt\right)+\left(1-\theta\left(\psi\left(\int_0^{M_2(u,v)}\kappa(t)dt\right)\right)\right)\int_0^{\rho(E,F)}\kappa(t)dt,$$
   where $u\in E$, $v\in F$, $\theta\in\mathscr{S}$, $\alpha,\beta:X\times X\to[1,\infty)$ and $\psi\in\Psi$.

Let $u_0\in E$ be any element and the sequence $\{u_n\}$ defined as $u_{n+1}=Tu_n$ for all $n\ge 0$. Then $\rho(u_n,Tu_n)\to\rho(E,F)$. If $\{u_{2n}\}$ has a convergent subsequence in E, the subsequence converges to a proximity point.

Proof. The proof follows easily by the similar arguments used in the proof of Theorem 2.3. □

Finally, we give an example of our best proximity results.

Example 2.7. *Let $X=[0,1]\times[0,1]$ be a metric space with the metric*

$$\rho(u,v)=\sum_{i=1}^{n}|u_i-v_i|,$$

where $u=(u_1,u_2)$, $v=(v_1,v_2)$. Let $E=\{(0,0),(0,1)\}$ and $F=\{(1,1),(1,0)\}$. Then we can see that E and F are closed subsets of X and $\rho(E,F)=1$. Let $\theta:[0,\infty)\to[0,1)$ be defined as

$$\theta(t)=\begin{cases}\frac{2}{2+t} & ;\quad if\ t>0\\ k & ;\quad if\ t=0\end{cases},$$

where $k<1$. Let us define the self-mapping $T:E\cup F\to E\cup F$ as:

$$\begin{aligned}T((0,0)) &= T((0,1))=(1,0),\\ T((1,1)) &= T((1,0))=(0,0).\end{aligned}$$

Let $\kappa:[0,\infty)\to[0,\infty)$ be defined as

$$\kappa(t)=2t,$$

for all $t\in[0,\infty)$. Then all the conditions of Theorem 2.1 are satisfied. Indeed, we have $T(E)\subseteq F$ and $T(F)\subseteq E$ by the definition of T. Now, we show that the inequality (2.2) is satisfied for each $u\in E$, $v\in F$ in the following cases:
Case 1 : Let $u=(0,0)$ and $v=(1,1)$. Then we get

$$\begin{aligned}\rho(Tu,Tv) &= \rho((1,0),(0,0))=1,\\ M_1(u,v) &= 2\end{aligned}$$

and

$$\int_0^1 2tdt = 1\le\theta\left(\int_0^2 2tdt\right)\int_0^2 2tdt+\left(1-\theta\left(\int_0^2 2tdt\right)\right)\int_0^1 2tdt=4\theta(4)+1-\theta(4)=2.$$

Case 2 : Let $u=(0,0)$ and $v=(1,0)$. Then we get

$$\begin{aligned}\rho(Tu,Tv) &= \rho((1,0),(0,0))=1,\\ M_1(u,v) &= 2\end{aligned}$$

and

$$\int_0^1 2tdt = 1\le\theta\left(\int_0^1 2tdt\right)\int_0^1 2tdt+\left(1-\theta\left(\int_0^1 2tdt\right)\right)\int_0^1 2tdt=\theta(1)+1-\theta(1)=1.$$

Case 3 : Let $u=(0,1)$ and $v=(1,1)$. Then we get

$$\begin{aligned}\rho(Tu,Tv) &= \rho((1,0),(0,0))=1,\\ M_1(u,v) &= 2\end{aligned}$$

and

$$\int_0^1 2tdt=1\le 2.$$

Case 4 : *Let* $u=(0,1)$ *and* $v=(1,0)$. *Then we get*

$$\begin{aligned} \rho(Tu,Tv) &= \rho((1,0),(0,0))=1, \\ M_1(u,v) &= 2 \end{aligned}$$

and

$$\int_0^1 2t dt = 1 \leq 2.$$

Consequently, $(0,0)$ *is a best proximity point of the self-mapping* T.

In Example 2.7, we note that if we take the functions $\alpha : X\times X \to [1,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ as

$$\alpha(u,v)=1 \text{ for all } (u,v)\in X\times X$$

and

$$\psi(t)=t \text{ for all } t\in[0,\infty),$$

respectively, then all the conditions of Theorem 2.3 are satisfied. Similarly, if we take the functions $\alpha,\beta : X\times X\to[1,\infty)$ and $\psi : [0,\infty)\to[0,\infty)$ as

$$\alpha(u,v)=\beta(u,v)=1 \text{ for all } (u,v)\in X\times X$$

and

$$\psi(t)=t \text{ for all } t\in[0,\infty),$$

respectively, then all the conditions of Theorem 2.5 are satisfied.
If we consider $M_2(u,v)$ instead of $M_1(u,v)$ in Example 2.7, then all the conditions of Theorem 2.2 (resp. Theorem 2.4 and Theorem 2.6) are satisfied. Consequently, $(0,0)$ is a best proximity point of the self-mapping T defined in Example 2.7.

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