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Modified Block-Pulse Functions Scheme for Solve of Two-Dimensional Stochastic Integral Equations

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Abstract

In this paper, two-dimensional modified block-pulse functions (2D-MBPFs) method is introduced for approximate solution of 2D-linear stochastic Volterra-Fredholm integral equations so the ordinary and stochastic operrational matrices of integration are utilized to reduce the computation of such equations into some algebraic equations. Convergence analysis of this method is discussed. Finally an illustrative example is given to show the accuracy of the proposed method so the results of it is compared with the block-pulse functions (BPFs) method.

1. Introduction

Mainly 2D-integral equations furnish the important implement for modeling the engineering and science problems [1, 2]. We have used the variant methods for solving 2D-linear stochastic integral equations in [3, 4, 5, 6, 7] that the BPFs method is one of these methods. The BPFs are very common in use, but it seems that their convergence is weak. Here the modified block-pulse functions (MBPFs) method is used for deriving approximation solution of 2D-linear stochastic Volterra-Fredholm integral equation of the second kind

$$g(x,y) = f(x,y) + \int_0^1 \int_0^1 V_1(x,y,s,t)g(s,t)dsdt + \int_0^y \int_0^x V_2(x,y,s,t)g(s,t)dsdt + \int_0^y \int_0^x V_3(x,y,s,t)g(s,t)dB(s)dB(t), \tag{1.1}$$

where $(x,y) \in [0,T_1) \times [0,T_2)$ and

$$s \leqslant x < t \leqslant y. \tag{1.2}$$

In (1.1), $g(x,y)$ is the unknown function and the condition (1.2) is necessary.
We organize the paper as follows:
The properties of 2D-MBPFs are introduced in the next section. In Section 3 we solve (1.1) by finding the ordinary and stochastic operational matrices. We depict the error analysis in Section 4. The certitude of the method is evinced by an example in Section 5. Eventually, we afford the brief conclusion in Section 6.

2. Two dimentional MBPFs

An $(n_1+1)\times(n_2+1)$-set of 2D-MBPFs $\omega_{a_1,a_2}(x,y)$ $(a_1 = 0,1,\dots,n_1)$; $(a_2 = 0,1,\dots,n_2)$ consists of $(n_1+1)\times(n_2+1)$ functions which are defined over district D by [8]

$$\omega_{a_1,a_2}(x,y) = \begin{cases} 1, & (x,y) \in D_{a_1,a_2} \\ 0, & otherwise, \end{cases} \tag{2.1}$$

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where

$$D_{a_1,a_2} = (x,y) : x \in I_{a_1,\varepsilon}, y \in I_{a_2,\varepsilon},$$

and

$$I_{a_1,\varepsilon} = \begin{cases} [0,k_1-\varepsilon), & a_1 = 0 \\ [a_1k_1-\varepsilon,(a_1+1)k_1-\varepsilon), & a_1 = 1(1)(n_1-1) \\ [1-\varepsilon,1) & a_1 = n_1, \end{cases}$$

$$I_{a_2,\varepsilon} = \begin{cases} [0,k_2-\varepsilon), & a_2 = 0 \\ [a_2k_2-\varepsilon,(a_2+1)k_2-\varepsilon), & a_2 = 1(1)(n_2-1) \\ [1-\varepsilon,1) & a_2 = n_1, \end{cases}$$

where n_1 and n_2 are arbitary positive integers and we have

$$k_1 = \frac{T_1}{n_1} \ , \ k_2 = \frac{T_2}{n_2}.$$

From (2.1), we can represent 2D-MBPFs as

$$\omega_{a_1,a_2}(x,y) = \omega_{a_1}(x)\omega_{a_2}(y),$$

where ω_{a_1} and ω_{a_2} are the one-dimensional MBPFs. Similar to the one-dimensional case, 2D-MBPFs have the elementary properties that are: disjointness, orthogonality and completeness. Also the set of 2D-MBPFs can be written as a vector $\Omega(x,y)$ of dimension $\varsigma_1 = (n_1+1)(n_2+1)\times 1$ as

$$\Omega(x,y) = [\omega_{0,0}(x,y),...,\omega_{0,n_2}(x,y),...,\omega_{n_1,0}(x,y),...,\omega_{n_1,n_2}(x,y)]^T, \tag{2.2}$$

where $(x,y) \in D$. For every ς_1-vector K from (2.2) we have

$$\Omega(x,y)\Omega^T(x,y)K = \tilde{K}\Omega(x,y), \tag{2.3}$$

where $\tilde{K} = diag(K)$ is a diagonal matrix of dimension $\varsigma_2 = (n_1+1)(n_2+1)\times(n_1+1)(n_2+1)$. Moreover, for every ς_2-matrix H we get

$$\Omega^T(x,y)H\Omega(x,y) = \hat{H}^T\Omega(x,y), \tag{2.4}$$

where $\hat{H}$ is an ς_1-vector with elements equal to the diagonal entries of matrix H.

2.1. Two dimensional MBPFs expansions

A function $f(x,y)$ defined over $L^2(D)$ can be expanded by the 2D-MBPFs as [8, 9]

$$f \simeq f_\varepsilon = \sum_{a_1=0}^{n_1}\sum_{a_2=0}^{n_2} f_{a_1,a_2}\omega_{a_1,a_2} = F^T\Omega,$$

where F is an ς_1-vector given by

$$F = [f_{0,0},...,f_{0,n_2},...,f_{n_1,0},...,f_{n_1,n_2}]^T,$$

and Ω is defined in (2.2). The modified block-pulse coefficients, f_{a_1,a_2} , are obtained as

$$f_{a_1,a_2} = \frac{1}{\ell(I_{a_1,\varepsilon})\times\ell(I_{a_2,\varepsilon})}\int_{I_{a_1,\varepsilon}}\int_{I_{a_2,\varepsilon}} f(x,y)dydx,$$

where $\ell(I_{a_1,\varepsilon})$ and $\ell(I_{a_2,\varepsilon})$ are length of intervals $I_{a_1,\varepsilon}$ and $I_{a_2,\varepsilon}$ respectively. Similarly for every function $f(x,y,s,t)$, we can write

$$f(x,y,s,t) \simeq \Omega(x,y)^T F_\varepsilon \Omega(s,t),$$

where F_ε is 2D-MBPF coefficient matrix of dimension ς_2.

2.2. Ordinary perational matrix of 2D-MBPFs

By the double integration of the vector Ω defined in (2.2) we have [8, 10, 11]

$$\int_0^y\int_0^x \Omega(s,t)dsdt \simeq P_\varepsilon\Omega(x,y) = [O_{\varepsilon,(n_1+1)\times(n_1+1)} \otimes O_{\varepsilon,(n_2+1)\times(n_2+1)}]\Omega(x,y), \tag{2.5}$$

where $(x,y)\in D$ and P_ε is ς_2-ordinary operational matrix of integration for 2D-MBPFs so O_ε is defined in [5]. In (2.5), $\otimes$ denotes the Kronecker product. By disjointness and orthogonality properties of 2D-MBPFs we have

$$\int_0^1\int_0^1 \Omega(s,t)\Omega^T(s,t)dsdt = \begin{pmatrix} (k_1-\varepsilon)(k_2-\varepsilon) & 0 & 0 & ... & 0 \\ 0 & k_1k_2 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & k_1k_2 & 0 \\ 0 & 0 & ... & 0 & \varepsilon \end{pmatrix} = R_\varepsilon, \tag{2.6}$$

where R_ε is the ς_2-known matrix.

2.3. Stochastic operational matrix of 2D-MBPFs

Similarly we obtain

$$\int_0^y \int_0^x \Omega(s,t)dB(s)dB(t) \simeq P_{\varepsilon,s}\Omega(x,y) = [O_{\varepsilon,s,(n_1+1)\times(n_1+1)} \otimes O_{\varepsilon,s,(n_2+1)\times(n_2+1)}]\Omega(x,y), \tag{2.7}$$

where $P_{\varepsilon,s}$ is the ς_2-stochastic operational matrix of integration for 2D-MBPFs where $O_{\varepsilon,s}$ is defined in [3]. In the next sections, it is assumed that $T_1 = T_2 = 1$.

3. Method of solution

Now, we solve (1.1) using 2D-MBPFs. By applying 2D-MBPFs approximates for functions

$$f(x,y)\ ,\ V_1(x,y,s,t)\ ,\ V_2(x,y,s,t)\ ,\ V_3(x,y,s,t)\ ,\ g(x,y),$$

we have

$$f = F_\varepsilon^T\Omega, \tag{3.1}$$

$$V_1 = \Omega^T(x,y)\Gamma_\varepsilon\Omega(s,t), \tag{3.2}$$

$$V_2 = \Omega^T(x,y)\Delta_\varepsilon\Omega(s,t), \tag{3.3}$$

$$V_3 = \Omega^T(x,y)\Theta_\varepsilon\Omega(s,t), \tag{3.4}$$

and

$$g = G_\varepsilon^T\Omega, \tag{3.5}$$

where the vectors F_ε and G_ε and matrices Γ_ε, Δ_ε and Θ_ε are the MBPFs coefficients of f, g, V_1, V_2 and V_3 respectively. In (3.1), F_ε is ς_1-known vector, also in (3.2), (3.3) and (3.4), Γ_ε, Δ_ε and Θ_ε are ς_2-known matrices but in (3.5), G_ε is ς_1-unknown vector. In (1.1), To approximate Fredholm integral case from (3.2), (3.5) and using operational matrix R_ε from (2.6) we get

$$\begin{aligned}
\int_0^1\int_0^1 V_1 g\,dsdt &= \int_0^1\int_0^1 \Omega^T(x,y)\Gamma_\varepsilon\Omega(s,t)\Omega^T(s,t)G_\varepsilon dsdt \\
&= \Omega^T(x,y)\Gamma_\varepsilon\left(\int_0^1\int_0^1 \Omega(s,t)\Omega^T(s,t)dsdt\right)G_\varepsilon \\
&= \Omega^T\Gamma_\varepsilon R_\varepsilon G_\varepsilon = (\Gamma_\varepsilon R_\varepsilon G_\varepsilon)^T\Omega = U_\varepsilon^T\Omega,
\end{aligned}$$

where U_ε is an ς_1-vector obtained from $\Gamma_\varepsilon R_\varepsilon G_\varepsilon$. Therefore for the approximation of the first 2D-integral we have

$$\int_0^1\int_0^1 V_1 g\,dsdt \simeq U_\varepsilon^T\Omega. \tag{3.6}$$

In addition from (2.3), (3.3) and (3.5) we get [12]

$$\begin{aligned}
\int_0^y\int_0^x V_2 g\,dsdt \simeq \int_0^y\int_0^x \Omega^T(x,y)\Delta_\varepsilon\Omega(s,t)\Omega^T(s,t)G_\varepsilon dsdt &= \Omega^T(x,y)\Delta_\varepsilon\left(\int_0^y\int_0^x \Omega(s,t)\Omega^T(s,t)G_\varepsilon dsdt\right) \\
&= \Omega^T\Delta_\varepsilon\left(\int_0^y\int_0^x \tilde{G}_\varepsilon\Omega(s,t)dsdt\right) = \Omega^T\Delta_\varepsilon\tilde{G}_\varepsilon\left(\int_0^y\int_0^x \Omega(s,t)dsdt\right),
\end{aligned}$$

where from (2.5) we arrive

$$\int_0^y\int_0^x V_2 g\,dsdt \simeq \Omega^T\Delta_\varepsilon\tilde{G}_\varepsilon P_\varepsilon\Omega,$$

in which $\Delta_\varepsilon\tilde{G}_\varepsilon P_\varepsilon$ is an ς_2-matrix. From (2.4) we can write

$$\int_0^y\int_0^x V_2 g\,dsdt \simeq \hat{W}_\varepsilon^T\Omega, \tag{3.7}$$

where $\hat{W}_\varepsilon$ is an ς_1-vector with components equal to the diagonal entries of matrix $\Delta_\varepsilon\tilde{G}_\varepsilon P_\varepsilon$. Similarly from (2.3), (3.4) and (3.5) we conclude

$$\begin{aligned}
\int_0^y\int_0^x V_3 g\,dB(s)dB(t) &\simeq \int_0^y\int_0^x \Omega^T(x,y)\Theta_\varepsilon\Omega(s,t)\Omega^T(s,t)G_\varepsilon dB(s)dB(t) = \Omega^T(x,y)\Theta_\varepsilon\left(\int_0^y\int_0^x \Omega(s,t)\Omega^T(s,t)G_\varepsilon dB(s)dB(t)\right) \\
&= \Omega^T\Theta_\varepsilon\left(\int_0^y\int_0^x \tilde{G}_\varepsilon\Omega(s,t)dB(s)dB(t)\right) = \Omega^T\Theta_\varepsilon\tilde{G}_\varepsilon\left(\int_0^y\int_0^x \Omega(s,t)dB(s)dB(t)\right),
\end{aligned}$$

by using (2.7) we can arrive

$$\int_0^y\int_0^x V_3 g\,dB(s)dB(t) \simeq \Omega^T\Theta_\varepsilon\tilde{G}_\varepsilon P_{\varepsilon,s}\Omega,$$

in which $\Theta_\varepsilon \tilde{G}_\varepsilon P_{\varepsilon,s}$ is an ς_2-matrix. From (2.4) we can write

$$\int_0^y \int_0^x V_3 g dB(s) dB(t) \simeq \hat{W}_{\varepsilon,s}^T \Omega, \tag{3.8}$$

where $\hat{W}_{\varepsilon,s}$ is an ς_1-vector with components equal to the diagonal entries of matrix $\Theta_\varepsilon \tilde{G}_\varepsilon P_{\varepsilon,s}$. Applying (3.1), (3.5), (3.6), (3.7) and (3.8) in (1.1) give

$$G_\varepsilon^T \Omega \simeq F_\varepsilon^T \Omega + \hat{U}_\varepsilon^T \Omega + \hat{W}_\varepsilon^T \Omega + \hat{W}_{\varepsilon,s}^T \Omega. \tag{3.9}$$

By replacing $\simeq$ with $=$, in (3.9) we can get

$$G_\varepsilon - \hat{U}_\varepsilon - \hat{W}_\varepsilon - \hat{W}_{\varepsilon,s} = F_\varepsilon, \tag{3.10}$$

where after solving System (3.10), we can find G_ε and get

$$g = G_\varepsilon^T \Omega.$$

Then

$$g \simeq g_\varepsilon = \frac{1}{\mu} \sum_{i=0}^{\mu-1} g_{\varepsilon_i},$$

where $\varepsilon_i = \dfrac{ik}{\mu}$, $i = 0(1)(\mu-1)$ is the estimation of the solution of (1.1) and μ is a positive integer.

4. Error analysis

In this section, we show that the convergence order of the proposed method is $\dfrac{1}{\mu n}$ by introducing several theorems. For convenience, we put $n_1 = n_2 = n$, so $k_1 = k_2 = \dfrac{1}{n}$.

Theorem 4.1. *Suppose that h is a differentiable function from $S \subset R^2$ into R, and for every $t \in S$*

$$\|h'\|_2 \le \xi,$$

where $\xi \in R$. Then

$$|h(d) - h(c)| \le \xi |d - c|,$$

for all $c, d \in S$.

Proof. See [10]. □

Theorem 4.2. *Assume that*

$$F_{n,\varepsilon} = \sum_{a=0}^{n} \sum_{b=0}^{n} \sum_{c=0}^{n} \sum_{d=0}^{n} F_{a,b,c,d} \omega_{a,b,c,d},$$

and

$$F_{a,b,c,d} = \frac{1}{\ell(I_{a,\varepsilon})\ell(I_{b,\varepsilon})\ell(I_{c,\varepsilon})\ell(I_{d,\varepsilon})} \times \int_0^1 \int_0^1 \int_0^1 \int_0^1 F \omega_{a,b,c,d} dt ds dy dx,$$

where $a, b, c, d = 0, 1, \dots, n$. Then the mean square error between F and $F_{n,\varepsilon}$ on $(x, y, s, t) \in D_{a,b,c,d}$ reaches its minimum, moreover we have

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 F^2 dt ds dy dx = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} F_{a,b,c,d}^2 \|\omega_{a,b,c,d}\|_2^2.$$

Proof. By using [11], we can easily prove this theorem. □

Theorem 4.3. *Assume f is continuous and differentiable over district $[-k, 1+k] \times [-k, 1+k]$ and $f_{n,\varepsilon}$; $\varepsilon_i = \dfrac{ik}{\mu}$ for $i = 0, 1, \dots, \mu - 1$ are correspondingly $2D-MBPFs(\varepsilon_0) = 2D-BPFs$, $2D-MBPFs(\varepsilon_1)$, ..., $2D-MBPFs(\varepsilon_{\mu-1})$ expansions of f based on $(n+1)^2$ 2D-MBPFs over district $[0,1) \times [0,1)$ and $\bar{f}_{n,\mu}(x,y) = \dfrac{1}{\mu} \sum_{i=0}^{\mu-1} f_{n,\varepsilon_i}(x,y)$, then for sufficient large n we have*

$$\|e_\varepsilon\|_2 \le \frac{\sqrt{2}N}{\mu n},$$

therefore

$$\|e_\varepsilon\|_2 = O\left(\frac{1}{\mu n}\right),$$

where N is bounded of $\|Df\|_2$.

Proof. See [13]. □

Theorem 4.4. *If F be an enough smooth function on $S=[0,1)^4$ with $\|F\|_2 \leq M$. Let*

$$\hat{F}_n = \hat{F}_{n,\varepsilon_0} = \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n} F_{a,b,c,d}\,\omega_{a,b,c,d},$$

be $4D-MBPFs(\varepsilon_0) = 4D-BPFs$ expansion of F and

$$e = F - \hat{F}_n,$$

then

$$\|e_\varepsilon\|_2 = O(\frac{1}{n}).$$

Proof. We have

$$e_{a,b,c,d} = F - F_{a,b,c,d}\phi_{a,b,c,d} = F - F_{a,b,c,d},$$

where ϕ is the set of 4D-BPFs of dimension $n_1n_2n_3n_4$ and

$$S_{a,b,c,d} = \{\frac{a-1}{n} \leq x < \frac{a}{n}, \frac{b-1}{n} \leq y < \frac{b}{n}, \frac{c-1}{n} \leq s < \frac{c}{n}, \frac{d-1}{n} \leq t < \frac{d}{n}\},$$

and $(x,y,s,t) \in S_{a,b,c,d}$. By using the mean value theorem we get

$$\|e_{a,b,c,d}\|_2^2 = \int_{(a-1)/n}^{a/n}\int_{(b-1)/n}^{b/n}\int_{(c-1)/n}^{c/n}\int_{(d-1)/n}^{d/n} \left(F - F_{a,b,c,d}\right)^2 dtdsdydx$$

$$= \frac{1}{n^4}\left(F(\gamma_1,\gamma_2,\gamma_3,\gamma_4) - F_{a,b,c,d}\right)^2; \qquad (\gamma_1,\gamma_2,\gamma_3,\gamma_4) \in S_{a,b,c,d}. \tag{4.1}$$

We know

$$F_{a,b,c,d} = \frac{1}{k^4}\int_{(a-1)k}^{ak}\int_{(b-1)k}^{bk}\int_{(c-1)k}^{ck}\int_{(d-1)k}^{dk} F_{a,b,c,d}\,dtdsdydx,$$

therefore by using mean value theorem we have

$$F_{a,b,c,d} = n^4 \times \frac{1}{n^4} \times F(\theta_1,\theta_2,\theta_3,\theta_4); \qquad (\theta_1,\theta_2,\theta_3,\theta_4) \in S_{a,b,c,d}. \tag{4.2}$$

From Theorem 4.1 and involving (4.2) into (4.1) we obtain

$$\|e_{a,b,c,d}\|_2^2 = \frac{1}{n^4}\left(V(\gamma_1,\gamma_2,\gamma_3,\gamma_4) - V(\theta_1,\theta_2,\theta_3,\theta_4)\right)^2 \leq \frac{1}{n^4} \times 4k^2 \times M^2 = \frac{4M^2}{n^6}. \tag{4.3}$$

So

$$\begin{aligned}\|e\|_2^2 &= \int_0^1\int_0^1\int_0^1\int_0^1 e^2 dtdsdydx \\ &= \int_0^1\int_0^1\int_0^1\int_0^1 \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n} e_{abcd}^2 dtdsdydx + 2\sum_{a<a'}^{n}\sum_{b<b'}^{n}\sum_{c<c'}^{n}\sum_{d<d'}^{n}\int_0^1\int_0^1\int_0^1\int_0^1 e_{abcd} \times e_{a'b'c'd'} dtdsdydx.\end{aligned}$$

Since for $a<a', b<b', c<c'$ and $d<d'$ we have

$$S_{a,b,c,d} \cap S_{a',b',c',d'} = \varnothing,$$

where (4.3) give

$$\|e\|_2^2 = \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n} \|e_{abcd}\|_2^2 \leq n^4 \times \frac{4M^2}{n^6} = \frac{1}{n^2} \times 4M^2,$$

namely

$$\|e\|_2 = O(\frac{1}{n}).$$

□

Theorem 4.5. *Assume $F(x,y,s,t)$ is continuous and differentiable over district $[-k,1+k]\times[-k,1+k]\times[-k,1+k]\times[-k,1+k]$, moreover suppose $F_{n,\varepsilon_i}(x,y,s,t)$; $\varepsilon_i = \frac{ik}{\mu}$ for $i=0,1,...,\mu-1$ are correspondingly $4D-MBPFs(\varepsilon_0) = 4D-BPFs$, $4D-MBPFs(\varepsilon_1)$, ..., $4D-MBPFs(\varepsilon_{\mu-1})$ expansions of F based on $(n+1)^4$ 4D-MBPFs over district $[0,1)^4$ and*

$$\bar{F}_{n,\mu} = \frac{1}{\mu}\sum_{i=0}^{\mu-1} F_{n,\varepsilon_i},$$

then for sufficient large values n

$$\|F - \bar{F}_{n,\mu}\|_\infty \lesssim \frac{1}{\mu}\max_{\varepsilon_i}\|F - \bar{F}_{n,\varepsilon_i}\|_\infty.$$

Proof. We consider partial differentials

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t},$$

in $D^4 = [\frac{i-1}{n}, \frac{i+1}{n})^4$ which are approximately equal to constants A_1, A_2, A_3 and A_4 respectively, where n is so large. Also we use function, $z = A_1x + A_2y + A_3s + A_4t + B$ instead of F in D^4. Now in the district $[\frac{i}{n}, \frac{i}{n} + \varepsilon_1)^4$ we have

$$\begin{aligned}
\bar{F}_{n,\mu}(x,y,s,t) &= \frac{1}{\mu}\sum_{j=1}^{\mu-1}\frac{1}{16}\times[(A_1+A_2+A_3+A_4)\rho_1+B+(A_1+A_2+A_3)\rho_1+A_4\rho_2+B+(A_1+A_2+A_4)\rho_1+A_3\rho_2+B+(A_1+A_2)\rho_1 \\
&+ (A_3+A_4)\rho_2+B+(A_1+A_3+A_4)\rho_1+A_2\rho_2+B+(A_1+A_3)\rho_1+(A_2+A_4)\rho_2+B+(A_1+A_4)\rho_1+(A_2+A_3)\rho_2+B+A_1\rho_1 \\
&+ (A_2+A_3+A_4)\rho_2+B+A_1\rho_2+(A_2+A_3+A_4)\rho_1+B+(A_1+A_4)\rho_2+(A_2+A_3)\rho_1+B+(A_1+A_3)\rho_2+(A_2+A_4)\rho_1+B \\
&+ (A_1+A_3+A_4)\rho_2+A_2\rho_1+B+(A_1+A_2)\rho_2+(A_3+A_4)\rho_1+B+(A_1+A_2+A_4)\rho_2+A_3\rho_1+B \\
&+ (A_1+A_2+A_3)\rho_2+A_4\rho_1+B+(A_1+A_2+A_3+A_4)\rho_2+B] \\
&= (A_1+A_2+A_3+A_4)(\frac{\frac{i}{n}+\frac{i+1}{n}}{2})+B-\frac{(A_1+A_2+A_3+A_4)k(\mu-1)}{2\mu}, \qquad (4.4)
\end{aligned}$$

where

$$\rho_1 = (\frac{i}{n} - \frac{jk}{\mu}),$$

and

$$\rho_2 = (\frac{i+1}{n} - \frac{jk}{\mu}).$$

Since $\frac{i+1}{n} = \frac{i}{n} + k$, we can reformulete (4.4) as

$$\bar{F}_{n,\mu} = (A_1+A_2+A_3+A_4)\frac{i}{n}+B+\frac{(A_1+A_2+A_3+A_4)k}{2\mu}.$$

Also we have

$$\max_{x,y,s,t\in[\frac{i}{n},\frac{i}{n}+\varepsilon_j)} |F-\bar{F}_{n,\mu}| \simeq \max_{x,y,s,t\in[\frac{i}{n},\frac{i}{n}+\varepsilon_j)} |A_1x+A_2y+A_3s+A_4t+B-\bar{F}_{n,\mu}| = \frac{(A_1+A_2+A_3+A_4)k}{2\mu}. \qquad (4.5)$$

Therefore, we get

$$\begin{aligned}
&\max_{\substack{\varepsilon_i \\ x,y,s,t\in D}} \|F-F_{n,\varepsilon_i}\|_\infty \geq \max_{\substack{\varepsilon_i \\ x,y,s,t\in D'}} |F-F_{n,\varepsilon_i}| \simeq |(A_1+A_2+A_3+A_4)\varpi_1+B-\frac{1}{16}[(A_1+A_2+A_3+A_4)\varpi_1+B \\
&+(A_1+A_2+A_3)\varpi_1+A_4\varpi_2+B+(A_1+A_2+A_4)\varpi_1+A_3\varpi_2+B+(A_1+A_2)\varpi_1+(A_3+A_4)\varpi_2+B+(A_1++A_3+A_4)\varpi_1+A_2\varpi_2+B \\
&+(A_1+A_3)\varpi_1+(A_2+A_4)\varpi_2+B+(A_1+A_4)\varpi_1+(A_2+A_3)\varpi_2+B+A_1\varpi_1+(A_2+A_3+A_4)\varpi_2+B+A_1\varpi_2+(A_2+A_3+A_4)\varpi_1+B \\
&+(A_1+A_4)\varpi_2+(A_2+A_3)\varpi_1+B+(A_1+A_3)\varpi_2+(A_2+A_4)\varpi_1+B+(A_1+A_3+A_4)\varpi_2+A_2\varpi_1+B+(A_1+A_2)\varpi_2+(A_3+A_4)\varpi_1+B \\
&+(A_1+A_2+A_4)\varpi_2+A_3\varpi_1+B+(A_1+A_2+A_3)\varpi_2+A_4\varpi_1+B+(A_1+A_2+A_3+A_4)\varpi_2+B]| \\
&= \frac{(A_1+A_2+A_3+A_4)}{2}k, \qquad (4.6)
\end{aligned}$$

where $\varpi_1 = \frac{i}{n}$, $\frac{i}{n}+k = \varpi_2$, $D = [\frac{i-1}{n}, \frac{i+1}{n})$ and $D' = [\frac{i}{n}, \frac{i}{n}+k)$. From (4.5) and (4.6) we get

$$\|F-\bar{F}_{n,\mu}\|_\infty \lesssim \frac{1}{\mu}\max_{\varepsilon_i}\|F-\bar{F}_{n,\varepsilon_i}\|_\infty.$$

□

Remark 4.6. *Let*

$$e_{n,\varepsilon} = F - \bar{F}_{n,\mu},$$

and

$$e_n = F - \bar{F}_n,$$

then from Theorem 4.2, Theorem 4.4 and Theorem 4.5 we have

$$\|e_{n,\varepsilon}\|_2 \leqslant \frac{2M}{\mu n},$$

also we can write

$$\lim_{n\to+\infty} F_{n,\varepsilon_i} = F.$$

Theorem 4.7. *If g be the exact solution of (1.1) and $\hat{g}_{n,\mu}(x,y)$ be the 2D-MBPFs approximate solution of it. Also*

(1) $\|g\|_2 \le \alpha \quad , \quad (s,t) \in [0,1)^2,$

(2) $\|V_i\|_2 \le \beta_i \ , \ i = 1,2,3 \ , \ (x,y,s,t) \in [0,1)^4,$

(3) $W_1(x,y) = \sup_{x\in[0,1)} x \times \sup_{y\in[0,1)} y,$

(4) $W_2(x,y) = \sup_{x\in[0,1)} |B(x)| \times \sup_{y\in[0,1)} |B(y)|,$

(5) $\left[\beta_1+\beta_2+\frac{2\beta_1+2\beta_2}{\mu n}+\left(\beta_3+\frac{2\beta_3}{\mu n}\right)\times W_2(x,y)\right] < 1,$

then

$$\|g-\hat{g}_n\|_2 = O(\frac{1}{\mu n}).$$

Proof. From (1.1), we get

$$g-\hat{g}_{n,\mu} = f-\hat{f}_{n,\mu}+\int_0^1\int_0^1 \left(V_1 g-\hat{V}_{1,n,\mu}\hat{g}_{n,\mu}\right) dsdt+\int_0^y\int_0^x \left(V_2 g-\hat{V}_{2,n,\mu}\hat{g}_{n,\mu}\right) dsdt$$
$$+\int_0^y\int_0^x \left(V_3(x,y,s,t)g(s,t)-\hat{V}_{3,n,\mu}\hat{g}_{n,\mu}\right) dB(s)dB(t),$$

so the mean value theorem give

$$\|g-\hat{g}_{n,\mu}\|_2 \le \|f-\hat{f}_{n,\mu}\|_2+\|V_1 g-\hat{V}_{1,n,\mu}\hat{g}_{n,\mu}\|_2+xy\|V_2 g-\hat{V}_{2,n,\mu}\hat{g}_{n,\mu}\|_2+B(x)B(y)\|V_3 g-\hat{V}_{3,n,\mu}\hat{g}_{n,\mu}\|_2. \tag{4.7}$$

By using Remark 4.6 and two first hypothesises, we obtain

$$\begin{aligned}\|V_1 g-\hat{V}_{1,n,\mu}\hat{g}_{n,\mu}\|_2 &\le \|V_1\|_2\|g-\hat{g}_{n,\mu}\|_2+\|V_1-\hat{V}_{1,n,\mu}\|_2\left(\|g-\hat{g}_{n,\mu}\|_2+\|g\|_2\right)\\ &\le \beta_1\|g-\hat{g}_{n,\mu}\|_2+\frac{2\beta_1}{\mu n}\left(\|g-\hat{g}_{n,\mu}\|_2+\alpha\right)=\left(\beta_1+\frac{2\beta_1}{\mu n}\right)\|g-\hat{g}_{\mu n}\|_2+\frac{2\beta_1}{\mu n}\alpha.\end{aligned} \tag{4.8}$$

Similarly we have

$$\|V_2 g-\hat{V}_{2,n,\mu}\hat{g}_{n,\mu}\|_2 = \left(\beta_2+\frac{2\beta_2}{\mu n}\right)\|g-\hat{g}_{n,\mu}\|_2+\frac{2\beta_2}{\mu n}\alpha, \tag{4.9}$$

and

$$\|V_3 g-\hat{V}_{3,n,\mu}\hat{g}_{n,\mu}\|_2 = \left(\beta_3+\frac{2\beta_3}{\mu n}\right)\|g-\hat{g}_{n,\mu}\|_2+\frac{2\beta_3}{\mu n}\alpha. \tag{4.10}$$

Substituting (4.8), (4.9) and (4.10) in (4.7) and Theorem 4.3 conclude

$$\|g-\hat{g}_{n,\mu}\|_2 \le \frac{\sqrt{2}N}{\mu n}+\left[\left(\beta_1+\frac{2\beta_1}{\mu n}\right)\|g-\hat{g}_{n,\mu}\|_2+\frac{2\beta_1}{\mu n}\alpha\right]+xy\left[\left(\beta_2+\frac{2\beta_2}{\mu n}\right)\|g-\hat{g}_{n,\mu}\|_2+\frac{2\beta_2}{\mu n}\alpha\right]$$
$$+B(x)B(y)\left[\left(\beta_3+\frac{2\beta_3}{\mu n}\right)\|g-\hat{g}_{n,\mu}\|_2+\frac{2\beta_3}{\mu n}\alpha\right].$$

By taking sup and Hypothesises 3 and 4, we have

$$\|g-\hat{g}_{n,\mu}\|_2 \le \frac{\sqrt{2}N}{\mu n}+\left[\left(\beta_1+\frac{2\beta_1}{\mu n}\right)\sup_{s\le x\ ,\ t\le y}\|g-\hat{g}_{n,\mu}\|_2+\frac{2\beta_1}{\mu n}\alpha\right]+W_1(x,y)\left[\left(\beta_2+\frac{2\beta_2}{\mu n}\right)\sup_{s\le x\ ,\ t\le y}\|g-\hat{g}_{n,\mu}\|_2+\frac{2\beta_2}{\mu n}\alpha\right]$$
$$+W_2(x,y)\left[\left(\beta_3+\frac{2\beta_3}{\mu n}\right)\sup_{s\le x\ ,\ t\le y}\|g-\hat{g}_{n,\mu}\|+\frac{2\beta_3}{\mu n}\alpha\right],$$

so

$$\|g-\hat{g}_{n,\mu}\|_2 \le \frac{\frac{\sqrt{2}N+2\beta_1\alpha+2\beta_2\alpha}{\mu n}+\frac{2\beta_3\alpha}{\mu n}\times W_2(x,y)}{1-\left[\beta_1+\beta_2+\frac{2\beta_1+2\beta_2}{\mu n}+\left(\beta_3+\frac{2\beta_3}{\mu n}\right)\times W_2(x,y)\right]},$$

and from the boundedness of Brownian motion we get

$$\|g-\hat{g}_{n,\mu}\|_2 = O(\frac{1}{\mu n}).$$

□

n	μ	$\bar{g}$	$\bar{e}$	(L,U)
2	1 (BPFs)	0.488705	0.188705	(0.487912, 0.489498)
	3 (MBPFs)	0.397704	0.097704	(0.390937, 0.404471)
3	1 (BPFs)	0.327094	0.027093	(0.326647, 0.327540)
	3 (MBPFs)	0.316383	0.016382	(0.304258, 0.328508)
4	1 (BPFs)	0.246046	0.053945	(0.245386, 0.246705)
	3 (MBPFs)	0.582630	0.034799	(0.222132, 0.343128)
5	1 (BPFs)	0.400235	0.100235	(0.399732, 0.400738)
	3 (MBPFs)	0.360730	0.060730	(0.356168, 0.365292)

Table 1: Results in $(0.1, 0.2)$

n	μ	$\bar{g}$	$\bar{e}$	(L,U)
2	1 (BPFs)	0.996151	0.296151	(0.987828, 1.004470)
	3 (MBPFs)	0.906776	0.206776	(0.870497, 0.943055)
3	1 (BPFs)	0.997234	0.297234	(0.976630, 1.017840)
	3 (MBPFs)	0.880689	0.180689	(0.833557, 0.927822)
4	1 (BPFs)	0.792689	0.092689	(0.788890, 0.796488)
	3 (MBPFs)	0.751917	0.051917	(0.733360, 0.770475)
5	1 (BPFs)	0.799899	0.099899	(0.799426, 0.800373)
	3 (MBPFs)	0.782178	0.082178	(0.776386, 0.787971)

Table 2: Results in $(0, 0.7)$

5. Numerical example

We consider a numerical example to illustrate the efficiency of the MBPFs method. Consider the 2D-linear stochastic Volterra-Fredholm integral equation

$$g(x,y) = f(x,y) + \int_0^1 \int_0^1 (xyst)g(s,t)dsdt + \int_0^y \int_0^x (xyst)g(s,t)dsdt + \int_0^y \int_0^x (xyst)g(s,t)dB(s)dB(t),$$

where

$$f(x,y) = x + y - \frac{xy}{3} - \frac{x^3y^3(x+y)}{6} - (x^3y^2 + x^2y^3)B(x)B(y) + B(x)(x^3y + 2x^2y)\int_0^y B(t)dt + B(y)(y^3x + 2y^2x)\int_0^x B(s)ds,$$

with the exact solution

$$g(x,y) = x + y.$$

The solution mean $(\bar{g}(x,y))$, error mean $(\bar{e}(x,y))$ and %95 confidence interval (L,U) at arbitrary points $(0.1, 0.2)$ and $(0, 0.7)$ for some values of n and μ are shown in Table 1 and Table 2. In this tables by the comparison between the computed results by the presented method and the BPFs method we will see that in the MBPFs method we achieve the good accuracy by increasing μ. You can see three-dimensional graphs of this example in Fig. 5.1 and Fig. 5.2.

6. Conclusion

In this paper, we have successfully developed the 2D-MBPFs numerical method for approximate a solution for 2D-linear stochastic Volterra-Fredholm integral equations. The numerical results represent that $\bar{e}$ in new method is lesser from $\bar{e}$ in BPF method.

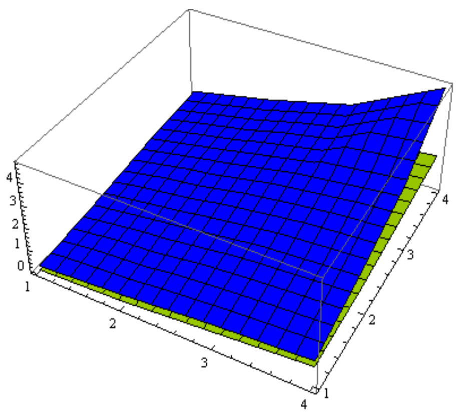

Figure 5.1: $(n = 3)$

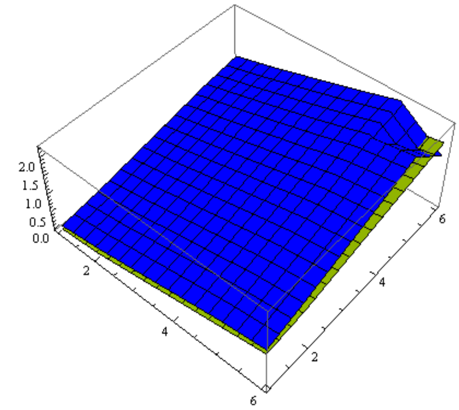

Figure 5.2: $(n = 5)$

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