Konuralp Journal of Mathematics, 8 (1) (2020) 50-56

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Generalized Open Sets vis-a-vis Δ-Sets

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Abstract

This paper concerns about the splitting of the collections of generalized open sets in topological spaces and their decompositions. Several characterizations of these sets are also discussed in this paper. Further, this paper also introduces a new type of normal space and its characterizations. Several properties of this space are discussed.

Keywords: *Δ-operator; Δ-sets; Hayashi-Samuel space; $\Psi - * -$normal space.*
2010 Mathematics Subject Classification: *Primary: 54D10; 54C10. Secondary: 54D9; 54C99.*

1. Introduction

The terms interior and closure in the study of topological spaces are nearly closed to the terms of open set and closed set respectively. The mathematicians like Hamlett [14], Levine [22], Ekici [13], Janković [19], Noiri and Al-Omari [1], Andrijevíc [2], Bandyopadhyay and Chattopadhyay [5] studied these two operators extensively and introduced generalized open sets in terms of these two operators. Some of these generalized open sets are: preopen set [23], b-open set [2], semi-open set [22], semi-preopen set [3], α-set [39], a-open set [13]. In the study of topological spaces, ideal played an important role and this type of study was introduced by Kuratowski [21] and Vaidyanathswamy [43]. A collection $\mathscr{I} \subseteq \wp(X)$ (power set of X) is said to be an ideal on X, if (i) $\emptyset \in X$, (ii) $A \subseteq B \in \mathscr{I}$ implies $A \in \mathscr{I}$ and, (iii) $A, B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$.

Let (X, τ) be a topological space and $\mathscr{I}$ be an ideal on X. Then the triplet $(X, \tau, \mathscr{I})$ is called an ideal topological space [21]. Some important ideals on the topological space (X, τ) are:

$\mathscr{I}_f$ - the ideal of finite subsets of X ,
$\mathscr{I}_c$ - the ideal of countable subsets of X ,
$\mathscr{I}_{cd}$ - the ideal of closed discrete sets in (X, τ),
$\mathscr{I}_n$ - the ideal of nowhere dense sets in (X, τ),
$\mathscr{I}_m$ - the ideal of meager sets in (X, τ),
$\mathscr{I}_s$- the ideal of scattered sets in (X, τ),
$\mathscr{I}_k$ - the ideal of relatively compact sets in (X, τ),
$\mathscr{I}_{L_0}$ - the ideal of Lebesgue null sets.

In the study of ideal topological space $(X, \tau, \mathscr{I})$, the operators $()^*$ [21] and Ψ [37], play remarkable role as like interior operator and closure operator of the topological space. Let A be a subset of X, then for the ideal topological space $(X, \tau, \mathscr{I})$,

$$A^*(\mathscr{I}) \text{ (or simply } A^*) = \{x \in X : U \cap A \notin \mathscr{I}, U \in \tau(x)\}, \text{ where } \tau(x) = \{U \in \tau : x \in U\} \text{ and } \Psi(A) = X \setminus (X \setminus A)^*.$$

It has already been proved that $*$ is not a closure operator and Ψ is not an interior operator (see [33, 35]). However, it is interesting that the functions $Cl^* : \wp(X) \to \wp(X)$ and $Int^* : \wp(X) \to \wp(X)$ defined by $Cl^*(A) = A \cup A^*$ and $Int^* = A \cap \Psi(A)$ are closure operator and interior operator respectively of a topology on X and this topology is denoted as $\tau^*(\mathscr{I})$ and will be called $*$-topology on X [14, 15, 17, 38, 24, 36, 41]. The ideal topological space $(X, \tau, \mathscr{I})$ has been constructed by two collections, τ and $\mathscr{I}$. Thus the condition $\tau \cap \mathscr{I} = \{\emptyset\}$ is a remarkable condition for the study of the ideal topological space $(X, \tau, \mathscr{I})$. An ideal $\mathscr{I}$, satisfying the condition $\tau \cap \mathscr{I} = \{\emptyset\}$, is called codense ideal according to the authors Dontchev, Ganster and Rose [10]. Newcomb [38] and Hamlett and Janković [14] call such ideal as τ-boundary, whereas, Dontchev [8] calls such space as Hayashi-Samuel space.

Mathematicians like Al-Omari, Noiri, Jafari, Hatir, Ekici and Modak have defined generalized open sets in terms of $*$ and Ψ-operators. Some of these sets are: Ψ-set [4], $\Psi - C$-set [32], pre-$\mathscr{I}$-open [9], $\mathscr{I}$-open [19], β-$\mathscr{I}$-open [16], α-$\mathscr{I}$-open [16], $\dot{\Psi}^*$ [24] sets. In this paper, we further consider the operators $*$ and Ψ for representation of generalized open sets of topological spaces. To do this, we define a set in terms of ideal and have reached a decomposition of this set. Further, we also defined a new type of normal space in ideal topological space and it will help us to characterize the generalized open sets.

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Definition 1.1. *A subset A of a topological space (X,τ) is said to be semi-open [22] (resp. regular open [42], preopen [23], semi-preopen [3](= β-open [12]), b-open [2] ,α [39] , δ [5]) set, if $A \subseteq Cl(Int(A))$ (resp. $A = Int(Cl(A))$, $A \subseteq Int(Cl(A))$, $A \subseteq Cl(Int(Cl(A)))$, $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$, $A \subseteq Int(Cl(Int(A)))$, $Int(Cl(A)) \subseteq Cl(Int(A))$).*

The collection of all semi-open (resp. regular open, preopen, semi-preopen (= β-open), b-open, α, δ) sets of (X,τ) is denoted as $SO(X,\tau)$ (resp. $RO(X,\tau)$, $PO(X,\tau)$, $SPO(X,\tau)$, $BO(X,\tau)$, τ^{α}, τ^{δ}).

Definition 1.2. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$, A is said to be a Ψ-set [4] (resp. $\mathscr{I}$-open [19], Ψ-C set [32], Ψ^*-set [33], $*^{\Psi}$-set [35]) if $A \subseteq Int(Cl(\Psi(A)))$ (rep. $A \subseteq Int(A^*)$, $A \subseteq Cl(\Psi(A))$, $A \subseteq (\Psi(A))^*$, $A \subseteq \Psi(A^*)$).*

The collection of all Ψ-sets (resp. $\mathscr{I}$-open sets, Ψ-C sets, Ψ^*-set, $*^{\Psi}$-set) in $(X,\tau,\mathscr{I})$ is denoted by τ^{Ψ} (resp. $\mathscr{I}O(X,\tau)$, $\Psi(X,\tau)$, $\Psi^*(X,\tau)$, $*^{\Psi}(X,\tau)$).
Note that $\mathscr{I}$-open sets related Hausdorff space has also been defined in [11].

Definition 1.3. *A subset A of an ideal topological space $(X,\tau,\mathscr{I})$ is said to be $\mathscr{I}$-dense [10] (resp. $*$-dense-in-itself [17], τ^*-dense [10]) if $A^* = X$ (resp. $A \subseteq A^*$, $Cl^*(A) = X$).*

Before starting the main section, we consider two propositions for preliminary purpose of this paper.

Proposition 1.4. *[35] Let $(X,\tau,\mathscr{I})$ be an ideal topological space. Then following hold:*
(a) $\mathscr{I}O(X,\tau) \subseteq *^{\Psi}(X,\tau)$.
(b) If the space is Hayashi-Samuel, then
(i) $\tau^{\alpha} \subseteq *^{\Psi}(X,\tau)$.
(ii) $\tau^*(\mathscr{I}) \subseteq *^{\Psi}(X,\tau)$.
(iii) $\tau \subseteq *^{\Psi}(X,\tau)$.
(iv) $\tau^{\Psi} \subseteq *^{\Psi}(X,\tau)$.
(v) $(\tau^*(\mathscr{I}))^{\alpha} \subseteq *^{\Psi}(X,\tau)$.
(vi) $*^{\Psi}(X,\tau) \subseteq PO(X,\tau)$.
(vii) $*^{\Psi}(X,\tau) \subseteq SPO(X,\tau)$.
(viii) $*^{\Psi}(X,\tau) \subseteq BO(X,\tau)$.

Proposition 1.5. *[33] Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. Then*
(a) for regular open set A, $A \in \Psi^*(X,\tau)$.
(b) for $A \in \Psi(X,\tau)$, $A \in \Psi^*(X,\tau)$.
(c) for $A \in \tau^{\Psi}$, $A \in \Psi^*(X,\tau)$ (τ^{Ψ} denotes the α-topology of $(X,\tau^*(\mathscr{I}))$).
(d) $A \in \Psi^*(X,\tau)$ when $A \in PO(X,\tau)$ and $A \in \tau^{\delta}$.
(e) for $A \in SO(X,\tau)$, $A \in \Psi^*(X,\tau)$.

2. Δ sets

Definition 2.1. *[34] Let $(X,\tau,\mathscr{I})$ be an ideal topological space. The operator $\Delta : \wp(X) \to \wp(X)$ is defined as:*

$$\Delta(A) = (\Psi(A))^* \cup \Psi(A^*), \text{ for } A \subseteq X.$$

The value of the operator Δ for $\mathscr{I} = \{\emptyset\}$, $\Delta(A) = Cl(Int(A)) \cup Int(Cl(A))$ and for $\mathscr{I} = \mathscr{I}_n$, $\Delta(A) = Cl(Int(Cl(Int(Cl(Int(A)))))) \cup Int(Cl(Int(Cl(Int(Cl(A)))))) = Cl(Int(A)) \cup Int(Cl(A))$. Thus for different ideals $\{\emptyset\}$ and $\mathscr{I}_n$, the values of Δ on a subset of X are same. In these two cases, Ψ and $*$ can be approximated by the interior operator and the closure operator respectively. Further, the value of Δ on A is a subset of A^*, when the space is Hayashi-Samuel.
Following is the definition of a new set which will be compared with Andrijević's b-open set.

Definition 2.2. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$, A is said to be a Δ-set, if $A \subseteq \Delta(A)$.*

The collection of all Δ-sets in $(X,\tau,\mathscr{I})$ is denoted as $\Delta(X,\tau)$. These collection is same as the collection of b-open sets when $\mathscr{I} = \{\emptyset\}$ and $\mathscr{I} = \mathscr{I}_n$.
This paper searches Δ-sets in any ideal topological space.
Following examples show that the collection of b-open sets and Δ-sets are different from each other.

Example 2.3. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X\}$ and $\mathscr{I} = \{\emptyset, \{a\}\}$. Let $A = \{a\}$. Then $A \subseteq Int(Cl(A)) \cup Cl(Int(A)) = X$. Therefore $A \in BO(X,\tau)$. Now $(\Psi(A))^* = \Psi(A^*) = \emptyset$. Therefore $A \nsubseteq \Delta(A)$. Thus $A \notin \Delta(X,\tau)$.*

Example 2.4. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, \{c\}, \{a,c\}, \{b,c\}, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Let $A = \{a,b\}$. Then $A^* = A$. Now $\Psi(A^*) = \Psi(A) = X \setminus \{c\}^* = X \setminus \emptyset = X$. Then $A \subseteq (\Psi(A))^* \cup \Psi(A^*) = \Delta(A)$. So $A \in \Delta(X,\tau)$ but $Cl(Int(A)) = Int(Cl(A)) = \emptyset$. So $A \notin BO(X,\tau)$.*

Theorem 2.5. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space. Then $\emptyset$ is a Δ-set.*

Theorem 2.6. *In a Hayashi-Samuel space $(X,\tau,\mathscr{I})$, the set X is a Δ-set.*

Proof. Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. Then $*^{\Psi}(X) = X = \Psi^*(X)$. So $X \subseteq *^{\Psi}(X) \cup \Psi^*(X)$. Therefore X is a Δ-set. □

The converse of the above theorem is not true and it follows from the following example:

Example 2.7. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, \{c\}, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. We find, $\Psi(X^*) = \Psi(\{a,b\}) = X \setminus \{c\}^* = X$ and $(\Psi(X))^* = X^* = \{a,b\}$. Therefore $\Delta(X) = (\Psi(X))^* \cup \Psi(X^*) = X$ but $(X,\tau,\mathscr{I})$ is not a Hayashi-Samuel space.*

Following theorem tells us about detail properties of Δ-set.

Theorem 2.8. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space. Then the following properties hold:*
(a) $\Psi^*(X,\tau) \subseteq \Delta(X,\tau)$.
(b) $*^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(c) $\Psi^*(X,\tau) \cup *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(d) $\mathscr{I}O(X,\tau) \subseteq \Delta(X,\tau)$.
(e) If the space is Hayashi-Samuel, then
(i) $\tau^{\alpha} \subseteq \Delta(X,\tau)$.
(ii) $\tau^*(\mathscr{I}) \subseteq \Delta(X,\tau)$.
(iii) $\tau \subseteq \Delta(X,\tau)$.
(iv) $\Psi(X,\tau) \subseteq \Delta(X,\tau)$.
(v) $\tau^{\Psi} \subseteq \Delta(X,\tau)$.
(vi) $(\tau^*(\mathscr{I}))^{\alpha} \subseteq \Delta(X,\tau)$.
(vii) $Cl^*(A) = A^*$*, if* $A \in \tau$.
(viii) $Cl^*(A) = A^*$*, if* $A \in \Psi^*(X,\tau)$.
(ix) $\Delta(X,\tau) \subseteq SPO(X,\tau^*(\mathscr{I}))$.
(x) $\Psi^*(X,\tau) \cup *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau) \subseteq SPO(X,\tau^*(\mathscr{I}))$.
(xi) $SPO(X,\tau^*(\mathscr{I})) \subseteq SPO(X,\tau)$.
(xii) $\Delta(X,\tau) \subseteq SPO(X,\tau^*(\mathscr{I})) \subseteq SPO(X,\tau)$.
(xiii) $SO(X,\tau) \subseteq \Psi(X,\tau) \subseteq \Delta(X,\tau) \subseteq SPO(X,\tau^*(\mathscr{I})) \subseteq SPO(X,\tau)$.
(xiv) For regular open set $A \subseteq X$, $A \in \Delta(X,\tau)$.
(xv) $\tau \subseteq \tau^{\alpha} \subseteq \mathscr{I}O(X,\tau) = PO(X,\tau^*(\mathscr{I})) = *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau) \subseteq SPO(X,\tau)$.
(xvi) $\tau \subseteq \tau^*(\mathscr{I}) \subseteq (\tau^*(\mathscr{I}))^{\alpha} \subseteq \mathscr{I}O(X,\tau^*(\mathscr{I})) \subseteq PO(X,\tau^*(\mathscr{I})) \subseteq \Delta(X,\tau)$.

Proof. (a) Let $A \in \Psi^*(X,\tau)$. Then $A \subseteq (\Psi(A))^* \subseteq (\Psi(A))^* \cup (\Psi(A^*)) = \Delta(A)$. Therefore $A \in \Delta(X,\tau)$. Hence $\Psi^*(X,\tau) \subseteq \Delta(X,\tau)$.
(b) Let $A \in *^{\Psi}(X,\tau)$. Then $A \subseteq (\Psi(A^*)) \subseteq (\Psi(A))^* \cup (\Psi(A^*)) = \Delta(A)$. Therefore $A \in \Delta(X,\tau)$. Hence $*^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(c) By the conditions (a) and (b), we get $\Psi^*(X,\tau) \subseteq \Delta(X,\tau)$ and $*^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$ respectively. These together imply that $\Psi^*(X,\tau) \cup *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(d) $\mathscr{I}O(X,\tau) \subseteq *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(e) (i) $\tau^{\alpha} \subseteq *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(ii) $\tau^*(\mathscr{I}) \subseteq *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(iii) $\tau \subseteq *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(iv) $\Psi(X,\tau) = \Psi^*(X,\tau) \subseteq \Delta(X,\tau)$.
(v) $\tau^{\Psi} \subseteq *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(vi) $(\tau^*(\mathscr{I}))^{\alpha} \subseteq *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau)$.
(vii) Let $A \in \tau$. Since the space is Hayashi-Samuel, therefore $A \subseteq A^*$. Hence $Cl^*(A) = A \cup A^* = A^*$.
$(viii)$ Obvious.
(ix) Let $A \in \Delta(X,\tau)$. Then $A \subseteq (\Psi(A))^* \cup \Psi(A^*) \subseteq (\Psi(A))^* \cup \Psi(A^*) \cup \Psi(A) = \Psi(A^*) \cup Cl^*(\Psi(A)) = Int(\Psi(A^*)) \cup Cl^*(Int(\Psi(A))) \subseteq Int^*(\Psi(A^*)) \cup Cl^*(Int^*(A^*)) \subseteq Int^*(A^{**}) \cup Cl^*(Int^*(A \cup A^*)) \subseteq Int^*(A^*) \cup Cl^*(Int^*(Cl^*(A))) \subseteq Int^*(A \cup A^*) \cup Cl^*(Int^*(Cl^*(A))) = Int^*(Cl^*(A)) \cup Cl^*(Int^*(Cl^*(A))) = Cl^*(Int^*(Cl^*(A)))$. Thus $A \in SPO(X,\tau^*(\mathscr{I}))$. Hence $\Delta(X,\tau) \subseteq SPO(X,\tau^*(\mathscr{I}))$.
(x) Combining (c) and $(e)(ix)$, we get $\Psi^*(X,\tau) \cup *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau) \subseteq SPO(X,\tau^*(\mathscr{I}))$.
(xi) Let $A \in SPO(X,\tau^*(\mathscr{I}))$. Then $A \subseteq Cl^*(Int^*(Cl^*(A))) \subseteq Cl^*Int^*Cl(A) = Cl^*[\Psi(Cl(A)) \cap Cl(A)] \subseteq Cl[\Psi(Cl(A))] = Cl[X \setminus (X \setminus Cl(A))^*] = Cl[X \setminus Cl(X \setminus Cl(A))] = Cl(Int(Cl(A)))$. Therefore, $A \in SPO(X,\tau)$. Hence $SPO(X,\tau^*(\mathscr{I}))$
$\subseteq SPO(X,\tau)$.
(xii) Combining $(e)(x)$ and $(e)(xi)$, we get $\Delta(X,\tau) \subseteq SPO(X,\tau^*(\mathscr{I})) \subseteq SPO(X,\tau)$.
$(xiii)$ We have $SO(X,\tau) \subseteq \Psi(X,\tau)$. Again from $(e)(iv)$, $\Psi(X,\tau) \subseteq \Delta(X,\tau)$ and from $(e)(xii)$, $\Delta(X,\tau) \subseteq SPO(X,\tau^*(\mathscr{I})) \subseteq SPO(X,\tau)$. Combining these results we get $SO(X,\tau) \subseteq \Psi(X,\tau) \subseteq \Delta(X,\tau) \subseteq SPO(X,\tau^*(\mathscr{I})) \subseteq SPO(X,\tau)$.
(xiv) Let A be a regular open subset of X. Then $A \in \Psi^*(X,\tau)$. Again by condition (a), $\Psi^*(X,\tau) \subseteq \Delta(X,\tau)$. These together imply $A \in \Delta(X,\tau)$.
(xv) From [39], $\tau \subseteq \tau^{\alpha}$.
We shall now prove $\tau^{\alpha} \subseteq \mathscr{I}O(X,\tau)$. For this, suppose that $A \in \tau^{\alpha}$. Then $A \subseteq Int(Cl(Int(A))) = Int((Int(A))^*)$ (since $Int(A)$ is an open set). Thus $A \subseteq Int((Int(A))^*) \subseteq (Int(A))^*$. Therefore $A \in \mathscr{I}O(X,\tau)$. Therefore $\tau^{\alpha} \subseteq \mathscr{I}O(X,\tau)$.
Again, $\mathscr{I}O(X,\tau) = PO(X,\tau^*(\mathscr{I})) = *^{\Psi}(X,\tau)$. By (b) and $(e)(xii)$ we get, $*^{\Psi}(X,\tau) \subseteq \Delta(X,\tau) \subseteq SPO(X,\tau)$. Combining all of these results we finally obtain, $\tau \subseteq \tau^{\alpha} \subseteq \mathscr{I}O(X,\tau) = PO(X,\tau^*(\mathscr{I})) = *^{\Psi}(X,\tau) \subseteq \Delta(X,\tau) \subseteq SPO(X,\tau)$.
(xvi) From [19], $\tau \subseteq \tau^*(\mathscr{I})$ and from [39], $\tau^*(\mathscr{I}) \subseteq (\tau^*(\mathscr{I}))^{\alpha}$. Using these results, $\tau \subseteq \tau^*(\mathscr{I}) \subseteq (\tau^*(\mathscr{I}))^{\alpha}$. We know from $(e)(xv)$, $\tau^{\alpha} \subseteq \mathscr{I}O(X,\tau)$. This implies $(\tau^*(\mathscr{I}))^{\alpha} \subseteq \mathscr{I}O(X,\tau^*(\mathscr{I}))$. Again from [31], we know that $\mathscr{I}O(X,\tau) \subseteq PO(X,\tau)$. So $\mathscr{I}O(X,\tau^*(\mathscr{I})) \subseteq PO(X,\tau^*(\mathscr{I}))$. From $(e)(xv)$, $PO(X,\tau^*(\mathscr{I})) \subseteq \Delta(X,\tau)$. Combining all of these results, we obtain $\tau \subseteq \tau^*(\mathscr{I}) \subseteq (\tau^*(\mathscr{I}))^{\alpha} \subseteq \mathscr{I}O(X,\tau^*(\mathscr{I})) \subseteq PO(X,\tau^*(\mathscr{I})) \subseteq \Delta(X,\tau)$. □

Following examples show that the converse of (a), (b), (d), e (i),(ii), (iv) of the Theorem 2.8 need not hold.

Example 2.9. *Let* $X = \{a,b\}$, $\tau = \{\emptyset,\{a\},X\}$ *and* $\mathscr{I} = \{\emptyset,\{a\}\}$. *Let* $A = \{a\}$. *Then* $A^* = \emptyset$ *and* $\Psi(A) = X \setminus \{b\}^* = X \setminus \{b\} = \{a\}$. *Therefore* $(\Psi(A))^* = \{a\}^* = \emptyset$. *Therefore* $A \notin \Psi^*(X,\tau)$. *Again* $\Psi(A^*) = X \setminus X^* = X \setminus \{b\} = \{a\}$. *Therefore* $\Delta(A) = (\Psi(A))^* \cup \Psi(A^*) = \{a\} = A$. *Therefore* $A \subseteq \Delta(A)$. *Thus* $A \in \Delta(X,\tau)$.

Example 2.10. *Let us consider the set of all real numbers* $\mathbb{R}$ *with usual topology* τ_u *and take* $\mathscr{I} = \{\emptyset\}$. *Let* $A = [0,1]$. *Then* $A^* = A = [0,1]$. *Therefore* $\Psi(A^*) = \Psi(A) = (0,1)$. *Therefore* $A \nsubseteq \Psi(A^*)$. *Thus* $A \notin *^{\Psi}(X,\tau)$. *Again* $(\Psi(A))^* = (0,1)^* = [0,1]$. *Then* $\Delta(A) = (\Psi(A))^* \cup (\Psi(A^*)) = [0,1] = A$. *Therefore* $A \subseteq \Delta(A)$. *Thus* $A \in \Delta(X,\tau)$.

Example 2.11. *Let* $X = \{a,b\}$, $\tau = \{\emptyset,\{a\},X\}$ *and* $\mathscr{I} = \{\emptyset,\{a\}\}$. *Let* $A = \{b\}$. *Then* $A^* = \{b\}$ *and* $Int(A^*) = \emptyset$. *therefore* $A \notin \mathscr{I}O(X,\tau)$. *Again* $\Psi(A^*) = X \setminus (X \setminus \{b\})^* = X \setminus \{a\}^* = X \setminus \emptyset = X$. *Therefore* $A \subseteq (\Psi(A))^* \cup \Psi(A^*)$. *So* $A \in \Delta(X,\tau)$.

Example 2.12. *Let $X=\{a,b\}$, $\tau=\{\emptyset,X\}$, $\mathscr{I}=\{\emptyset,\{a\}\}$. Then $\{b\}^*=\{a,b\}=X$, $\Psi(\{b\}^*)=X\setminus(X\setminus X)^*=X$. Therefore $\{b\}\subseteq(\Psi(\{b\}))^*\cup\Psi(\{b\}^*)=\Delta(\{b\})$. So $\{b\}$ is a Δ-set. Now $Int(Cl(Int(\{b\})))=\emptyset$. Thus $\{b\}$ is not an α-set.*

Example 2.13. *Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X\}$ and $\mathscr{I}=\{\emptyset,\{c\}\}$. Let $B=\{b,c\}$. Then $B^*=X$. Therefore $\Psi(B^*)=\Psi(X)=X$. So $B\subseteq(\Psi(B))^*\cup\Psi(B^*)=\Delta(B)$. Therefore $B\in\Delta(X,\tau)$. Now $\tau^*(\mathscr{I})=\{\emptyset,\{a,b\},X\}$. Thus $B\notin\tau^*(\mathscr{I})$.*

Example 2.14. *In Example 2.13, we see that the set $B=\{b,c\}$ is a Δ-set but it is not a Ψ-C set, since $\Psi(B)=X\setminus\{a\}^*=X\setminus X=\emptyset$ implies $Cl(\Psi(B))=\emptyset$. Therefore $B\nsubseteq Cl(\Psi(B))$.*

Theorem 2.15. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. Then for any $A\subseteq X$,*
(i) $\Psi(A)\in\Delta(X,\tau)$.
(ii) $\Psi(A^*)\in\Delta(X,\tau)$.
(iii) $(\Psi(A))^*\in\Delta(X,\tau)$, if $A\in\Psi^*(X,\tau)$.
(iv) $A^*\in\Delta(X,\tau)$, if $A\in *^{\Psi}(X,\tau)$.

Proof. Let $A\subseteq X$.
(i) Let us write $G=\Psi(A)$. Then G is an open set, therefore $\Psi(A)=G\subseteq\Psi(G)\subseteq(\Psi(G))^*\subseteq(\Psi(G))^*\cup\Psi(G^*)=\Delta(G)$. Thus $\Psi(A)\subseteq\Delta(\Psi(A))$. Hence $\Psi(A)\in\Delta(X,\tau)$.
(ii) Replacing A by A^* in (i), we get $\Psi(A^*)\in\Delta(X,\tau)$.
(iii) Let us write $H=(\Psi(A))^*$. Since $A\in\Psi^*(X,\tau)$, therefore $A\subseteq(\Psi(A))^*$, that is, $A\subseteq H$. This implies $\Psi(A)\subseteq\Psi(H)$. Then $(\Psi(A))^*\subseteq(\Psi(H))^*$. Thus $H\subseteq(\Psi(H))^*\subseteq(\Psi(H))^*\cup\Psi(H^*)=\Delta(H)$. Hence $H=(\Psi(A))^*\in\Delta(X,\tau)$.
(iv) Let us write $K=A^*$. Since $A\in *^{\Psi}(X,\tau)$, therefore $A\subseteq\Psi(A^*)$, that is, $A\subseteq\Psi(K)$. This implies $A^*\subseteq(\Psi(K))^*$, that is, $K\subseteq(\Psi(K))^*\subseteq(\Psi(K))^*\cup\Psi(K^*)=\Delta(K)$. Hence $K=A^*\in\Delta(X,\tau)$. □

From Theorem 2.8 , we get the following diagrams:

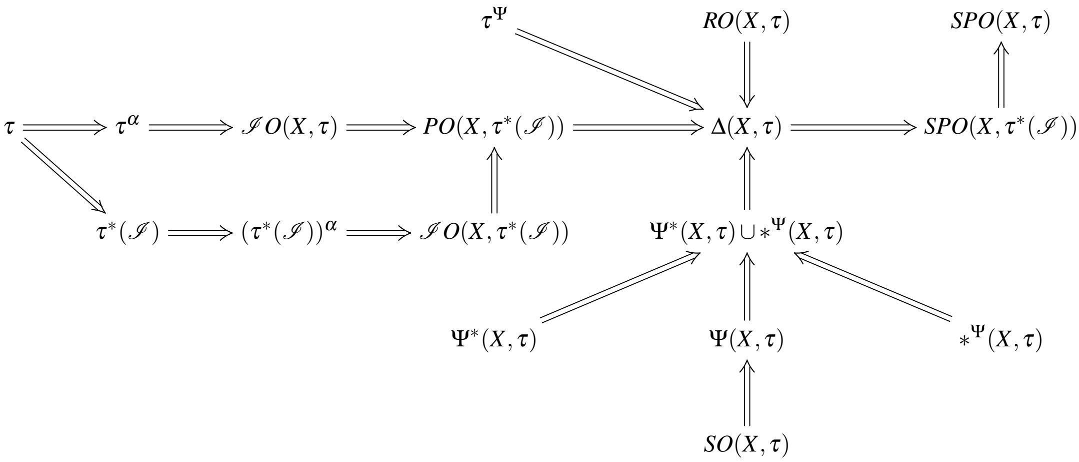

Diagram 1

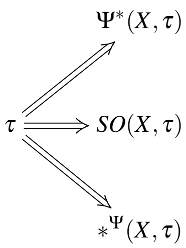

Diagram 2

Theorem 2.16. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. Then arbitrary union of Δ-sets is a Δ-set.*

Proof. Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $\{U_\alpha:\alpha\in\Lambda\}$ (Λ is an index set) be an arbitrary collection of Δ-sets. Since, both the operators $*$ and Ψ are monotonic, hence $U_\alpha\subseteq\Psi^*(U_\alpha)\subseteq\Psi^*(\bigcup_{\alpha\in\Lambda}U_\alpha)$ and $U_\alpha\subseteq *^{\Psi}(U_\alpha)\subseteq *^{\Psi}(\bigcup_{\alpha\in\Lambda}U_\alpha)$, for any $\alpha\in\Lambda$. Hence, $U_\alpha\subseteq\Psi^*(\bigcup_{\alpha\in\Lambda}U_\alpha)\cup *^{\Psi}(\bigcup_{\alpha\in\Lambda}U_\alpha)=\Delta(\bigcup_{\alpha\in\Lambda}U_\alpha)$, for any $\alpha\in\Lambda$. Hence, $\bigcup_{\alpha\in\Lambda}U_\alpha\subseteq\Delta(\bigcup_{\alpha\in\Lambda}U_\alpha)$. □

Observe that, intersection of two Δ-sets may not a Δ-set and it follows from the following example:

Example 2.17. *Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X\}$ and $\mathscr{I}=\{\emptyset,\{a\}\}$. Let $A=\{a,b\}$ and $B=\{a,c\}$. Then $A^*=B^*=X$ and $\Psi(A)=\Psi(B)=\emptyset$. Therefore $\Psi(A^*)=\Psi(B^*)=X$ and $(\Psi(A))^*=(\Psi(B))^*=\emptyset$. Therefore $\Delta(A)=(\Psi(A))^*\cup\Psi(A^*)=X$ and $\Delta(B)=(\Psi(B))^*\cup\Psi(B^*)=X$. Thus $A\subseteq\Delta(A)$ and $B\subseteq\Delta(B)$. Thus $A,B\in\Delta(X,\tau)$. Now $A\cap B=\{a\}$. Then $\{a\}^*=\emptyset$ and $\Psi(\{a\})=X\setminus\{b,c\}^*=X\setminus X=\emptyset$. Therefore $(\Psi(\{a\}))^*=\emptyset^*=\emptyset$ and $\Psi(\{a\}^*)=\Psi(\emptyset)=X\setminus X^*=X\setminus X=\emptyset$. Therefore $\Delta(\{a\})=(\Psi(\{a\}))^*\cup(\Psi(\{a\}^*))=\emptyset$. Therefore $\Delta(A\cap B)=\emptyset$. Thus $A\cap B\nsubseteq\Delta(A\cap B)$. Therefore $A\cap B\notin\Delta(X,\tau)$.*

Therefore we conclude from Example 2.17 that, $\Delta(X,\tau)$ does not form a topology on X.

Theorem 2.18. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. Then*
(i) $A\in\Delta(X,\tau)$, if $A\in PO(X,\tau)$ and $A\in C(\tau)$ ($C(\tau)$ denotes the collection of closed subsets of X).
(ii) $A\in\Delta(X,\tau)$, if $A\in PO(X,\tau)$ and $A\in\tau^{\delta}$.

Proof. Obvious. □

Theorem 2.19. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. If $A\in SPO(X,\tau)$ and $A\in C(\tau)$, then $A\in\Delta(X,\tau)$.*

Proof. Let $A\in SPO(X,\tau)$ and $A\in C(\tau)$. Then $A\subseteq Cl(Int((Cl(A)))\subseteq Cl[\Psi(Int(Cl(A)))]$ (since, the space is Hayashi-Samuel) $\subseteq [\Psi(Int(Cl(A)))]^*\subseteq[\Psi(Cl(A))]^*=(\Psi(A))^*\subseteq\Delta(A)$. Thus $A\in\Delta(X,\tau)$. □

Theorem 2.20. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space and $U\in\tau, A\in *^{\Psi}(X,\tau)$. Then $U\cap A\in\Delta(X,\tau)$.*

Proof. From [19], $U\cap A^*\subseteq(U\cap A)^*$. Then $U\cap A\subseteq\Psi(U)\cap\Psi(A^*)=\Psi(U\cap A^*)\subseteq\Psi((U\cap A)^*)\subseteq\Delta(U\cap A)$. □

Corollary 2.21. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space and $A,B\subseteq X$. Then*
(i) $A\cap\Psi(B)\in\Delta(X,\tau)$, if $A\in *^{\Psi}(X,\tau)$.
(ii) $\Psi(A\cap B)\in\Delta(X,\tau)$.
(iii) $A\cap\Psi(B^*)\in\Delta(X,\tau)$, if $A\in *^{\Psi}(X,\tau)$.
(iv) $\Psi(A^*\cap B^*)\in\Delta(X,\tau)$.

Corollary 2.22. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space and $A\in\tau^{\alpha}, B\in\tau$. Then $A\cap B\in\Delta(X,\tau)$.*

Corollary 2.23. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space and $A,B\subseteq X$. If $A\in\tau$, then*
(i) $A\cap\Psi(B)\in\Delta(X,\tau)$.
(ii) $A\cap\Psi(B^*)\in\Delta(X,\tau)$.

Corollary 2.24. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space and $A\in\tau^{\alpha}\cap C(\tau)$ and $B\in *^{\Psi}(X,\tau)$. Then $A\cap B\in\Delta(X,\tau)$.*

Following is the one way for equalities of $\Delta(X,\tau)$ with different types of generalized open sets in topological spaces. For this, we consider following:
A topological space (X,τ) is said to be resolvable [18], if it can be expressed as the union of two disjoint dense subsets; otherwise it is called irresolvable [18]. We shall call a topological space (X,τ) open hereditarily irresolvable [6] (in short o.h.i.) if every open subset of X is irresolvable in its relative topology . We recall following theorem from [33]:

Theorem 2.25. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space, where (X,τ) is an o.h.i. space, then $\Psi(X,\tau)=SPO(X,\tau)=\Delta(X,\tau)=SPO(X,\tau^*(\mathscr{I}))=\Psi^*(X,\tau)$.*

3. $\Psi-*-$normal space

Definition 3.1. *An ideal topological space $(X,\tau,\mathscr{I})$ is said to be $\Psi-*-$normal, if for each $A\subseteq X$, $(\Psi(A))^*\cap\Psi(A^*)=\emptyset$.*

A space $(X,\tau,\mathscr{I})$ is $\Psi-*-$normal if and only if $(\Psi(A))^*\subseteq(X\setminus A^*)^*$. Further, a Hayashi-Samuel space $(X,\tau,\mathscr{I})$ is $\Psi-*-$normal if and only if $\Psi(A^*)\subseteq Int((X\setminus A)^*)$.
Following example shows that every ideal topological space is not a $\Psi-*-$normal space.

Example 3.2. *Let $X=\{a,b\}$, $\tau=\{\emptyset,\{a\},X\}$ and $\mathscr{I}=\{\emptyset,\{a\}\}$. Let $A=\{b\}$. Then $A^*=\{b\}^*=\{b\}$ and $\Psi(A)=X\setminus\{a\}^*=X$. Then $(\Psi(A))^*=(\Psi(\{b\}))^*=X^*=\{b\}$ and $(\Psi(A^*))=(\Psi(\{b\}^*))=\Psi(\{b\})=X\setminus\{a\}^*=X$. Then $(\Psi(A))^*\cap\Psi(A^*)=\{b\}\cap X=\{b\}\neq\emptyset$. Therefore the space $(X,\tau,\mathscr{I})$ is not $\Psi-*-$normal.*

Following theorem is the existence of $\Psi-*-$normal space:

Theorem 3.3. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $\mathscr{I}=\wp(X)$. Then the space $(X,\tau,\mathscr{I})$ is a $\Psi-*-$normal space.*

Proof. Let $A\subseteq X$. Then $A\in\mathscr{I}$. Therefore $A^*=\emptyset$ and $\Psi(A)=X\setminus(X\setminus A)^*=X\setminus\emptyset=X$. Therefore $(\Psi(A))^*=X^*=\emptyset$ and $\Psi(A^*)=\Psi(\emptyset)=X\setminus X^*=X\setminus\emptyset=X$. Therefore $(\Psi(A))^*\cap\Psi(A^*)=\emptyset$. Hence, the space $(X,\tau,\mathscr{I})$ is $\Psi-*-$normal. □

Lemma 3.4. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. Then for $A\subseteq X$, $\Psi(A^*)=X\setminus(\Psi(X\setminus A))^*$.*

Proof. Let $A\subseteq X$. Then $(\Psi(X\setminus A))^*=[X\setminus(X\setminus(X\setminus A))^*]^*=(X\setminus A^*)^*=X\setminus[X\setminus(X\setminus A^*)^*]=X\setminus\Psi(A^*)$. Therefore $\Psi(A^*)=X\setminus(\Psi(X\setminus A))^*$. □

Another version of this lemma is:

Lemma 3.5. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. Then for $A\subseteq X$, $(\Psi(A))^*=X\setminus\Psi((X\setminus A)^*)$.*

Corollary 3.6. *An ideal topological space $(X,\tau,\mathscr{I})$ is a $\Psi-*-$normal space if and only if $(\Psi(A))^*\subseteq(\Psi(X\setminus A))^*$, for each $A\subseteq X$.*

Corollary 3.7. *An ideal topological space $(X,\tau,\mathscr{I})$ is a $\Psi-*-$normal space if and only if $\Psi(A^*)\subseteq(\Psi((X\setminus A)^*))^*$, for each $A\subseteq X$.*

Theorem 3.8. *Let $(X,\tau,\mathscr{I})$ be a $\Psi-*-$normal space. Then $A\in\Delta(X,\tau)$ if and only if either $A\in\Psi^*(X,\tau)$ or $A\in *^{\Psi}(X,\tau)$.*

Proof. Let $A \in \Delta(X,\tau)$. Then $A \subseteq (\Psi(A))^* \cup \Psi(A^*)$. Since the space is $\Psi-*-$normal, therefore $(\Psi(A))^* \cap \Psi(A^*) = \emptyset$. Therefore either $A \subseteq (\Psi(A))^*$ or $A \subseteq \Psi(A^*)$. Therefore either $A \in \Psi^*(X,\tau)$ or $A \in *^{\Psi}(X,\tau)$.
Conversely, let $A \in \Psi^*(X,\tau)$ or $A \in *^{\Psi}(X,\tau)$. Then $A \subseteq (\Psi(A))^*$ or $A \subseteq \Psi(A^*)$. Therefore $A \subseteq (\Psi(A))^* \cup \Psi(A^*) = \Delta(A)$. Hence $A \in \Delta(X,\tau)$. □

Corollary 3.9. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space, where $\mathscr{I} = \{\emptyset\}$ or $\mathscr{I} = \mathscr{I}_n$. Then for $A \in \Delta(X,\tau)$, either $A \in SO(X,\tau)$ or $A \in PO(X,\tau)$.*

Corollary 3.10. *Let $(X,\tau,\mathscr{I})$ be a $\Psi-*-$normal Hayashi-Samuel space. Then for $A \in \Delta(X,\tau)$, either $A \in PO(X,\tau)$ or $A \in \Psi(X,\tau)$.*

Proof. Let $A \in \Delta(X,\tau)$. By Theorem 3.8, either $A \in \Psi^*(X,\tau)$ or $A \in *^{\Psi}(X,\tau)$. By Theorem 2.13 of [33], $\Psi^*(X,\tau) \subseteq \Psi(X,\tau)$ and from Theorem 1.4 (vi), $*^{\Psi}(X,\tau) \subseteq PO(X,\tau)$. Hence, either $A \in PO(X,\tau)$ or $A \in \Psi(X,\tau)$. □

Corollary 3.11. *Let $(X,\tau,\mathscr{I})$ be a $\Psi-*-$normal Hayashi-Samuel space. Then for $A \in \Delta(X,\tau)$, either $A \in PO(X,\tau^*(\mathscr{I}))$ or $A \in \Psi(X,\tau)$.*

Corollary 3.12. *Let $(X,\tau,\mathscr{I})$ be a $\Psi-*-$normal Hayashi-Samuel space. Then either $PO(X,\tau^*(\mathscr{I})) = \Delta(X,\tau)$ or $\Delta(X,\tau) = \Psi(X,\tau)$.*

4. Conclusion

1. The collection $\Delta(X,\tau)$ lies in between $\Psi^*(X,\tau) \cup *^{\Psi}(X,\tau)$ and $SPO(X,\tau)$, $\Psi^*(X,\tau) \cup *^{\Psi}(X,\tau)$ and $SPO(X,\tau^*(\mathscr{I}))$, $SO(X,\tau)$ and $SPO(X,\tau)$, $\Psi(X,\tau)$ and $SPO(X,\tau)$, $SO(X,\tau)$ and $SPO(X,\tau^*(\mathscr{I}))$, $\Psi(X,\tau)$ and $SPO(X,\tau^*(\mathscr{I}))$, τ and $SPO(X,\tau)$, τ^α and $SPO(X,\tau)$, $\mathscr{I}O(X,\tau)$ and $SPO(X,\tau)$, $PO(X,\tau^*(\mathscr{I}))$ and $SPO(X,\tau)$, τ and $SPO(X,\tau)$, $\tau^*(\mathscr{I})$ and $SPO(X,\tau)$, $(\tau^*(\mathscr{I}))^\alpha$ and $SPO(X,\tau)$, $\mathscr{I}O(X,\tau^*(\mathscr{I}))$ and $SPO(X,\tau)$, $PO(X,\tau^*(\mathscr{I}))$ and $SPO(X,\tau)$, τ and $SPO(X,\tau^*(\mathscr{I}))$, $\tau^*(\mathscr{I})$ and $SPO(X,\tau^*(\mathscr{I}))$, $(\tau^*(\mathscr{I}))^\alpha$ and $SPO(X,\tau^*(\mathscr{I}))$, $\mathscr{I}O(X,\tau^*(\mathscr{I}))$ and $SPO(X,\tau^*(\mathscr{I}))$, $PO(X,\tau^*(\mathscr{I}))$ and $SPO(X,\tau^*(\mathscr{I}))$.
2. We get decomposition of the following sets: $\Psi-C$ set, semi-preopen set of $(X,\tau^*(\mathscr{I}))$, Ψ^*-set and semi-preopen set, when the space is o.h.i.
3. $\Psi-*-$normal space characterizes the following collections: $*^{\Psi}(X,\tau)$, $\Psi^*(X,\tau)$, $SO(X,\tau)$, $\Psi(X,\tau)$, $PO(X,\tau)$ and $PO(X,\tau^*(\mathscr{I}))$.
4. All of the results of this paper have also been true if we replace the ideal $\mathscr{I}$ of $(X,\tau,\mathscr{I})$ by grill $\mathscr{G}$ [7] on X (see [20, 24, 25, 27, 28, 29, 30, 40, 41]).

One can consider all of the results of this paper in the filter topological space $(X,\tau,\mathfrak{F})$, where $()^*$ is defined as: $A^* = \{x \in X : U \cap A \in \mathfrak{F}, U \in \tau(x)\}$ (like [25, 26, 27, 28, 29, 30]). In this connection, we say that a filter is a dual ideal and is defined as: $\{A \subseteq X : X \setminus A \in \mathscr{I}, \mathscr{I}$ is a proper ideal on $X\}$.

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