Konuralp Journal of Mathematics, 8 (1) (2020) 62-69

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

On the Statistical Limits of Strongly Measurable Fuzzy Valued Functions

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Abstract

We introduce the notions of statistical limit, statistical Cesàro summability of strongly measurable fuzzy valued functions and give slowly decreasing-slowly oscillating type Tauberian conditions under which statistical limits and statistical Cesàro summability of fuzzy valued functions imply ordinary limits in fuzzy number space.

Keywords: *Cesàro summability; fuzzy set theory; statistical limits; Tauberian theorems*
2010 Mathematics Subject Classification: *03E72,40A35,40G15,40E05*

1. Introduction

The notion of statistical convergence was first introduced by Fast [6] as a generalization of ordinary convergence of sequences and excited the attention of many researchers. Authors studied and applied the notion in different fields of mathematics such as trigonometric series, number theory, probability theory, approximation theory, summability theory and fuzzy set theory. Among them, Móricz [11] generalized the notion of statistical convergence to nondiscrete setting and defined the concept statistical limit of a measurable function at ∞ as the following: A Lebesgue measurable function f on interval (a,∞) has statistical limit at ∞ if there exists a number ℓ such that for every $\varepsilon > 0$,

$$\lim_{b\to\infty} \frac{1}{b-a} |\{x \in (a,b) : |f(x)-\ell| > \varepsilon\}| = 0,$$

where by $|\{.\}|$ we denote the Lebesgue measure of the set $\{.\}$, and is denoted by $\underset{x\to\infty}{\text{st-lim}}\, f(x) = \ell$.
In [11], Móricz introduced the notions of statistical limit, statistical limit inferior, statistical limit superior of a measurable function at ∞, studied corresponding properties and as an application he proved that Fourier integral $s_v(f,x)$ of a function $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ has statistical limit as $v \to \infty$ uniformly on $\mathbb{R}$. Besides in [12, 13], Móricz investigated the statistical limits of Cesàro and logarithmic averages of real- or complex-valued functions and introduced statistical Cesàro and statistical logarithmic summability of functions. In these studies, Tauberian conditions of slow decrease type under which statistical Cesàro and statistical logarithmic summability of functions imply ordinary limit at ∞ have also been given.
The main goal of this paper is to extend the concepts of statistical limit and statistical Cesàro summability to fuzzy analysis. In Section 2 we introduce statistical limits of strongly measurable fuzzy valued functions at infinity and give some basic properties of the statistical limits. In the sequel we give slowly decreasing-slowly oscillating type Tauberian conditions under which statistical limits of fuzzy valued functions imply ordinary limits. In Section 3, we first compare the statistical limits and the Cesàro limits in fuzzy setting, and introduce statistical Cesàro summability of fuzzy valued functions. Then, we give slowly decreasing-slowly oscillating type conditions for statistical Cesàro summability of fuzzy valued functions to imply ordinary limit in fuzzy number space. Before to continue with main results of the paper we give some preliminaries concerning fuzzy number space.
Let $\mathscr{K}_c(\mathbb{R}^n)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$. If $A, B \in \mathscr{K}_c(\mathbb{R}^n)$ and $k \in \mathbb{R}$ then the operations of addition and scalar multiplication are defined as

$$A+B = \{a+b : a \in A, b \in B\} \text{ and } kA = \{ka : a \in A\}.$$

The Hausdorff metric on $\mathscr{K}_c(\mathbb{R}^n)$ is defined by

$$d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\},$$

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where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.
A fuzzy number is a mapping $u : \mathbb{R}^n \to [0,1]$ which satisfies the following four conditions:
(i) u is normal, i.e. there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$.
(ii) u is fuzzy convex, i.e. $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$, for all $x, y \in \mathbb{R}^n$ and for all $\lambda \in [0,1]$.
(iii) u is upper semi-continuous.
(iv) The set $[u]_0 := \overline{\{x \in \mathbb{R}^n : u(x) > 0\}}$ is compact.[16]
The set of all fuzzy numbers is denoted by E^n and E^n is called fuzzy number space. If $u \in E^n$, then α-level set $[u]_\alpha$ of u, defined by

$$[u]_\alpha := \begin{cases} \{x \in \mathbb{R}^n : u(x) \geq \alpha\} & , \quad (0 < \alpha \leq 1), \\ \overline{\{x \in \mathbb{R}^n : u(x) > 0\}} & , \quad (\alpha = 0), \end{cases}$$

is a nonempty compact convex subset of $\mathbb{R}^n$.
Let $r \in \mathbb{R}^n$. We say that $\bar{r}$ is a crisp fuzzy number if

$$\bar{r}(x) := \begin{cases} 1 & , \quad \text{if } x = r \\ 0 & , \quad \text{otherwise.} \end{cases}$$

The operations addition and scalar multiplication on fuzzy numbers are defined by

$$u + v = w \quad \Longleftrightarrow \quad [w]_\alpha = [u]_\alpha + [v]_\alpha, \text{ for all } \alpha \in [0,1]$$

and

$$[ku]_\alpha = k[u]_\alpha, \text{ for all } \alpha \in [0,1].$$

The operations addition and scalar multiplication on fuzzy numbers have the following properties.

Lemma 1.1. *[10]*

(i) $\overline{0} \in E^n$ *is neutral element with respect to* $+$*, i.e.,* $u + \overline{0} = \overline{0} + u = u$*, for all* $u \in E^n$*.*
(ii) For any $a, b \in \mathbb{R}$ *with* $a, b \geq 0$ *or* $a, b \leq 0$*, and any* $u \in E^n$*, we have* $(a+b)u = au + bu$*. For general* $a, b \in \mathbb{R}$*, the above property does not hold.*
(iii) For any $a \in \mathbb{R}$ *and any* $u, v \in E^n$*, we have* $a(u+v) = au + av$*.*
(iv) For any $a, b \in \mathbb{R}$ *and any* $u \in E^n$*, we have* $a(bu) = (ab)u$*.*

The metric D on E^n is defined as follows:

$$D(u,v) := \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha).$$

From [10], we have the following lemma.

Lemma 1.2. *Let* $u, v, w, z \in E^n$ *and* $k \in \mathbb{R}$*.*

(i) (E^n, D) *is a complete metric space.*
(ii) $D(ku, kv) = |k| D(u,v)$*.*
(iii) $D(u+v, w+v) = D(u,w)$*.*
(iv) $D(u+v, w+z) \leq D(u,w) + D(v,z)$*.*
(v) $|D(u,\overline{0}) - D(v,\overline{0})| \leq D(u,v) \leq D(u,\overline{0}) + D(v,\overline{0})$*.*

We recall the concepts of measurability and integrability for fuzzy valued function.

Definition 1.3. *[7] Let* $T = [a,b] \subset \mathbb{R}$*. A function* $s : T \to E^n$ *is strongly measurable if for all* $\alpha \in [0,1]$ *the set valued function* $s_\alpha : T \to \mathscr{K}_c(\mathbb{R}^n)$ *defined by*

$$s_\alpha(x) = [s(x)]_\alpha$$

is Lebesgue measurable, when $\mathscr{K}_c(\mathbb{R}^n)$ *is endowed with the topology generated by Hausdorff metric* d*.*

Theorem 1.4. *[7] If fuzzy valued function* s *is strongly measurable, then it is measurable with respect to the topology generated by* D*.*

Definition 1.5. *[7] Let* $s : T \to E^n$*. The integral of* s *over* T *is defined by the following:*

$$\left[\int_T s(x)dx\right]_\alpha = \int_T [s(x)]_\alpha dx = \left\{\int_T f(x)dx \mid f : T \to \mathbb{R}^n \text{ is a measurable selection of } s_\alpha\right\},$$

for $\alpha \in (0,1]$*.*

A function $s : T \to E^n$ is called integrably bounded if there exists an integrable function $h : T \to \mathbb{R}^+$ such that $D(s(t), \bar{0}) \leq h(t)$, for all $t \in T$.
A strongly measurable and integrably bounded function $s : T \to E^n$ is said to be integrable over T if

$$\int_T s(x)dx \in E^n.$$

Theorem 1.6. *[7] If* $s : T \to E^n$ *is strongly measurable and integrably bounded, then* s *is integrable.*

Definition 1.7. *A fuzzy valued function* $s : T \to E^n$ *is said to be continuous at* $x_0 \in T$ *if for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $D(s(x), s(x_0)) < \varepsilon$*, whenever* $x \in T$ *with* $|x - x_0| < \delta$*. If* s *is continuous at each* $x \in T$*, then we say* s *is continuous on* T*.*

Theorem 1.8. *[7] If $s : T \to E^n$ is continuous, then s is integrable.*

Theorem 1.9. *[7] If $s : T \to E^n$ is continuous, $g(x) = \int_a^x s(t)dt$ is a Lipschitz continuous on T.*

Theorem 1.10. *[7] Let $f, g : T \to E^n$ be integrable and $\lambda \in \mathbb{R}$. Then,*

(i) $\int_T (f(x) + g(x))dx = \int_T f(x)dx + \int_T g(x)dx$;

(ii) $\int_T \lambda f(x)dx = \lambda \int_T f(x)dx$;

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, *where* $a < c < b$.

(iv) The function $F : T \to \mathbb{R}_+$ defined by $F(x) = D(f(x), g(x))$ is integrable on T and

$$D\left(\int_T f(x)dx, \int_T g(x)dx\right) \leq \int_T D(f(x), g(x))dx.$$

Lemma 1.11. *[10] Suppose $\mu \in E^n$ and define $s : T \to E^n$ by $s(x) = \mu$, for all $x \in [a, b]$. Then,*

$$\int_a^b s(x)dx = (b-a)\mu.$$

If $u \in E^1$, then α*-level set* $[u]_\alpha$ of u is closed, bounded and non-empty interval and we can write $[u]_\alpha := [u^-(\alpha), u^+(\alpha)]$. The partial ordering relation on E^1 is defined as follows:

$$u \preceq v \Longleftrightarrow [u]_\alpha \preceq [v]_\alpha \Longleftrightarrow u^-(\alpha) \leq v^-(\alpha) \text{ and } u^+(\alpha) \leq v^+(\alpha), \text{ for all } \alpha \in [0, 1].$$

Combining the results of Lemma 6 in [1], Lemma 5 in [2], Lemma 3.4, Theorem 4.9 in [9] and Lemma 14 in[14], following lemma is obtained.

Lemma 1.12. *Let $u, v, w, e \in E^1$ and $\varepsilon > 0$. The following statements hold:*

(i) $D(u, v) \leq \varepsilon$ *if and only if* $u - \overline{\varepsilon} \preceq v \preceq u + \overline{\varepsilon}$

(ii) If $u \preceq v + \overline{\varepsilon}$ *for every* $\varepsilon > 0$, *then* $u \preceq v$.

(iii) If $u \preceq v$ *and* $v \preceq w$, *then* $u \preceq w$.

(iv) If $u \preceq w$ *and* $v \preceq e$, *then* $u + v \preceq w + e$.

(v) If $u + w \preceq v + w$ *then* $u \preceq v$.

A fuzzy valued function $s : T \to E^1$ has the parametric representation

$$[s(x)]_\alpha = [s_\alpha^-(x), s_\alpha^+(x)],$$

where $s_\alpha^+, s_\alpha^- : T \to \mathbb{R}$, for all $\alpha \in [0, 1]$.

Theorem 1.13. *[8] Fuzzy valued function $s : T \to E^1$ is strongly measurable if and only if s_α^+ and s_α^- are measurable for all $\alpha \in [0, 1]$.*

Taking into account Remark 4.2 and Example 4.2 in [7] together, we have following lemma.

Lemma 1.14. *Fuzzy valued function $s : T \to E^1$ is integrable if and only if s_α^+, s_α^- are integrable over T and*

$$\left[\int_T s(x)dx\right]_\alpha = \left[\int_T s_\alpha^-(x)dx, \int_T s_\alpha^+(x)dx\right], \tag{1.1}$$

for all $\alpha \in [0, 1]$.

By Lemma 1.14, we obtain following useful lemma.

Lemma 1.15. *Let $f, g : T \to E^1$ be integrable and $f(x) \preceq g(x)$, for all $x \in T$. Then, $\int_T f(x)dx \preceq \int_T g(x)dx$.*

Definition 1.16. *[15] A fuzzy valued function $s : [0, \infty) \to E^1$ is said to be slowly decreasing if for every $\varepsilon > 0$ there exist $x_0 \geq 0$ and $\lambda > 1$ such that*

$$s(t) \succeq s(x) - \overline{\varepsilon}, \tag{1.2}$$

whenever $x_0 \leq x < t \leq \lambda x$.

Definition 1.17. *[4] A fuzzy valued function $s : [0, \infty) \to E^n$ is said to be slowly oscillating if for every $\varepsilon > 0$ there exist $x_0 \geq 0$ and $\lambda > 1$ such that*

$$D(s(t), s(x)) \leq \varepsilon, \tag{1.3}$$

whenever $x_0 \leq x < t \leq \lambda x$.

2. Statistical limits of strongly measurable fuzzy valued functions

In this section, we consider fuzzy valued function $s:[a,\infty)\to E^n$ where $a\geq 0$, unless otherwise stated.

Definition 2.1. *A fuzzy valued function s has limit at ∞, if there exists a fuzzy number μ such that $\lim_{x\to\infty} D(s(x),\mu)=0$. In this case, we write $\lim_{x\to\infty} s(x)=\mu$.*

Very recently(simultaneously with this paper), Belen [5] has defined statistical limits of continuous fuzzy valued functions and obtained related results. Definition of statistical limit in this paper is for strongly measurable fuzzy valued functions and more general than that of Belen since every continuous fuzzy valued function is strongly measurable(see [7, Lemma 3.2]). Our definition is as follows.

Definition 2.2. *A strongly measurable fuzzy valued function s has statistical limit at ∞, if there exists a fuzzy number μ such that for every $\varepsilon>0$,*

$$\lim_{b\to\infty}\frac{1}{b-a}\left|\{x\in(a,b):D(s(x),\mu)>\varepsilon\}\right|=0, \tag{2.1}$$

where by $|\{.\}|$ we denote the Lebesgue measure of the set $\{.\}$. In this case, we write $st\text{-}\lim_{x\to\infty} s(x)=\mu$.

Remark 2.3. *In (2.1), the set $\{x\in(a,b):D(s(x),\mu)>\varepsilon\}$ is Lebesgue measurable by Theorem 1.4.*

Theorem 2.4. *If statistical limit of a fuzzy valued function at ∞ exists, then it is unique.*

Proof. Let $\text{st-}\lim_{x\to\infty} s(x)=\mu$ and $\text{st-}\lim_{x\to\infty} s(x)=\nu$. Since $D(\mu,\nu)\leq D(s(x),\mu)+D(s(x),\nu)$, we have $D(\mu,\nu)=0$. So $\mu=\nu$. □

Theorem 2.5. *Let s be strongly measurable fuzzy valued function. Then,*

$$\lim_{x\to\infty} s(x)=\mu \Rightarrow st-\lim_{x\to\infty} s(x)=\mu. \tag{2.2}$$

Proof. Suppose that $\lim_{x\to\infty} s(x)=\mu$. Then, for $\varepsilon>0$ there exists $c>a$ such that $D(s(x),\mu)\leq\varepsilon$, whenever $x>c$. So

$$\lim_{b\to\infty}\frac{1}{b-a}\left|\{x\in(a,b):D(s(x),\mu)>\varepsilon\}\right|\leq\lim_{b\to\infty}\frac{c-a}{b-a}=0.$$

This means that $\text{st-}\lim_{x\to\infty} s(x)=\mu$. □

The converse of Theorem 2.5 does not hold in general. As a counter example, we can give the following.

Example 2.6. *Let $\mu,\nu\in E^n$ with $\mu\neq\nu$ and define $s:T\to E^n$ by*

$$s(x):=\begin{cases}\mu & , \quad if x\in\mathbb{Q}\\ \nu & , \quad otherwise.\end{cases}$$

Obviously, $st\text{-}\lim_{x\to\infty} s(x)=\nu$. However, $\lim_{x\to\infty} s(x)$ does not exist.

Theorem 2.7. *Let s_1,s_2 be strongly measurable fuzzy valued functions on some interval $[a,\infty)$, $\mu,\nu\in E^n$ and $c\in\mathbb{R}$. If $st\text{-}\lim_{x\to\infty} s_1(x)=\mu$ and $st\text{-}\lim_{x\to\infty} s_2(x)=\nu$, then $st\text{-}\lim_{x\to\infty}(s_1(x)+s_2(x))=\mu+\nu$ and $st\text{-}\lim_{x\to\infty}(cs_1(x))=c\mu$.*

Proof. We have the following inclusion: for every $\varepsilon>0$ and $b>a$,

$$\{x\in(a,b):D(s_1(x)+s_2(x),\mu+\nu)>\varepsilon\}\subseteq\left\{x\in(a,b):D(s_1(x),\mu)>\frac{\varepsilon}{2}\right\}\cup\left\{x\in(a,b):D(s_2(x),\nu)>\frac{\varepsilon}{2}\right\}.$$

So $\text{st-}\lim_{x\to\infty}(s_1(x)+s_2(x))=\mu+\nu$. Similarly, we have the following equation: for every $\varepsilon>0$ and $b>a$,

$$\{x\in(a,b):D(cs_1(x),c\mu)>\varepsilon\}=\left\{x\in(a,b):D(s_1(x),\mu)>\frac{\varepsilon}{|c|}\right\},$$

where $c\neq 0$. For $c=0$, validity of $\text{st-}\lim_{x\to\infty}(cs_1(x))=c\mu$ is obvious. So $\text{st-}\lim_{x\to\infty}(cs_1(x))=c\mu$. □

Theorem 2.8. *If a strongly measurable fuzzy valued function $s:[0,\infty)\to E^1$ is slowly decreasing, then $st\text{-}\lim_{x\to\infty} s(x)=\mu$ implies $\lim_{x\to\infty} s(x)=\mu$.*

Proof. Let fuzzy valued function $s:[0,\infty)\to E^1$ be slowly decreasing and $\text{st-}\lim_{x\to\infty} s(x)=\mu$. Then, for given an $\varepsilon>0$ there exist $x_0\geq 0$ and $\lambda>1$ such that slow decrease condition (1.2) is satisfied. Besides, replacing absolute value with metric D in [12, Lemma 1] we obtain fuzzy analogue of Lemma 1 in[12]. Then, since $\text{st-}\lim_{x\to\infty} s(x)=\mu$, there is a sequence $b_n\uparrow\infty$ of positive real numbers such that

$$D(s(b_n),\mu)\leq\varepsilon, \qquad n=1,2,\ldots \tag{2.3}$$

and for some n_0 we have

$$b_{n+1}<\lambda b_n, \qquad n=n_0+1,n_0+2,\ldots. \tag{2.4}$$

Now consider $t \in (b_n, b_{n+1}]$ for $n > n_0$. In view of (2.4) and monotonicity of sequence (b_n) we get

$$b_n < t \leq b_{n+1} < \lambda b_n < \lambda t.$$

So by slow decrease condition (1.2) and by (2.3), for every $n > n_0$ and $t \in (b_n, b_{n+1}]$ we have

$$s(t) \succeq s(b_n) - \bar{\varepsilon} \succeq \mu - 2\bar{\varepsilon}. \tag{2.5}$$

Again for every $n > n_0$ and $t \in (b_n, b_{n+1}]$ we have

$$s(t) \preceq s(b_{n+1}) + \bar{\varepsilon} \preceq \mu + 2\bar{\varepsilon}. \tag{2.6}$$

Then, combining (2.5) and (2.6) we get

$$D(s(t),\mu) \leq 2\varepsilon, \quad for\ every \quad t \in \bigcup_{n=n_0+1}^{\infty} (b_n, b_{n+1}] = (b_{n_0+1}, \infty)$$

by (i) of Lemma 1.12. This proves $\lim_{x\to\infty} s(x) = \mu$. □

Theorem 2.9. *If a strongly measurable fuzzy valued function $s : [0,\infty) \to E^n$ is slowly oscillating, then $st\text{-}\lim_{x\to\infty} s(x) = \mu$ implies $\lim_{x\to\infty} s(x) = \mu$.*

Proof. Let $\text{st-}\lim_{x\to\infty} s(x) = \mu$ and s be slowly oscillating. Then, as in the proof of Theorem 2.8, for given $\varepsilon > 0$ and $\lambda > 1$ there exists a sequence $b_n \uparrow \infty$ such that (2.3) and (2.4) are satisfied. By condition (2.4) and by condition of slow oscillation we have

$$D(s(t), s(b_n)) \leq \varepsilon, \quad \text{whenever} \quad x_0 \leq b_n < t < b_{n+1} \tag{2.7}$$

for large enough n, say $n > n_1$. From (2.3) and (2.7) it follows that

$$D(s(t),\mu) \leq D(s(t), s(b_n)) + D(s(b_n),\mu) \leq 2\varepsilon,$$

for every $t \in \bigcup_{n=n_1+1}^{\infty} (b_n, b_{n+1}] = (b_{n_1+1}, \infty)$. This means that $\lim_{x\to\infty} s(x) = \mu$. □

3. Statistical Cesàro summability of fuzzy valued functions

By $L_{loc}([a,\infty), E^n)$, we denote the set of fuzzy valued functions $s : [a,\infty) \to E^n$ such that s is integrable on every bounded interval $[a,x]$, $x > a$. Cesàro means of fuzzy valued functions are studied by many authors [3, 4, 15] and defined by the following.

Definition 3.1. *[15] Let $s \in L_{loc}([a,\infty), E^n)$ and $\mu \in E^n$. s is said to be Cesàro summable to μ if*

$$\lim_{b\to\infty} \frac{1}{b-a} \int_a^b s(x)dx = \mu, \tag{3.1}$$

and s is said to be strongly p-Cesàro summable μ if

$$\lim_{b\to\infty} \frac{1}{b-a} \int_a^b D(s(x),\mu)^p dx = 0. \quad (0 < p < \infty) \tag{3.2}$$

Theorem 3.2. *Let $s \in L_{loc}([a,\infty), E^n)$, where $a \geq 0$.*

(i) If s is strongly p-Cesàro summable to some $\mu \in E^n$, then the statistical limit of s at ∞ exists and equals to μ.
(ii) If the statistical limit of s at ∞ exists and equals to $\mu \in E^n$, and s is bounded, then s is strongly p-Cesàro summable to μ, for every $0 < p < \infty$.

Proof. We assume that s is strongly p-Cesàro summable to μ. Given $\varepsilon > 0$, By Markov's inequality,

$$\varepsilon^p \left|\{a < x < b : D(s(x),\mu) > \varepsilon\}\right| \leq \int_a^b D(s(x),\mu)^p dx,$$

for all $0 < p < \infty$ and $a < b < \infty$. So we obtain (2.1) from (3.2).
Conversely, we assume that the statistical limit of s at ∞ equals to μ and there exists $B > 0$ such that $D(s(x),\bar{0}) \leq B$, for all x. Then,

$$\int_a^b D(s(x),\mu)^p dx = \int_{\{a<x<b: D(s(x),\mu)\leq\varepsilon\}} + \int_{\{a<x<b: D(s(x),\mu)>\varepsilon\}} \leq (b-a)\varepsilon^p + (B + D(\mu,\bar{0}))^p \left|\{a < x < b : D(s(x),\mu) > \varepsilon\}\right|.$$

By (2.1), we obtain that

$$\lim_{b\to\infty} \frac{1}{b-a} \int_a^b D(s(x),\mu)^p dx \leq \varepsilon^p.$$

Since ε is arbitrary, we have (3.2). □

For convenience we take $a=0$ in the rest of the paper. Then, for $s\in L_{loc}([0,\infty),E^n)$, Cesàro average $\sigma(x)$ of s is

$$\sigma(x)=\frac{1}{x}\int_0^x s(u)du,\quad x\in(0,\infty).$$

We say that s is statistically Cesàro summable to a fuzzy number μ, if $\text{st-}\lim_{x\to\infty}\sigma(x)=\mu$.

Analogous of Corollary 1 in [15] may be given for $s\in L_{loc}([0,\infty),E^1)$ as the following. The proof is similar and hence omitted.

Theorem 3.3. *If $s\in L_{loc}([0,\infty),E^1)$ is Cesàro summable to a fuzzy number μ and is slowly decreasing, then $\lim_{x\to\infty}s(x)=\mu$.*

Note that in case of $p=1$, condition (3.2) implies condition (3.1) and so does statistical Cesàro summability. Then in view of (ii) of Theorem 3.2 we conclude that statistical limit of fuzzy valued functions implies statistical Cesàro summability under the condition of boundedness. In this section, we give the conditions (called Tauberian conditions) for statistical Cesàro summability of fuzzy valued functions to imply ordinary limit at ∞. That is, we aim to replace Cesàro summability with statistical Cesàro summability in Theorem 3.3 in view of the results of Móricz[12].

Lemma 3.4. *If $s:[0,\infty)\to E^1$ is a fuzzy valued function such that slow decrease condition (1.2) is satisfied for $\varepsilon:=1$, where $x_0>0$ and $\lambda>1$, then there exists a constant $B_1>0$ such that*

$$s(t)\succeq s(x)-B_1\ln\left(\frac{t}{x}\right)\quad whenever\quad x_0\le x<\frac{t}{\lambda}. \tag{3.3}$$

Proof. Let $s:[0,\infty)\to E^1$ be a fuzzy valued function such that slow decrease condition (1.2) is satisfied only for $\varepsilon:=1$, where $x_0>0$ and $\lambda>1$ and let $x_0\le x<\frac{t}{\lambda}$ be given. Then, consider the sequence

$$t_0:=t,\quad t_p:=\frac{t_{p-1}}{\lambda},\quad p=1,2,\dots,q+1,$$

where q is defined by the condition $t_{q+1}\le x<t_q$. Since (1.2) is satisfied for $\varepsilon:=1$, we get

$$s(t)\succeq s(t_1)-1\succeq s(t_2)-2\succeq\cdots\succeq s(t_q)-q\succeq s(x)-q-1.$$

Then, since $q<\frac{1}{\ln\lambda}\ln\left(\frac{t}{x}\right)$ we get

$$s(t)\succeq s(x)-1-\frac{1}{\ln\lambda}\ln\left(\frac{t}{x}\right),\quad whenever\quad x_0\le x<\frac{t}{\lambda}. \tag{3.4}$$

Then, in view of $x<\frac{t}{\lambda}$ we have $\ln\lambda<\ln\left(\frac{t}{x}\right)$, and as result we conclude

$$s(t)\succeq s(x)-B_1\ln\left(\frac{t}{x}\right),\quad whenever\quad x_0\le x<\frac{t}{\lambda},$$

where $B_1:=2/\ln\lambda$. □

Lemma 3.5. *Let $s\in L_{loc}([0,\infty),E^1)$. Under the assumptions of Lemma 3.4, there exists a constant $B_2>0$ such that*

$$\frac{1}{t}\int_{x_0}^t s(t)dx\succeq\frac{1}{t}\int_{x_0}^t s(x)dx-B_2,\quad whenever\quad t>\lambda x_0. \tag{3.5}$$

Proof. Let the fuzzy valued function s satisfy slow decrease condition only for $\varepsilon:=1$, where this time assume $x_0>0$. Then, by (3.3), we get the following:

$$\begin{aligned}\int_{x_0}^t s(t)dx &= \int_{x_0}^{\frac{t}{\lambda}} s(t)dx+\int_{\frac{t}{\lambda}}^t s(t)dx\\ &\succeq \int_{x_0}^{\frac{t}{\lambda}} s(x)dx-B_1\int_{x_0}^{\frac{t}{\lambda}}\ln\left(\frac{t}{x}\right)dx+\int_{\frac{t}{\lambda}}^t s(x)dx-\int_{\frac{t}{\lambda}}^t dx\\ &\succeq \int_{x_0}^t s(x)dx-B_1\int_0^{\frac{t}{\lambda}}\ln\left(\frac{t}{x}\right)dx-\int_0^t dx\\ &= \int_{x_0}^t s(x)dx-B_1t\left(\frac{\ln\lambda+1}{\lambda}\right)-t\\ &= \int_{x_0}^t s(x)dx-t\left\{B_1\left(\frac{\ln\lambda+1}{\lambda}\right)+1\right\}.\end{aligned}$$

If we take

$$B_2:=B_1\left(\frac{\ln\lambda+1}{\lambda}\right)+1,$$

this proves (3.5). □

Lemma 3.6. *If $s\in L_{loc}([0,\infty),E^1)$ is slowly decreasing, then Cesàro mean σ is also slowly decreasing.*

Proof. Let $s \in L_{loc}([0,\infty),E^1)$ and s be slowly decreasing. We aim to show that Cesàro mean σ of s is also slowly decreasing. Let some $0<\varepsilon<1$ be given. Then, consider $0<x_0\le x<t\le\lambda x$ that in slow decrease condition (1.2), where

$$1<\lambda\le 1+\frac{\varepsilon}{\max\{1,B_2\}}$$

and B_2 is from (3.5).
By the following equality

$$\begin{aligned}\sigma(t)+\left(1-\frac{x}{t}\right)\frac{1}{x}\int_0^{x_0}s(u)du+\left(1-\frac{x}{t}\right)\frac{1}{x}\int_{x_0}^{x}s(u)du+\left(1-\frac{x}{t}\right)\frac{x_0}{x}s(x) &= \sigma(t)+\left(1-\frac{x}{t}\right)\sigma(x)+\left(1-\frac{x}{t}\right)\frac{x_0}{x}s(x)\\ &= \sigma(x)+\frac{1}{t}\int_x^t s(u)du+\left(1-\frac{x}{t}\right)\frac{x_0}{x}s(x),\end{aligned}$$

we have

$$\sigma(t)+\left(1-\frac{x}{t}\right)\frac{1}{x}\int_0^{x_0}s(u)du+\left(1-\frac{x}{t}\right)\frac{1}{x}\int_{x_0}^{x}s(u)du+\left(1-\frac{x}{t}\right)\frac{x_0}{x}s(x)=\sigma(x)+\frac{1}{t}\int_x^t s(u)du+\left(1-\frac{x}{t}\right)\frac{x_0}{x}s(x). \tag{3.6}$$

Then, by Lemma 3.5 and from slow decrease condition, we get

$$\sigma(t)+\left(1-\frac{x}{t}\right)\frac{1}{x}\int_0^{x_0}s(u)du+\left(1-\frac{x}{t}\right)\left\{\frac{1}{x}\int_{x_0}^{x}s(x)du+B_2\right\}+\left(1-\frac{x}{t}\right)\frac{x_0}{x}s(x)\succeq\sigma(x)+\frac{1}{t}\int_x^t(s(x)-1)du+\left(1-\frac{x}{t}\right)\frac{x_0}{x}s(x),$$

which yields

$$\sigma(t)+\left(1-\frac{x}{t}\right)\frac{1}{x}\int_0^{x_0}s(u)du+\left(1-\frac{x}{t}\right)B_2\succeq\sigma(x)-\left(1-\frac{x}{t}\right)+\left(1-\frac{x}{t}\right)\frac{x_0}{x}s(x). \tag{3.7}$$

At this point there exists $x_1>\lambda x_0$ such that

$$\left(1-\frac{x}{t}\right)\frac{x_0}{x}s(x)\succeq-\bar{\varepsilon},\quad \textit{whenever}\quad x>x_1 \tag{3.8}$$

holds since by (3.4) we have

$$\frac{s(x)}{x}\succeq\frac{s(x_0)-1}{x}-\frac{1}{x\ln\lambda}\ln\left(\frac{x}{x_0}\right)\to 0,\quad \textit{as}\quad x\to\infty.$$

Besides there exists x_2 such that

$$\left(1-\frac{x}{t}\right)\frac{1}{x}\int_0^{x_0}s(u)du\preceq\bar{\varepsilon},\quad \textit{whenever}\quad x>x_2, \tag{3.9}$$

since

$$\lim_{x\to\infty}\left(1-\frac{x}{t}\right)\frac{1}{x}\int_0^{x_0}s(u)du=\bar{0}.$$

Also, from the fact $\frac{1}{\lambda}\le\frac{x}{t}$ we have

$$\left(1-\frac{x}{t}\right)B_2\le\left(1-\frac{1}{\lambda}\right)B_2\le(\lambda-1)B_2\le\varepsilon. \tag{3.10}$$

Again from the fact that $\frac{1}{\lambda}\le\frac{x}{t}$, we get

$$-\left(1-\frac{x}{t}\right)\ge-\left(1-\frac{1}{\lambda}\right)\ge-(\lambda-1)\ge-\varepsilon. \tag{3.11}$$

Then, inserting the expressions (3.8)–(3.11) in (3.7), we get

$$\sigma(t)\succeq\sigma(x)-4\bar{\varepsilon},\quad \textit{whenever}\quad x_3\le x<t\le\lambda x,$$

where $x_3=\max\{x_1,x_2\}$. This proves that σ is slowly decreasing. □

In view of Theorem 2.8, Theorem 3.3 and Lemma 3.6 we give the following result.

Theorem 3.7. *If $s\in L_{loc}([0,\infty),E^1)$ is slowly decreasing, then $st\text{-}\lim_{x\to\infty}\sigma(x)=\mu$ implies $\lim_{x\to\infty}s(x)=\mu$.*

Replacing absolute value with metric D in Lemma 3 and Lemma 5 in [12] we obtain the following lemmas in fuzzy setting.

Lemma 3.8. *If $s:[0,\infty)\to E^n$ is a fuzzy valued function such that slow oscillation condition (1.3) is satisfied for $\varepsilon:=1$, where $x_0>0$ and $\lambda>1$, then there exists a constant $B_3>0$ such that*

$$D(s(t),s(x))\le B_3\ln\left(\frac{t}{x}\right),\quad \textit{whenever}\quad x_0\le x<\frac{t}{\lambda}.$$

Lemma 3.9. *Let $s \in L_{loc}([0,\infty),E^n)$. Under the assumptions of Lemma 3.8, there exists a constant $B_4 > 0$ such that*

$$\frac{1}{t}\int_{x_0}^{t} D(s(t),s(x))dx \leq B_4, \quad \textit{whenever} \quad t > \lambda x_0.$$

Lemma 3.10. *If $s \in L_{loc}([0,\infty),E^n)$ is slowly oscillating, then Cesàro mean σ is also slowly oscillating.*

Proof. As in the proof of Lemma 3.6, for given $0 < \varepsilon < 1$ consider $0 < x_0 \leq x < t \leq \lambda x$ that in slow oscillation condition (1.3), where

$$1 < \lambda \leq 1 + \frac{\varepsilon}{\max\{1,B_4\}}$$

and B_4 is from Lemma 3.9. Adding $2\left(1-\frac{x}{t}\right)\left(1-\frac{x_0}{x}\right)s(x)$ to both sides of the equation (3.6), we get

$$\sigma(t) + \frac{t-x}{tx}\int_0^{x_0} s(u)du + \frac{t-x}{tx}\int_{x_0}^{x} s(u)du + \frac{1}{t}\int_x^t s(x)du + \frac{t-x}{tx}(x-x_0)s(x) = \sigma(x) + \frac{1}{t}\int_x^t s(u)du + \frac{t-x}{t}s(x) + \frac{t-x}{tx}\int_{x_0}^{x} s(x)du.$$

Then, by the properties given in Lemma 1.2 and Theorem 1.10, we have

$$\begin{aligned} D(\sigma(t),\sigma(x)) &= D\left(\frac{t-x}{tx}\int_0^{x_0} s(u)du + \frac{t-x}{tx}\int_{x_0}^{x} s(u)du + \frac{1}{t}\int_x^t s(x)du + \frac{t-x}{tx}(x-x_0)s(x), \frac{1}{t}\int_x^t s(u)du + \frac{t-x}{t}s(x) + \frac{t-x}{tx}\int_{x_0}^{x} s(x)du\right) \\ &\leq \frac{t-x}{tx}x_0 D(s(x),\bar{0}) + \frac{t-x}{tx}\int_0^{x_0} D(s(u),\bar{0})du + \frac{t-x}{tx}\int_{x_0}^{x} D(s(u),s(x))du + \frac{1}{t}\int_x^t D(s(u),s(x))du \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

By Lemma 3.8, there exists $x_1 > \lambda x_0$ such that $J_1 \leq \varepsilon$ for $x > x_1$ in view of the fact that

$$\frac{D(s(x),\bar{0})}{x} \leq \frac{D(s(x),s(x_0))}{x} + \frac{D(s(x_0),\bar{0})}{x} \leq B_3\frac{\ln(x/x_0)}{x} + \frac{D(s(x_0),\bar{0})}{x} \to 0 \quad (\text{as } x \to \infty).$$

Besides, since

$$\lim_{x\to\infty} \frac{t-x}{tx}\int_0^{x_0} D(s(u),\bar{0})du = 0,$$

there exists x_2 such that $J_2 \leq \varepsilon$ for $x > x_2$.
Furthermore, from the fact that $\frac{1}{\lambda} \leq \frac{x}{t}$ and by Lemma 3.9 we have $J_3 \leq (\lambda-1)B_4 \leq \varepsilon$, for $x > \lambda x_0$.
Again from the fact that $\frac{1}{\lambda} \leq \frac{x}{t}$ and by slow oscillation condition we have $J_4 \leq \varepsilon$.
Hence combining all findings we have

$$D(\sigma(t),\sigma(x)) \leq J_1 + J_2 + J_3 + J_4 \leq 4\varepsilon, \quad \text{whenever} \quad x_3 \leq x < t \leq \lambda x,$$

where $x_3 = \max\{x_1,x_2\}$, and this completes the proof. □

Analogous of Corollary 2.1 in [4] may be given for $s \in L_{loc}([0,\infty),E^n)$ as the following. The proof is similar and hence omitted.

Theorem 3.11. *If $s \in L_{loc}([0,\infty),E^n)$ is Cesàro summable to a fuzzy number μ and is slowly oscillating, then $\lim_{x\to\infty} s(x) = \mu$*

In view of Theorem 2.9, Lemma 3.10 and Theorem 3.11 we give the following result.

Theorem 3.12. *If $s \in L_{loc}([0,\infty),E^n)$ is slowly oscillating, then $st\text{-}\lim_{x\to\infty}\sigma(x) = \mu$ implies $\lim_{x\to\infty} s(x) = \mu$.*

Acknowledgement

We would like to thank reviewers for their careful reading and constructive criticism of the earlier version of this paper which improved the presentation and its readability.
*This work was presented partly in *4th International Conference on Pure and Applied Sciences (Renewable Energy), ICPAS2017, November 23-25, 2017, Istanbul-Turkey* with title "On some statistical weighted mean summability methods of fuzzy-number-valued functions".

References

[1] S. Aytar, M. Mammadov and S. Pehlivan, *Statistical limit inferior and limit superior for sequences of fuzzy numbers*, Fuzzy Set Syst **157**, No. 7 (2006) 976–985.
[2] S. Aytar and S. Pehlivan, *Statistical cluster and extreme limit points of sequences of fuzzy numbers*, Inform. Sci. **177**, No. 16 (2007) 3290–3296
[3] B. Bede and S. G. Gal, *Almost periodic fuzzy-number-valued functions*, Fuzzy Set Syst **147**, (2004) 385–403.
[4] C. Belen, *Tauberian theorems for weighted mean summability method of improper Riemann integrals of fuzzy-number-valued functions*, Soft Comput **22**, No. 12 (2018) 3951–3957
[5] C. Belen, *Tauberian theorems for statistical limit and statistical summability by weighted means of continuous fuzzy valued functions*, J. Math. Ext. (2020), preprint
[6] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2**, (1951) 241–244.
[7] O. Kaleva, *Fuzzy differential equations*, Fuzzy Set Syst **24**, (1987) 301–317.
[8] Y. K. Kim and B. M. Ghil, *Integrals of fuzzy-number-valued functions*, Fuzzy Set Syst **86**, (1997) 213–222
[9] H. Li and C. Wu, *The integral of a fuzzy mapping over a directed line*, Fuzzy Set Syst **158**, (2007) 2317–2338.
[10] J. Li, A. Zhao and J. Yan, *The Cauchy problem of fuzzy differential equations under generalized differentiability*, Fuzzy Set Syst **200**, (2012) 1–24.
[11] F. Móricz, *Statistical limits of measurable functions*, Analysis **24**, (2004) 1–18.
[12] F. Móricz, *Statistical extensions of some classical Tauberian theorems in nondiscrete setting*, Colloq. Math. **107**, No. 1 (2007) 45–56.
[13] F. Móricz and Németh Z., *Statistical extension of classical Tauberian theorems in the case of logarithmic summability*, Anal. Math. **40**, No. 3 (2014) 231–242.
[14] Ö. Talo and F. Başar, *On the Slowly Decreasing Sequences of Fuzzy Numbers*, Abstr. Appl. Anal. **2013**, (2013) 1–7.
[15] E. Yavuz, Ö. Talo and H. Çoşkun, *Cesàro summability of integrals of fuzzy-number-valued functions*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **67**, No. 2 (2018) 38–49.
[16] L. A. Zadeh, *Fuzzy sets*, Inform. Control **8**, (1965) 338–353.