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Almost Conformal η-Ricci Solitons in Three-Dimensional Lorentzian Concircular Structures

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Abstract

The object of the present paper is to study the properties of three-dimensional Lorentzian concircular structure ($(LCS)_3$-)manifolds admitting the almost conformal η-Ricci solitons and gradient shrinking η-Ricci solitons. It is proved that an $(LCS)_3$-manifold with either an almost conformal η-Ricci soliton or a gradient shrinking η-Ricci soliton is a quasi-Einstein manifold. Also, the example of an almost conformal η-Ricci soliton in an $(LCS)_3$-manifold is provided in the region where $(LCS)_3$-manifold is expanding.

Keywords: *η-Ricci solitons; $(LCS)_3$-manifold; Quasi-Einstein manifold; Einstein manifold.*
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1. Introduction

To study the Schönfliess conjecture, the Poincaré conjecture, Thurston's geometrization conjecture, the 1/4-pinching theorem, Yau's uniformization conjecture or isoperimetric inequalities, the geometric evolution equations play a major role during the last years. In recent years, the pioneering works of R. Hamilton [22] towards the solutions of the Poincaré conjecture in dimension three have produced a flourishing activity in the research of self similar solutions, or solitons of the Ricci flow. A Ricci flow is an intrinsic geometric flow (Ricci flow or Riemann flow or mean curvature flow). The study of the geometry of solitons, in particular their classification in dimension three, has been essential in providing a positive answer to the conjecture; however in higher dimension and in the complete, possibly noncompact case, the understanding of the geometry and the classification of solitons seem to remain a desired goal for a not too proximate future. In the generic case, a soliton structure on a Riemannian manifold (M,g) is the choice of a smooth vector field V on M and a real constant λ satisfying the structural requirement

$$\mathscr{L}_V g+2S+2\lambda g=0, \tag{1.1}$$

where S is the Ricci tensor and g is the Riemannian metric of M. $\mathscr{L}_V g$ denotes the Lie derivative of g in the direction of V. In what follows we shall refer to λ as the soliton constant (real constant). A Ricci soliton [22] on a Riemannian manifold (M,g) is a triplet (g,V,λ) satisfying equation (1.1). The soliton (g,V,λ) on (M,g) is called expanding, steady or shrinking if $\lambda>0$, $\lambda=0$ or $\lambda<0$, respectively. When V is the gradient of a potential function $\psi\in C^\infty(M)$, the soliton (g,V,λ) is called a gradient Ricci soliton [29] and the equation (1.1) takes the form

$$\nabla\nabla\psi+S+\lambda g=0, \tag{1.2}$$

where ∇ represents the Levi-Civita connection of the metric g on M. Both equations (1.1) and (1.2) can be considered as perturbations of the Einstein equation $S=\lambda g$ and reduce to this latter in case V or $\nabla\psi$ are Killing vector fields. When $V=0$ or ψ is constant we call the underlying Einstein manifold a trivial Ricci soliton. It is well know that, if the potential vector field ψ is zero or Killing then the Ricci soliton is an Einstein real hypersurfaces on non-flat complex-space-forms [8]. In 2009, J. T. Cho and M. Kimura [18] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non flat complex-space-forms admitting η-Ricci solitons. An η-Ricci soliton (g,V,λ,μ) on a Riemannian manifold (M,g) is defined by

$$\mathscr{L}_V g+2S+2\lambda g+2\mu\eta\otimes\eta=0, \tag{1.3}$$

where μ is a real constant and η is a 1-form defined as $\eta(X)=g(X,V)$. In particular, if $\mu=0$ then the η-Ricci soliton reduces to Ricci soliton. The properties of Ricci-soliton and η-Ricci solitons have been studied by several authors. For more details, we refer ([1], [3]-[5],

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[8]-[16], [23]-[25], [29], [30], [34]-[43]) and the references therein.
In [21], A. E. Fischer introduced a new concept called conformal Ricci flow, which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. These new equations are given by

$$\frac{\partial}{\partial t}g = -2S - \left(p + \frac{2}{n}\right)g, \tag{1.4}$$

where $R(g) = -1$ and p is a non-dynamical scalar field (time dependent scalar field), $R(g)$ is the scalar curvature of the manifold and n is the dimension of the manifold M. The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy the time dependent scalar field p is called a conformal pressure and, as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint. The equilibrium points of the conformal Ricci flow equations are Einstein metrics with Einstein constant $\frac{-1}{n}$. Thus the conformal pressure p is zero at an equilibrium point and positive otherwise.
In 2015, N. Basu and A. Bhattacharyya [2] introduced the notion of conformal Ricci soliton and the equation is follows as:

$$\mathcal{L}_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]g = 0. \tag{1.5}$$

It is an interesting and natural to see the condition in case of conformal η-Ricci soliton. Dutta et al. [20] also studied the properties of conformal Ricci soliton in Lorentzian α-Sasakian manifolds. From equations (1.3) and (1.5) we are introducing the notion of conformal η-Ricci soliton by the following equation

$$\mathcal{L}_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]g + 2\mu\eta\otimes\eta = 0. \tag{1.6}$$

In particular, if $\mu = 0$ then the data (g, V, λ) is a conformal-Ricci soliton [2]. Thus we can say that the conformal η-Ricci soliton is a generalization of conformal Ricci soliton.
The concept of almost Ricci soliton was first introduced by S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti in 2010 [29]. R. Sharma has also done excellent work in almost Ricci soliton [34]. A Riemannian manifold (M, g) is an almost Ricci soliton [34] if there exists a complete vector field X and a smooth soliton function λ such that $\lambda : M \longrightarrow \mathbb{R}$ satisfying

$$R_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) = \lambda g_{ij}, \tag{1.7}$$

where R_{ij} and $X_{ij} + X_{ji}$ stand for the Ricci tensor and the Lie derivative $\mathcal{L}_X g$ in local coordinates, respectively. It will called shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.
The notion of η-Ricci soliton has been studied by A. M Blaga ([3], [4]) and many others. Therefore, motivated by these studies in the present paper we study the notion of almost conformal η-Ricci soliton in $(LCS)_3$-manifolds. A data (g, V, λ, μ) on (M, g) is said to be almost conformal η-Ricci soliton if it satisfies equation (1.6), where $\lambda : M \longrightarrow \mathbb{R}$ is a smooth function.
A gradient Ricci soliton on a Riemannian manifold (M, g) is defined by [7]

$$S + Hess\,\psi = \rho g, \tag{1.8}$$

where $Hess\,\psi = \nabla\nabla\psi$, ρ is a constant and a smooth function ψ on M, called a potential function of the Ricci soliton. In particular, a gradient shrinking Ricci soliton [6] satisfies the equation

$$S + Hess\,\psi - \frac{1}{2\tau}g = 0, \tag{1.9}$$

where $\tau = T - t$ and T is the maximal time of the solution. Again for a conformal Ricci soliton if the vector field V is the gradient of a function f, then we call it as a conformal gradient shrinking Ricci soliton [34]. For conformal gradient shrinking Ricci soliton, the equation is

$$S + Hess\,\psi - \left(\frac{1}{2\tau} - \frac{2}{n} - p\right)g = 0, \tag{1.10}$$

where $\tau = T - t$ and T is the maximal time of the solution and ψ is the Ricci potential function.
In 2003, Shaikh [31] introduced the notion of the Lorentzian almost paracontact manifolds with a structure of the concircular type, in short $(LCS)_n$-manifolds. Since then, many authors studied the properties of $(LCS)_n$-manifolds, for instance ([23], [32], [33], [39], [40]). In the present paper we studied the properties of almost conformal and conformal gradient shrinking η-Ricci solitons in $(LCS)_3$-manifolds.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdroff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, ..., +)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (*resp.* non-sapcelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (*resp.* $\leq 0, = 0, > 0$) ([27], [28]). The category to which a given vector field is belonging, called its causal character.

Definition 2.1. *In a Lorentzian manifold (M,g), a vector field P defined by $g(X,P)=A(X)$ for any $X\in\chi(M)$ is said to be a concircular vector field if $(\nabla_X A)Y=\alpha\{g(X,Y)+\omega(X)A(Y)\}$, where α is a non-zero scalar and ω is a closed 1-form. Here $\chi(M)$ denotes the collection of all differentiable vector fields of M.*

Let M be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristics vector field of M. Then we have

$$g(\xi,\xi)=-1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X,\xi)=\eta(X), \tag{2.2}$$

the equation of the following form holds:

$$(\nabla_X\eta)Y=\alpha[g(X,Y)+\eta(X)\eta(Y)]\Longleftrightarrow\nabla_X\xi=\alpha\{X+\eta(X)\xi\} \tag{2.3}$$

for all vector fields $X,Y\in\chi(M)$, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X\alpha=(X\alpha)=d\alpha(X)=\rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho=-(\xi\alpha)$. If we put

$$\phi X=\frac{1}{\alpha}\nabla_X\xi, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\phi X=X+\eta(X)\xi, \tag{2.6}$$

$$g(\phi X,Y)=g(X,\phi Y), \tag{2.7}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with unit timelike concircular vector field ξ, an associated 1-form η and $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold or briefly called $(LCS)_n$-manifold [31]. Especially if we take $\alpha=1$, then we can obtain the *LP*-Sasakian structure (see, [19], [27]). An $(LCS)_n$-manifold, $n\geq 4$, is a generalized Robertson-Walker spacetime [17]. On an $(LCS)_3$-manifold the following relations hold [16]

$$\phi^2X=X+\eta(X)\xi, \tag{2.8}$$

$$\eta(\xi)=-1,\quad \phi\xi=0,\quad \eta\circ\phi=0, \tag{2.9}$$

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y), \tag{2.10}$$

$$\eta(R(X,Y)Z)=(\alpha^2-\rho)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)], \tag{2.11}$$

$$R(X,Y)Z=\phi R(X,Y)Z+(\alpha^2-\rho)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\xi, \tag{2.12}$$

$$S(X,Y)=\left[\frac{r}{2}-(\alpha^2-\rho)\right]g(X,Y)-\left[\frac{r}{2}-3(\alpha^2-\rho)\right]\eta(X)\eta(Y), \tag{2.13}$$

$$QX=\left[\frac{r}{2}-(\alpha^2-\rho)\right]X-\left[\frac{r}{2}-3(\alpha^2-\rho)\right]\eta(X)\xi, \tag{2.14}$$

$$(\nabla_X\phi)Y=\alpha\{g(X,Y)\xi+2\eta(X)\eta(Y)+\eta(Y)X\}, \tag{2.15}$$

$$(X\rho)=d\rho(X)=\beta\eta(X) \tag{2.16}$$

for all vector fields $X,Y,Z\in\chi(M)$. Using (2.12) and (2.13), for constants α and ρ, we have

$$R(X,Y)\xi=(\alpha^2-\rho)[\eta(Y)X-\eta(X)Y], \tag{2.17}$$

$$R(\xi,Y)Z=(\alpha^2-\rho)[g(Y,Z)\xi-\eta(Z)Y], \tag{2.18}$$

$$S(X,\xi)=2(\alpha^2-\rho)\eta(X), \tag{2.19}$$

$$Q\xi=2(\alpha^2-\rho)\xi, \tag{2.20}$$

where R is the curvature tensor, while Q is the Ricci operator given by $S(X,Y)=g(QX,Y)$.

Definition 2.2. *An $(LCS)_n$-manifold is said to be an η-Einstein if its non-vanishing Ricci tensor S of the type $(0,2)$ is of the form*

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y) \tag{2.21}$$

for all $X,Y\in\chi(M)$, where a and $b(\neq 0)$ are scalar constants on M. If a and $b(\neq 0)$ are scalar functions on (M,g) and satisfies equation (2.21), then the manifold to be quasi Einstein.

Again from the definition of Lie derivative and equations (2.1), (2.2), (2.3), we have

$$(\mathscr{L}_\xi g)(X,Y)=2\alpha[g(X,Y)+\eta(X)\eta(Y)], \quad \forall X,Y\in\chi(M). \tag{2.22}$$

Putting the above value in the conformal η-Ricci solition (1.6) and taking $n=3$, we get

$$S(X,Y)=-\frac{1}{2}\left[2(\lambda+\alpha)-\left(p+\frac{2}{3}\right)\right]g(X,Y)-(\mu+\alpha)\eta(X)\eta(Y), \tag{2.23}$$

which is equivalent to

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{2.24}$$

where $a=-\frac{1}{2}\left[2(\lambda+\alpha)-\left(p+\frac{2}{3}\right)\right]$ and $b=-(\mu+\alpha)$. Since α is a scalar function on (M,g), therefore we can state the following:

Theorem 2.3. *If an $(LCS)_3$-manifold admits a conformal η-Ricci soliton (g,ξ,λ,μ), then it becomes a quasi Einstein manifold.*

Also, we have

$$QX=-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{3}\right)\right]X-\alpha X-(\mu+\alpha)\eta(X)\xi \tag{2.25}$$

or

$$QX=aX+b\eta(X)\xi. \tag{2.26}$$

Again for an almost conformal η-Ricci soliton

$$S(X,Y)=-\lambda g(X,Y)+\frac{1}{2}\left(p+\frac{2}{3}\right)g(X,Y)-\alpha g(X,Y)-(\mu+\alpha)\eta(X)\eta(Y),$$

equivalently

$$S(X,Y)=(B-\lambda-\alpha)g(X,Y)-(\mu+\alpha)\eta(X)\eta(Y), \tag{2.27}$$

where $V=\xi$, $n=3$ and equations (1.6), (2.22) are used. Here $B=\frac{1}{2}\left(p+\frac{2}{3}\right)$. Equation (2.27) shows that the manifold under consideration is a class of quasi Einstein manifold (for details, see [12]-[16]). Thus we can state the following theorem as:

Theorem 2.4. *A three-dimensional $(LCS)_3$-manifold equipped with an almost conformal η-Ricci soliton (g,ξ,λ,μ) is a quasi Einstein manifold.*

3. Conformal η-Ricci soliton and almost conformal η-Ricci soliton

Let the vector field V is point-wise collinear with ξ on (M,g), i. e., $V=f\xi$, where f is a smooth function on an $(LCS)_3$-manifold. Then the equation (1.6) and covariant derivative of V give

$$g(\nabla_X V,Y)+g(X,\nabla_Y V)+2S(X,Y)+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)+2\mu\eta(X)\eta(Y)=0,$$

which becomes

$$fg(\nabla_X\xi,Y)+(Xf)\eta(Y)+(Yf)\eta(X)+fg(\nabla_Y\xi,X)+2S(X,Y)+\{2\lambda-\left(p+\frac{2}{n}\right)\}g(X,Y)+2\mu\eta(X)\eta(Y)=0. \tag{3.1}$$

In consequence of equations (2.2) and (2.3), the last expression assumes the form

$$2\alpha fg(X,Y)+2\alpha f\eta(X)\eta(Y)+(Xf)\eta(Y)+(Yf)\eta(X)+2S(X,Y)+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)+2\mu\eta(X)\eta(Y)=0. \tag{3.2}$$

Replacing Y with ξ in (3.2) and using (2.1), (2.2) and (2.19) in it, we get

$$Xf=\{2\lambda-2\mu-(p+\frac{2}{n})+\xi f+4(\alpha^2-\rho)\}\eta(X). \tag{3.3}$$

Again putting $X=\xi$ in (3.3) and then using the equations (2.1) and (2.2) we find that

$$\xi f=\frac{1}{2}\left(p+\frac{2}{n}\right)+\mu-2(\alpha^2-\rho)-\lambda. \tag{3.4}$$

Using (3.4) in (3.3) we have

$$Xf=-\left(\frac{1}{2}\left(p+\frac{2}{n}\right)+\mu-2(\alpha^2-\rho)-\lambda\right)\eta(X). \tag{3.5}$$

Applying exterior derivative of either sides of (3.5) and let λ and α are constants, then we have

$$\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+2(\alpha^2-\rho)-\mu=0 \tag{3.6}$$

because $d\eta\neq 0$ (in general). Using (3.6) in (3.5), we conclude that $Xf=0$. Since X is an arbitrary vector field of M, therefore f to be constant on M. Hence from (3.2), we have

$$S(X,Y)=-\left[\lambda+\alpha f-\frac{1}{2}\left(p+\frac{2}{n}\right)\right]g(X,Y)-(\mu+f\alpha)\eta(X)\eta(Y). \tag{3.7}$$

This expression reflects that the manifold under consideration is a certain class of η-Einstein manifold. Putting $X=Y=e_i$, where $\{e_i, i=1,2,3\}$ is an orthonormal basis of the tangent space TM at each point of the manifold (M,g), and summing over i, we get

$$r=-\frac{3}{2}\left[2\lambda+2\alpha f-\left(p+\frac{2}{n}\right)\right]+(\mu+f\alpha). \tag{3.8}$$

In case of conformal η-Ricci soliton, the scalar curvature of the manifold is -1, that is, $r=-1$. So putting this value in (3.8), we get

$$\lambda=\frac{1}{2}p+\frac{2}{3}-\frac{2}{3}\alpha f+\frac{1}{3}\mu. \tag{3.9}$$

Thus we can state the following theorem as:

Theorem 3.1. *Let an $(LCS)_3$-manifold admits a conformal η-Ricci soliton and the vector field V is point-wise collinear with ξ, then V is a constant multiple of ξ. Also, the value of $\lambda=\frac{1}{2}p+\frac{2}{3}-\frac{2}{3}\alpha f+\frac{1}{3}\mu$, provided α and ρ are constants.*

Theorem 3.2. *Let (g,V,λ,μ) be a conformal η-Ricci soliton on an $(LCS)_3$-manifold and V is point-wise collinear with ξ. Then (g,V,λ,μ) on (M,g) is shrinking, steady or expanding if $3p+2\mu+4<,=,$ or $>4\alpha f$, respectively.*

In consequence of equation (3.7) and Theorem 3.1, we conclude the following corollary as:

Corollary 3.3. *If the vector field V is point-wise collinear with ξ on an $(LCS)_3$-manifold bearing the conformal η-Ricci soliton (g,ξ,λ,μ), then the manifold to be η-Einstein.*

Again for almost conformal η-Ricci soliton we consider that λ is a smooth function. Then applying exterior derivative in (3.5) we get

$$\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+2(\alpha^2-\rho)-\mu=0 \tag{3.10}$$

and

$$d\lambda=0. \tag{3.11}$$

So λ is a constant function and from (3.5) and (3.10) we can say that f is constant. Hence we can conclude the following theorem:

Theorem 3.4. *If an $(LCS)_3$-manifold admits an almost conformal η-Ricci soliton and if V is point-wise collinear with ξ, then V is constant multiple of ξ as well as λ becomes constant function, i.e., almost conformal η-Ricci soliton becomes conformal η-Ricci soliton.*

Now from (2.13) and (2.22), the equation (1.6) takes the form

$$2\alpha[g(X,Y)+\eta(X)\eta(Y)]+2\left[\left(\frac{r}{2}-(\alpha^2-\rho)\right)g(X,Y)-\left(\frac{r}{2}-3(\alpha^2-\rho)\right)\eta(X)\eta(Y)\right]+\left[2\lambda-\left(p+\frac{2}{3}\right)\right]g(X,Y)+2\mu\eta(X)\eta(Y)=0.$$

For conformal η-Ricci soliton we have $r=-1$, so the above equation becomes

$$\left[2\alpha+2\left(-\frac{1}{2}-(\alpha^2-\rho\right)+\left(2\lambda-\left(p+\frac{2}{3}\right)\right)\right]g(X,Y)+\left[2\alpha-2\left(\frac{-1}{2}-3(\alpha^2-\rho)\right)+2\mu\right]\eta(X)\eta(Y)=0. \tag{3.12}$$

Now taking $X=Y=\xi$ in (3.12) and using the equations (2.1) and (2.2) in it, we get $\left[2\alpha+2\left(-\frac{1}{2}-(\alpha^2-\rho)\right)+\left(2\lambda-\left(p+\frac{2}{3}\right)\right)\right]-\left[2\alpha-2\left(\frac{-1}{2}-3(\alpha^2-\rho)\right)+2\mu\right]=0$, which gives

$$\lambda=4(\alpha^2-\rho)+\frac{1}{2}p+\frac{4}{3}+\mu. \tag{3.13}$$

Since $\alpha^2 \neq \rho$, so we discuss the following:
(i) Suppose $4(\alpha^2-\rho)+\frac{p+2\mu}{2} > -\frac{4}{3}$ and therefore the equation (3.13) shows that $\lambda > 0$. Thus the conformal η-Ricci soliton (g,ξ,λ,μ) is expanding.
(ii) Suppose $4(\alpha^2-\rho)+\frac{p+2\mu}{2} < -\frac{4}{3}$ and therefore (3.13) reveals that $\lambda < 0$. Thus the conformal η-Ricci soliton (g,ξ,λ,μ) is shrinking.
(iii) Suppose $4(\alpha^2-\rho)+\frac{p+2\mu}{2} = -\frac{4}{3}$ and therefore (3.13) allows that $\lambda = 0$. Thus the conformal η-Ricci soliton (g,ξ,λ,μ) is steady.
Thus we state our result in the form of theorem as:

Theorem 3.5. *A conformal η-Ricci soliton (g,ξ,λ,μ) on an $(LCS)_3$-manifold is said to be expanding, shrinking or steady if $4(\alpha^2-\rho)+\frac{p+2\mu}{2} >, <$ or $= -\frac{4}{3}$, respectively.*

Example 3.6. *Consider the three-dimensional differentiable manifold $M = \{(x,y,z) \in \mathbb{R}^3 \mid z \neq 0\}$, where (x,y,z) are the Cartesian coordinates in $\mathbb{R}^3$ and let the vector fields on M are given by*

$$e_1 = z^3\frac{\partial}{\partial x}, \quad e_2 = z^3\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent vector fields at each point of M. Let g be the Lorentzian metric defined by

$$g(e_1,e_1) = g(e_2,e_2) = -g(e_3,e_3) = 1,\ g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0.$$

Let η be the 1-form defined by $\xi = e_3$, $\eta(X) = g(X,e_3)$ for any vector field X on M and ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = 0.$$

Then by using the linearity of ϕ and g, we have $\phi^2X = X+\eta(X)\xi$, with $\xi = e_3$. Further $g(\phi X,\phi Y) = g(X,Y)+\eta(X)\eta(Y)$ holds for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure (ϕ,ξ,η,g) defines an $(LCS)_3$-structure in $\mathbb{R}^3$. The Lie brackets corresponding to the vector fields e_1, e_2, e_3 are

$$[e_1,e_2] = 0, \quad [e_1,e_3] = -\frac{3}{z}e_1, \quad [e_2,e_3] = -\frac{3}{z}e_2.$$

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y,Z) = Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]),$$

which is know as Koszul's formula. From above results, we can easily find

$$\nabla_{e_1}e_1 = -\frac{3}{z}e_3,\ \nabla_{e_1}e_2 = 0,\ \nabla_{e_1}e_3 = -\frac{3}{z}e_1,\ \nabla_{e_2}e_2 = -\frac{3}{z}e_2,\ \nabla_{e_2}e_1 = 0,\ \nabla_{e_2}e_3 = -\frac{3}{z}e_2,\ \nabla_{e_3}e_1 = 0,\ \nabla_{e_3}e_2 = 0,\ \nabla_{e_3}e_3 = 0.$$

Using above relations, for any vector X on M, we have $\nabla_X\xi = \alpha[X+\eta(X)\xi)]$, where $\alpha = -\frac{3}{z}$. Hence the structure (ϕ,ξ,η,g,α) defines an $(LCS)_3$-structure in $\mathbb{R}^3$ and the manifold M equipped with the $(LCS)_3$-structure is known as $(LCS)_3$-manifold of dimension 3.

Example 3.7. *On the $(LCS)_3$-manifold $(M,g,\xi,\eta,\phi,\alpha)$ considered in Example 3.6, the data (g,ξ,λ,μ) for $\lambda = \frac{3(z+2)}{z^2}$ and $\mu = \frac{3(z+10)}{z^2}$ defines an almost η-Ricci soliton.*

Indeed, the curvature and the Ricci tensors of M are computed as follows:

$$R(e_1,e_2)e_2 = \frac{6}{z^2}e_1, \quad R(e_1,e_3)e_3 = -\frac{12}{z^2}e_1, \quad R(e_2,e_1)e_1 = \frac{6}{z^2}e_2, \quad R(e_2,e_3)e_3 = -\frac{12}{z^2}e_2, \quad R(e_3,e_1)e_1 = \frac{12}{z^2}e_3, \quad R(e_3,e_2)e_2 = \frac{12}{z^2}e_3.$$

Other components of the curvature tensors can be evaluate by symmetric properties. From the above expression of the curvature tensors we can also obtain Ricci tensors as:

$$S(e_1,e_1) = S(e_2,e_2) = -\frac{6}{z^2}, \quad S(e_3,e_3) = -\frac{24}{z^2}.$$

Also $\alpha = -\frac{3}{z}$, $\rho = \frac{3}{z^2}$. By the definition of almost η- Ricci soliton and the equations (1.3) and (2.22), we obtain

$$S(X,Y) = -(\alpha+\lambda)g(X,Y)-(\alpha+\mu)\eta(X)\eta(Y). \tag{3.14}$$

Now, from (3.14) we obatin $S(e_1,e_1) = S(e_2,e_2) = -(\alpha+\lambda)$ and $S(e_3,e_3) = \lambda-\mu$, therefore $\lambda = \frac{3(z+2)}{z^2}$ and $\mu = \frac{3(z+10)}{z^2}$.
For the almost conformal η-Ricci soliton now using the equations (2.23), (2.24) and (2.27), we get $a = -\frac{6}{z^2}$, $b = -\frac{30}{z^2}$, $\mu = -\frac{3(10+z)}{z^2}$ and $\lambda = -\frac{3(2+z)}{z^2}+\frac{1}{2}p+\frac{1}{3}$. Hence the statement of the Theorem 2.3 and Theorem 2.4 are verified.

4. Conformal gradient shrinking η-Ricci soliton

A conformal gradient shrinking η-Ricci soliton equation on an $(LCS)_3$-manifold is given by

$$S+\nabla\nabla\psi=\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)g+\mu\eta\otimes\eta.$$

The above equation reduces to

$$\nabla_Y D\psi+QY=\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)Y+\mu\eta(Y)\xi. \tag{4.1}$$

From (4.1) it follows that

$$\nabla_X\nabla_Y D\psi+(\nabla_X Q)(Y)+Q(\nabla_X Y)=\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)\nabla_X Y+\mu[(\nabla_X\eta)(Y)\xi+\eta(\nabla_X Y)\xi+\eta(Y)\nabla_X\xi].$$

Now,

$$\begin{aligned}R(X,Y)D\psi=\nabla_X\nabla_Y D\psi-\nabla_Y\nabla_X D\psi-\nabla_{[X,Y]}D\psi=\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)\{\nabla_X Y-\nabla_Y X-[X,Y]\}-\mu\eta([X,Y])\xi\\-\alpha[\eta(X)Y-\eta(Y)X]-\nabla_X(QY)+\nabla_Y(QX)+Q[X,Y],\end{aligned} \tag{4.2}$$

where R is the curvature tensor. Since ∇ is the Levi-Civita connection, therefore the above equation with the help of the equations (2.1)-(2.3) assume the form

$$R(X,Y)D\psi=(\nabla_Y Q)X-(\nabla_X Q)Y+\mu\alpha[\eta(Y)X-\eta(X)Y]. \tag{4.3}$$

Differentiating equation (2.14) covariantly with respect to W and then using $W=\xi$, $n=3$ and equations (2.1), (2.2), we get

$$(\nabla_\xi Q)X=\frac{dr(\xi)}{2}(X-\eta(X)\xi), \tag{4.4}$$

provided α and ρ are non-zero constants. In the similar way from equation (2.20), we can obtain that

$$(\nabla_X Q)\xi=0. \tag{4.5}$$

Therefore the equations (4.4) and (4.19) give

$$g((\nabla_\xi Q)X-(\nabla_X Q)\xi,\xi)=g\left(\frac{dr(\xi)}{2}(X-\eta(X)\xi),\xi\right)=dr(\xi)\eta(X). \tag{4.6}$$

Putting this value in (4.3), we get

$$g(R(\xi,X)D\psi,\xi)=-dr(\xi)\eta(X). \tag{4.7}$$

Again from the equations (2.18), (4.7) and by hypothesis $r=-1$ on an $(LCS)_3$-manifold, we notice that

$$(\alpha^2-\rho)(g(X,D\psi)+\eta(X)\eta(D\psi))=0.$$

Since, $\alpha^2\neq\rho$, we have from the above equation $g(X,D\psi)=-\eta(X)g(D\psi,\xi)$, which implies

$$D\psi=-(\xi\psi)\xi. \tag{4.8}$$

Now from (4.1), we have $g(\nabla_Y D\psi,X)+g(QY,X)=\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)g(Y,X)+\mu\eta(Y)\eta(X)$. That is,

$$S(X,Y)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)g(Y,X)-\mu\eta(Y)\eta(X)=g(\nabla_Y(\xi\psi)\xi,X)=\alpha(\xi\psi)g(X,Y)+\alpha(\xi\psi)\eta(Y)\eta(X)+Y(\xi\psi)\eta(X). \tag{4.9}$$

Putting $X=\xi$ in (4.9) we get

$$S(Y,\xi)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)\eta(Y)+\mu\eta(Y)=-Y(\xi\psi),$$

where equations (2.1) and (2.2) are used. Using (2.19) in the above equation, we get

$$2(\alpha^2-\rho)\eta(Y)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)\eta(Y)+\mu\eta(Y)=-Y(\xi\psi). \tag{4.10}$$

Now interchanging X and Y in (4.9), we obtain

$$S(X,Y)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)g(X,Y)-\mu\eta(X)\eta(Y)=g(\nabla_X(\xi\psi)\xi,Y)=\alpha(\xi\psi)g(Y,X)+\alpha(\xi\psi)\eta(X)\eta(Y)+X(\xi\psi)\eta(Y). \tag{4.11}$$

Adding (4.9) and (4.11) we get

$$2S(X,Y)-2\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)g(X,Y)-2\mu\eta(X)\eta(Y)=2\alpha(\xi\psi)g(X,Y)+2\alpha(\xi\psi)\eta(X)\eta(Y)+Y(\xi\psi)\eta(X)+X(\xi\psi)\eta(Y). \tag{4.12}$$

Putting the value of $Y(\xi\psi)$ from equation (4.10) in (4.12), we find

$$S(X,Y)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)g(X,Y)-\mu\eta(X)\eta(Y)=\alpha(\xi\psi)g(X,Y)+\left(\alpha(\xi\psi)+\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)-2(\alpha^2-\rho)-\mu\right)\eta(X)\eta(Y), \tag{4.13}$$

which is equivalent to

$$QY-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)Y-\mu\eta(Y)\xi=\alpha(\xi\psi)Y+\alpha(\xi\psi)\eta(Y)\xi-2(\alpha^2-\rho)\eta(Y)\xi+\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)\eta(Y)\xi-\mu\eta(Y)\xi.$$

Hence from (4.1) we can write

$$\nabla_Y D\psi=-\alpha(\xi\psi)[Y+\eta(Y)\xi]+[2(\alpha^2-\rho)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)+\mu]\eta(Y)\xi. \tag{4.14}$$

Now,

$$\begin{aligned}R(X,Y)D\psi&=\nabla_X\nabla_Y D\psi-\nabla_Y\nabla_X D\psi-\nabla_{[X,Y]}D\psi=\nabla_X\left(-\alpha(\xi\psi)[Y+\eta(Y)\xi]+[2(\alpha^2-\rho)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)+\mu]\eta(Y)\xi\right)\\&\quad-\nabla_Y\left(-\alpha(\xi\psi)[X+\eta(X)\xi]+[2(\alpha^2-\rho)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)+\mu]\eta(X)\xi\right)-\nabla_{[X,Y]}D\psi\\&=-\alpha\{X(\xi\psi)(Y+\eta(Y)\xi)-Y(\xi\psi)(X+\eta(X)\xi)\}+\left(2(\alpha^2-\rho)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)+\mu-\alpha(\xi\psi)\right)\eta(\nabla_XY-\nabla_YX)\\&\quad-\alpha(\xi\psi)[\nabla_XY-\nabla_YX]+\alpha\left(2(\alpha^2-\rho)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)+\mu-\alpha(\xi\psi)\right)\{\eta(Y)X-\eta(X)Y\}-\nabla_{[X,Y]}D\psi,\end{aligned} \tag{4.15}$$

where we considered α as a constant and equations (2.1), (2.2), (2.3) and (4.14) are used. Also we have from equation (4.14)

$$\nabla_{[X,Y]}D\psi=-\alpha(\xi\psi)([X,Y]+\eta([X,Y])\xi)+[2(\alpha^2-\rho)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)+\mu])\eta([X,Y])\xi. \tag{4.16}$$

In view of the equation (4.16) and properties of the Levi-Civita connection, equation (4.15) takes the form

$$R(X,Y)D\psi=\alpha[2(\alpha^2-\rho)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)+\mu-\alpha(\xi\psi)]\{\eta(Y)X-\eta(X)Y\}-\alpha\{X(\xi\psi)(Y+\eta(Y)\xi)-Y(\xi\psi)(X+\eta(X)\xi)\}. \tag{4.17}$$

The inner product of (4.17) with ξ gives

$$2(\alpha^2-\rho)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)+\mu-\alpha(\xi\psi)=0.$$

From (4.10) we obtain

$$\alpha(\xi\psi)\eta(Y)+Y(\xi\psi)=0. \tag{4.18}$$

Using (4.18) in (4.12) we have

$$S(X,Y)-\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)g(X,Y)-\mu\eta(X)\eta(Y)=\alpha(\xi\psi)g(X,Y). \tag{4.19}$$

Setting $Y=X=e_i$, $\{e_i, i=1,2,3\}$ be an orthonormal basis of the tangent space at each point of the manifold, in above equation for $1\leq i\leq 3$, we find

$$\xi\psi=-\frac{1}{3\alpha}\left(-1-3\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)+\mu\right)=-K(say),$$

where K is some constant on M. So from (4.8) we get

$$D\psi=-(\xi\psi)\xi=K\xi. \tag{4.20}$$

Therefore $g(D\psi,X)=g(K\xi,X)$, which gives $d\psi(X)=K\eta(X)$. Applying exterior derivative on above relation we get $Kd\eta=0$ as $d^2\psi(X)=0$. So from (4.20) we have found that ψ is constant as $d\eta\neq 0$. Finally from (4.19) we get

$$S(X,Y)=\left(\frac{1}{2\tau}-\frac{2}{3}-p\right)g(X,Y)+\mu\eta(X)\eta(Y)=\{2(\alpha^2-\rho)+\mu\}g(X,Y)+\mu\eta(X)\eta(Y).$$

Hence M is an η-Einstein manifold. Thus we can state the following theorem:

Theorem 4.1. *If an $(LCS)_3$-manifold admits a conformal gradient shrinking η-Ricci soliton, then the manifold is an η-Einstein manifold.*

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