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Some New Inequalities of Hermite-Hadamard Type for Differentiable Godunova-Levin Functions via Fractional Integrals

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Abstract

In this paper, we present new inequalities of the Hermite -– Hadamard type related to fractional integrals for Godunova– Levin type functions. These inequalities are obtained with the help the of definitions of the Godunova-–Levin functions, the Holder and Power mean type inequalities.

Keywords: *convex function, Godunova-Levin function, s–Godunova–Levin function, Hermite–Hadamard inequalitiy, Hölder inequality, Riemann–Liouville fractional integral, power mean inequalitiy.*
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1. Introduction

In optimization theory, convex analysis occupies a special place ([1, 2]). Especially, a lot of research has been devoted address issues of the theory of convexity in last several decades(e.g. see [2]–[12] and references therein).
In convex analysis, the integral inequality of Hermite–Hamarad type plays very important role. For this reason, almost existing studies have been devoted to obtain upper bounds of this inequality for various classes of convex functions. This double inequality is stated as follows in literature (see [13]):
Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following double inequality;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

A large number of classes of convex function are built on the basis of the classical convexity of function in existing studies. One of these classes is the Godunova–Levin function ([14]).

Definition 1.1. *([14]) We say that $f : I \to R$ is a Godunova–Levin function or that f belongs to the class $Q(I)$, if f is non–negative and*

$$f(tx+(1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}, \forall x, y \in I, t \in (0,1). \tag{1.1}$$

The following two definitions were introduced by S. S. Dragomir et al. [15].

Definition 1.2. *([15]). A function $f : I \to \mathbb{R}$ is said to be $s-$Godunova-Levin functions of second kind, if*

$$f(tx+(1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}, \forall x, y \in I, t \in (0,1), s \in [0,1]. \tag{1.2}$$

Definition 1.3. *([15]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$; if it is nonnegative and for all $x; y \in I$ and $t \in [0;1]$; satisfies the following inequality:*

$$f(tx+(1-t)y) \leq f(x)+f(y). \tag{1.3}$$

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It is obvious, for $s=0$ that $s-$Godunova–Levin type function yield P function.
Godunova–Levin type inequalities, for different classes of convex functions have been considered in several papers (e.g. [16] - [22]). Akdemir et al. obtained inequalities for functions whose $p-$th powers are $h-$concave by using the Godunova–Levin inequality [16]. M.E. Özdemir et al. established two new convex dominant functions $(g,Q(I))$ and $(g,P(I))$ and obtained new Hadamard–type inequalities associated with these functions [17]. M. D. Noor et al. introduced a new class of preinvex $s-$Godunova–Levin functions and obtained new Hadamard type inequalities for functions whose first derivatives belong to this class [18]. M. A. Noor et al. introduced some new classes of Godunova–Levin functions ($s-$Godunova-Levin and $log-s-$Godunova–Levin functions first sense) and obtained Hermite–Hadamard type inequalities are derived for these classes [19]. M. E. Özdemir obtained new Hadamard–type inequalities for functions whose first derivatives are $s-$Godunova–Levin functions in the second sense [20].
In the literature (see [23]), the definition of a Riemann-–Liouville fractional integral is given in the following way

Definition 1.4. *Let $f\in L_1[a,b]$. The Riemann Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha>0$ with $a\geq 0$ are defined by*

$$J^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt\ , \qquad x>a$$

and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt\ , \qquad x<b$$

Here is $\Gamma(\alpha)=\int\limits_0^{\infty}e^{-u}u^{\alpha-1}du$ and if $\alpha=0$ then $J^0_{a^+}f(x)=J^0_{b^-}f(x)=f(x)$.

In this paper, we obtained new inequlities of Hermite type associated with fractional integrals for functions whose second derivatives are Gudonova–Levin functions. These inequalities are obtained with the help of the definition of Godunova–Levin function, the Hölder and Power mean type inequalities.
The results of this paper were obtained with the basis of the lemma which was formulated by B. Bayraktar [24]:

Lemma 1.5. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I°. If $f''\in L[a,b]$, where $a,b\in I$, then for all $\alpha>1$ the following equality holds*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}}\times U=\frac{(b-a)^2}{2}(I_1+I_2) \tag{1.4}$$

where

$$\begin{aligned} U &= \frac{(\alpha+1)}{b-a}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]-\left[J^{\alpha-1}_{a^+}f(b)+J^{\alpha-1}_{b^-}f(a)\right],\\ I_1 &= \int\limits_0^1 t(1-t)^{\alpha}f''(at+(1-t)b)dt,\\ I_2 &= \int\limits_0^1 t(1-t)^{\alpha}f''((1-t)a+tb)dt. \end{aligned}$$

2. Some New Inequalities

Theorem 2.1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on I°. If $f''\in L[a,b]$, where $a,b\in I$ and $|f''|$ is a $s-$Godunova–Levin convex function of second kind, then for all $\alpha>1$ and $t\in(0,1)$, the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}}\times U\right|\leq\frac{(b-a)^2}{2}\left[\frac{1}{(\alpha-s+1)(\alpha-s+2)}+B(2-s,\alpha+1)\right]\left(\left|f''(a)\right|+\left|f''(b)\right|\right) \tag{2.1}$$

where

$$U=\frac{\alpha+1}{b-a}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]-\left[J^{\alpha-1}_{a^+}f(b)+J^{\alpha-1}_{b^-}f(a)\right]$$

and B is Euler Beta function:

$$B(2-s,\alpha+1)=\int\limits_0^1 x^{1-s}(1-x)^{\alpha}dx \quad (2-s>0,\alpha+1>0).$$

Proof. From Lemma 1.5 and from the triangle inequality, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}}\times U\right|\leq\frac{(b-a)^2}{2}\left(|I_1|+|I_2|\right). \tag{2.2}$$

Since $|f''|$ is a $s-$Godunova – Levin function with inequality (1.2), we get

$$|I_1| \leq \int_0^1 t(1-t)^{\alpha}\left[\frac{|f''(a)|}{t^s}+\frac{|f''(b)|}{(1-t)^s}\right]dt = B(2-s,\alpha+1)\left|f''(a)\right|+\frac{1}{(\alpha-s+1)(\alpha-s+2)}\left|f''(b)\right|$$

Similarly

$$|I_2| \leq \int_0^1 t(1-t)^{\alpha}\left[\frac{|f''(a)|}{(1-t)^s}+\frac{|f''(b)|}{t^s}\right]dt = \frac{1}{(\alpha-s+1)(\alpha-s+2)}\left|f''(a)\right|+B(2-s,\alpha+1)\left|f''(b)\right|.$$

Taking into account the symmetry property of the Beta function, we write

$$|I_1|+|I_2| \leq \left[\frac{1}{(\alpha-s+1)(\alpha-s+2)}+B(\alpha+1,2-s)\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right]. \tag{2.3}$$

Multiplying both sides of inequality (2.3) by the expression $\frac{(b-a)^2}{2}$ and taking into account the inequality (2.2) we obtain (2.1). The proof is completed. □

The following corollary of Theorem 2.1 is obvious.

Corollary 1. *Under the conditions of Theorem 2.1, for $s=1$ and the Godunova–Levin functions we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}}\times U\right| \leq \frac{(b-a)^2}{2\alpha}\left(\left|f''(a)\right|+\left|f''(b)\right|\right) \tag{2.4}$$

where

$$U=\frac{\alpha+1}{b-a}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]-\left[J^{\alpha-1}_{a^+}f(b)+J^{\alpha-1}_{b^-}f(a)\right].$$

Remark 2.2. *If we choose $\alpha=2$ in (2.4) for Godunova–Levin functions, we get the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{4}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]. \tag{2.5}$$

Remark 2.3. *In Teorem 2.1 if we choose $\alpha=2$ and $s=0$ in (2.1) for P(I) functions we get inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]. \tag{2.6}$$

Theorem 2.4. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on I°. If $f''\in L[a,b]$, where $a,b\in I$ and $|f''|^q$ is a s– Godunova – Levin function then for all $\alpha,q>1$, $\frac{1}{q}+\frac{1}{p}=1$ and $t\in(0,1)$, the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}}\times U\right| \leq \frac{(b-a)^2}{2}\times 2^{-\frac{1}{p}}\times D \tag{2.7}$$

where

$$\begin{aligned}
U &= \frac{(\alpha+1)}{b-a}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]-\left[J^{\alpha-1}_{a^+}f(b)+J^{\alpha-1}_{b^-}f(a)\right],\\
D &= \left[\left|f''(a)\right|^q B(2-s,\alpha q+1)+\xi\left|f''(b)\right|^q\right]^{\frac{1}{q}}\\
&\quad+\left[\xi\left|f''(a)\right|^q+\left|f''(b)\right|^q B(2-s,\alpha q+1)\right]^{\frac{1}{q}},\\
\xi &= \frac{1}{(\alpha q-s+1)(\alpha q-s+2)}
\end{aligned}$$

and B is Euler Beta function.

Proof. From Lemma 1.5 and the triangle inequality we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}}\times U\right| \leq \frac{(b-a)^2}{2}\left(|I_1|+|I_2|\right). \tag{2.8}$$

Using the well-known Hölder inequality, since $|f''|^q$ is a s– Godunova – Levin function, we have

$$\begin{aligned}
|I_1| &= \left|\int_0^1 t(1-t)^{\alpha}f''(at+(1-t)b)dt\right| \leq \int_0^1 t^{\frac{1}{p}}t^{\frac{1}{q}}(1-t)^{\alpha}\left|f''(at+(1-t)b)\right|dt\\
&\leq \left(\int_0^1 tdt\right)^{\frac{1}{p}}\left[\int_0^1 t(1-t)^{\alpha q}\left[\frac{|f''(a)|^q}{t^s}+\frac{|f''(b)|^q}{(1-t)^s}\right]dt\right]^{\frac{1}{q}}\\
&= 2^{-\frac{1}{p}}\times\left[\left|f''(a)\right|^q\int_0^1 t^{1-s}(1-t)^{\alpha q}dt+\left|f''(b)\right|^q\int_0^1 t(1-t)^{\alpha q-s}dt\right]^{\frac{1}{q}}
\end{aligned}$$

and so, we finde the inequality

$$|I_1| \leq 2^{-\frac{1}{p}} \times \left[\left|f''(a)\right|^q B(2-s,\alpha q+1)+\xi\left|f''(b)\right|^q\right]^{\frac{1}{q}}. \tag{2.9}$$

For the second integral, we can write

$$\begin{aligned}|I_2| &= \left|\int_0^1 t(1-t)^{\alpha} f''((1-t)a+tb)dt\right| = \left|\int_0^1 t^{\alpha}(1-t) f''(ta+(1-t)b)dt\right| = \left|\int_0^1 t^{\alpha}(1-t)^{\frac{1}{p}}(1-t)^{\frac{1}{q}} f''(ta+(1-t)b)dt\right| \\ &\leq \left(\int_0^1 (1-t)\,dt\right)^{\frac{1}{p}} \left[\int_0^1 t^{\alpha q}(1-t)\left|f''(ta+(1-t)b)\right|^q dt\right]^{\frac{1}{q}} \\ &\leq 2^{-\frac{1}{p}} \left[\left|f''(a)\right|^q \int_0^1 t^{\alpha q-s}(1-t)\,dt+\left|f''(b)\right|^q \int_0^1 t^{\alpha q}(1-t)^{1-s}\,dt\right]^{\frac{1}{q}}\end{aligned}$$

and so, we finde the inequality

$$|I_2| \leq 2^{-\frac{1}{p}} \times \left[\xi\left|f''(a)\right|^q+\left|f''(b)\right|^q B(\alpha q+1,2-s)\right]^{\frac{1}{q}}. \tag{2.10}$$

Adding the last inequalites (2.9) and (2.10) and taking into account that the Beta symmetric function, we get

$$|I_1|+|I_2| \leq 2^{-\frac{1}{p}} \times D. \tag{2.11}$$

Multiplying both sides of inequality (2.11) by the expression $\frac{(b-a)^2}{2}$ and taking into account inequality (2.8) we obtain (2.7). The proof thus completed. □

It is easy to show the validity of the following corollary of Theorem 2.4

Corollary 2. *Under the conditions of Theorem 2.4, with $s=1$ for the Godunova–Levin functions we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}} \times U\right| \leq \frac{(b-a)^2}{4}\left[\frac{2}{\alpha q(\alpha q+1)}\right]^{\frac{1}{q}} \times D \tag{2.12}$$

where

$$\begin{aligned} U &= \frac{(\alpha+1)}{b-a}\left[J_{a^+}^{\alpha} f(b)+J_{b^-}^{\alpha} f(a)\right]-\left[J_{a^+}^{\alpha-1} f(b)+J_{b^-}^{\alpha-1} f(a)\right], \\ D &= \left[\alpha q\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\left[f''(a)^q+\alpha q\left|f''(b)\right|^q\right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 3. *If we choose $\alpha=2$ in (2.12), for Godunova–Levina functions, we get*

$$\xi=\frac{1}{2q(2q+1)} \text{ and } B(1,2q+1)=B(2q+1,1)=\frac{1}{2q+1},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{4} \times \varphi(q) \times F \tag{2.13}$$

where

$$\begin{aligned} \varphi(q) &= \left(\frac{1}{q(2q+1)}\right)^{\frac{1}{q}}, \\ F &= \left[\left|f''(a)\right|^q+2q\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\left[2q\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}. \end{aligned}$$

Here, since the $\lim_{q\to 1^+}\varphi(q)=\frac{1}{3}$ and $\lim_{q\to\infty}\varphi(q)=1$ then $\frac{1}{3}\leq\ \varphi(q)<1$ for all $q>1$. For $q\to 1^+$ in (2.13) we get the inequality (2.5).

Corollary 4. *If we choose $s=0$ and $\alpha=2$, in (2.7) since $\frac{1}{p}=1-\frac{1}{q}$, then for $P(I)$ functions, we get*

$$\xi=\frac{1}{(2q+1)(2q+2)} \text{ and } B(2,2q+1)=B(2q+1,2)=\frac{1}{(2q+1)(2q+2)},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2} \times \varphi(q) \times \left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}} \tag{2.14}$$

where

$$\varphi(q)=\left(\frac{1}{(q+1)(2q+1)}\right)^{\frac{1}{q}}.$$

Here, since the $\lim_{q\to 1^+}\varphi(q)=\frac{1}{6}$ and $\lim_{q\to\infty}\varphi(q)=1$ then $\frac{1}{6}\leq\ \varphi(q)<1$ for all $q>1$. For $q\to 1^+$ in (2.14) we get the inequality (2.6).

Theorem 2.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. If $f'' \in L[a,b]$, where $a,b \in I$ and $|f''|^q$ is a $s-$Godunova–Levin convex function then for all $\alpha > 1$, $q \geq 1$ and $t \in (0,1)$ the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}} \times U\right| \leq \frac{(b-a)^2}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \times V \tag{2.15}$$

where

$$U = \frac{(\alpha+1)}{b-a}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] - \left[J_{a^+}^{\alpha-1} f(b) + J_{b^-}^{\alpha-1} f(a)\right]$$

and

$$\begin{aligned} V &= \left[\left|f''(a)\right|^q B(q-s+1,\alpha+1) + \left|f''(b)\right|^q B(q+1,\alpha-s+1)\right]^{\frac{1}{q}} \\ &\quad + \left[\left|f''(a)\right|^q B(q+1,\alpha-s+1) + \left|f''(b)\right|^q B(q-s+1,\alpha+1)\right]^{\frac{1}{q}}. \end{aligned}$$

Proof. From Lemma 1.5 and the triangle inequality we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}} \times U\right| \leq \frac{(b-a)^2}{2}\left(|I_1| + |I_2|\right). \tag{2.16}$$

Using the well–known Power–mean integral inequality, since $|f''|^q$ is a $s-$Godunova–Levin convex function, we have

$$\begin{aligned} |I_1| &\leq \int_0^1 t(1-t)^{\alpha}\left|f''(at+(1-t)b)\right| dt \leq \left(\int_0^1 (1-t)^{\alpha} dt\right)^{1-\frac{1}{q}} \left[\int_0^1 (1-t)^{\alpha} t^q \left(\frac{|f''(a)|^q}{t^s} + \frac{|f''(b)|^q}{(1-t)^s}\right) dt\right]^{\frac{1}{q}} \\ &= \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\left|f''(a)\right|^q \int_0^1 (1-t)^{\alpha} t^{q-s} dt + \left|f''(b)\right|^q \int_0^1 (1-t)^{\alpha-s} t^q dt\right]^{\frac{1}{q}} \\ &= \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\left|f''(a)\right|^q B(q-s+1,\alpha+1) + \left|f''(b)\right|^q B(q+1,\alpha-s+1)\right]^{\frac{1}{q}}. \end{aligned}$$

Thus, for $|I_1|$, we get

$$|I_1| \leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\left|f''(a)\right|^q B(q-s+1,\alpha+1) + \left|f''(b)\right|^q B(q+1,\alpha-s+1)\right]^{\frac{1}{q}}.$$

Similarly, for the second integral, we can write

$$|I_2| \leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\left|f''(a)\right|^q B(q+1,\alpha-s+1) + \left|f''(b)\right|^q B(q-s+1,\alpha+1)\right]^{\frac{1}{q}}$$

And adding the last inequalites, we get

$$|I_1| + |I_2| \leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \tag{2.17}$$

$$\times \left\{ \begin{array}{c} \left[|f''(a)|^q B(q-s+1,\alpha+1) + |f''(b)|^q B(q+1,\alpha-s+1)\right]^{\frac{1}{q}} \\ + \left[|f''(a)|^q B(q+1,\alpha-s+1) + |f''(b)|^q B(q-s+1,\alpha+1)\right]^{\frac{1}{q}} \end{array} \right\}.$$

Finally, multiplying both sides of inequality (2.17) by the expression $\frac{(b-a)^2}{2}$ and taking into account (2.16) we obtain the desired inequality (2.15). The proof is completed. □

Corollary 5. *Under the conditions of Theorem 2.5, for Godunova–Levin functions we obtain*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}} \times U\right| \leq \frac{(b-a)^2}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\frac{\Gamma(q)\Gamma(\alpha)}{\Gamma(q+\alpha+1)}\right]^{\frac{1}{q}} \times V \tag{2.18}$$

where

$$U = \frac{(\alpha+1)}{b-a}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] - \left[J_{a^+}^{\alpha-1} f(b) + J_{b^-}^{\alpha-1} f(a)\right]$$

and

$$V = \left[\alpha\left|f''(a)\right|^q + q\left|f''(b)\right|^q\right]^{\frac{1}{q}} + \left[q\left|f''(a)\right|^q + \alpha\left|f''(b)\right|^q\right]^{\frac{1}{q}}.$$

Proof. If we take $s=1$, then we have

$B(q-s+1,\alpha+1)=B(q,\alpha+1)$ and $B(q+1,\alpha-s+1)=B(q+1,\alpha)$.

Using the well-known properties of Beta and Gamma functions

$$B(q,\alpha+1)=\frac{\Gamma(q)\Gamma(\alpha+1)}{\Gamma(q+\alpha+1)}=\frac{\alpha\Gamma(q)\Gamma(\alpha)}{\Gamma(q+\alpha+1)} \text{ and } B(q+1,\alpha)=\frac{q\Gamma(q)\Gamma(\alpha)}{\Gamma(q+\alpha+1)},$$

we have

$$|I_1|\leq\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{\Gamma(q)\Gamma(\alpha)}{\Gamma(q+\alpha+1)}\right]^{\frac{1}{q}}\left[\alpha\left|f''(a)\right|^q+q\left|f''(b)\right|^q\right]^{\frac{1}{q}}.$$

Similarly, for the second integral, we can write

$$|I_2|\leq\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{\Gamma(q)\Gamma(\alpha)}{\Gamma(q+\alpha+1)}\right]^{\frac{1}{q}}\left[q\left|f''(a)\right|^q+\alpha\left|f''(b)\right|^q\right]^{\frac{1}{q}}.$$

Summing the last two inequalities and multiplying both sides of the resulting inequality by the expression $\frac{(b-a)^2}{2}$, we obtain the inequality (2.18). The proof is completed. □

Remark 2.6. *If we take $\alpha=2$ and $q=1$ in (2.18), we reach the inequality (2.5).*

It is not difficult to show that the following corollary of Theorem 2.5 is true.

Corollary 6. *If we choose $s=0$ and $\alpha=2$, in (2.15), then, for $P(I)$ functions, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{3}\times\psi(q)\times\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}} \tag{2.19}$$

where

$$\psi(q)=\left[\frac{6}{(q+1)(q+2)(q+3)}\right]^{\frac{1}{q}}.$$

Here, since the $\lim_{q\to1^+}\psi(q)=\frac{1}{4}$ and $\lim_{q\to\infty}\psi(q)=1$ then $\frac{1}{4}\leq\psi(q)<1$ for all $q>1$.

Remark 2.7. *For $q\to1^+$ in (2.19) and P(I) functions we get (2.6).*

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