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Yang–Baxter Equation in Lattice Effect Algebras

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Abstract

In this paper, at first lattice effect algebras and the Yang-Baxter equation are given and some of their properties are presented. We examine set-theoretical solutions of the Yang-Baxter equation in lattice effect algebras.

Keywords: *Yang-Baxter equation, Lattice effect algebra, , bounded involutive lattice.*
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1. Introduction

The Yang-Baxter equation was firstly given by C.N. Yang in theoretical physics and by R.J Baxter in statistical mechanics. It gave rise to be one of the main equations in mathematical physics.
The Yang-Baxter equation has been worked on as the one of the main equations in theoretical physics [15], in statistical mechanics ([1], [2], [16]), in integrable systems. It has been caused many applications like quantum groups, quantum computing, knot theory, braided categories, etc. [10]. One of the significant unsolved problems is to discover all the solutions R of the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where V is a vector space, $R : V \otimes V \to V \otimes V$ is a linear map and R_{ij} denotes the map $V \otimes V \otimes V \to V \otimes V \otimes V$ that acts as R on the (i, j) tensor factor and as the identity on the remaining factor. In recent years, many solutions have been found and the related algebraic structures have been intensively studied [9]. Drinfeld, in [6] , posed the question of finding the simplest solutions, that is, the solutions R that are induced by a linear extension of a mapping $\Re : X \times X \to X \times X$, where X is a basis for V . In this case, one says that $\Re$ is a set theoretic solution of the quantum Yang-Baxter equation. We want to sight the Yang-Baxter equation with regard to lattice effect algebras.

In the recent times, the solutions for the Yang-Baxter equation in Wasjberg Algebras, MTL-algebras and BL-algebras were given $[12-14]$.

Effect algebras, offered by D.J. Foulis and M.K. Bennett [8] in 1994, were identified as abstract models of the set of quantum effects (self-adjoint operators between the zero and identity operator as regards the usual ordering). It is an unsharp (or fuzzy) generalization of an orthomodular lattice. Lattice effect algebras may consider as a common framework for orthomodular lattices and MV-algebras. Lately, effect algebras and their generalizations have been extremely studied in various ways ([4], [5], [7]).

In this paper, making use of the axioms of the effect algebras, we give some solutions of the set-theoretical Yang-Baxter equation in lattice effect algebras.

2. Preliminaries

The reader is referred to [3] for the notions and notations. We present some of the prelimary definitions and results.

Definition 2.1. *By an effect algebra we shall mean an algebra $(E;\oplus,0,1)$ of type (2, 0) satisfying the following axioms:*
(1) if $e_{a_1}\oplus e_{a_2}$ is defined, then $e_{a_2}\oplus e_{a_1}$ is defined and $e_{a_1}\oplus e_{a_2} = e_{a_2}\oplus e_{a_1}$;
(2) if $e_{a_2}\oplus e_{a_3}$ and $e_{a_1}\oplus(e_{a_2}\oplus e_{a_3})$ are defined, then $e_{a_1}\oplus e_{a_2}$ and $(e_{a_1}\oplus e_{a_2})\oplus e_{a_3}$ are defined and $e_{a_1}\oplus(e_{a_2}\oplus e_{a_3}) = (e_{a_1}\oplus e_{a_2})\oplus e_{a_3}$;
(3) for every $e\in E$, there exists a unique $e'\in E$, orthosupplement of e, such that $e\oplus e'$ is defined and $e\oplus e' = 1$;
(4) if $e\oplus 1$ is defined, then $e = 0$.

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$(E;\oplus,0,1)$, or simply E, is a lattice effecet algebra.

The following useful properties of effect algebras can be found in [7] or [8] :
(e1) $e''=e$;
(e2) $1'=0$ and $0'=1$;
(e3) $e\oplus 0=e$;
(e4) $e_{a_1}\leq e_{a_2}\Rightarrow e'_{a_2}\leq e'_{a_1}$;
(e5) $e_{a_1}\leq e_{a_2}\Rightarrow e_{a_1}\perp(e_{a_1}\oplus e'_{a_2})'$ and $e_{a_1}\oplus(e_{a_1}\oplus e'_{a_2})'=e_{a_2}$.

If a partial operation $\odot$ on any effect algebra E is defined by $e_{a_1}\odot e_{a_2}=(e'_{a_1}\oplus e'_{a_2})'$, then following are hold:
(e6) if $e_{a_1}\odot e_{a_2}$ is defined, then $e_{a_2}\odot e_{a_1}$ is defined and $e_{a_1}\odot e_{a_2}=e_{a_2}\odot e_{a_1}$;
(e7) if $e_{a_2}\odot e_{a_3}$ and $e_{a_1}\odot(e_{a_2}\odot e_{a_3})$ are defined, then $e_{a_1}\odot e_{a_2}$ and $(e_{a_1}\odot e_{a_2})\odot e_{a_3}$ are defined and $e_{a_1}\odot(e_{a_2}\odot e_{a_3})=(e_{a_1}\odot e_{a_2})\odot e_{a_3}$.

In every lattice effect algebra,

$$(e_{a_1}\odot e_{a_2})' \quad = \quad e'_{a_1}\oplus e'_{a_2}$$

holds for all $e_{a_1},e_{a_2}\in E$.

3. Solutions to the Yang-Baxter Equation in Lattice Effect Algebras

In this paper, we present some results in connection with the (set-theoretical) Yang- Baxter equation in lattice effect algebras.
Let V be a vector space over a field F, which is algebraically closed and of characteristic zero.

Definition 3.1. *[11] A linear automorphism ψ of $V\otimes V$ is a solution of the Yang-Baxter equation, if the following equality holds in the automorphism group of $V\otimes V\otimes V$:*

$$(\psi\otimes id_V)\circ(id_V\otimes\psi)\circ(\psi\otimes id_V)=(id_V\otimes\psi)\circ(\psi\otimes id_V)\circ(id_V\otimes\psi). \tag{3.1}$$

In the following equations ψ^{nm} means ψ acting on the n-th and m-th component.

ψ is a solution of the Yang-Baxter equation if

$$\psi^{12}\circ\psi^{23}\circ\psi^{12}=\psi^{23}\circ\psi^{12}\circ\psi^{23}. \tag{3.2}$$

Definition 3.2. *[11] ψ is a solution of the quantum Yang-Baxter equation if*

$$\psi^{12}\circ\psi^{13}\circ\psi^{23}=\psi^{23}\circ\psi^{13}\circ\psi^{12}. \tag{3.3}$$

Let T be the twist map $T:V\otimes V\to V\otimes V$ defined by $T(u\otimes v)=v\otimes u$. Then, ψ satisfies (3.2) if and only if $\psi\circ T$ satisfies (3.3) if and only if $T\circ\psi$ satisfies (3.3).
A connection between the set-theoretical Yang-Baxter equation and lattice effect algebras is constituted by the following definition.

Definition 3.3. *[11] Let X be a set and $\psi:X^2\to X^2,\psi(p,q)=(p^*,q^*)$ be a map. The map ψ is a solution for the set-theoretical Yang-Baxter equation if it satisfies (3.2), which is also equivalent to (3.3), where*

$$\begin{aligned}
\psi^{12}:X^3\to X^3,\;\; \psi^{12}(e_{a_1},e_{a_2},e_{a_3})=(e^*_{a_1},e^*_{a_2},e_{a_3}),\\
\psi^{23}:X^3\to X^3,\;\; \psi^{23}(e_{a_1},e_{a_2},e_{a_3})=(e_{a_1},e^*_{a_2},e^*_{a_3}),\\
\psi^{13}:X^3\to X^3,\;\; \psi^{13}(e_{a_1},e_{a_2},e_{a_3})=(e^*_{a_1},e_{a_2},e^*_{a_3}).
\end{aligned}$$

Now we construct solutions to the set theoretical Yang-Baxter equation by using lattice effect algebras.

Lemma 3.4. *Let $(E;\oplus,0,1)$ be a lattice effect algebra. Then the following are solutions to the set-theoretical Yang-Baxter equation:*
(a) $\psi(e_{a_1},e_{a_2})=(e_{a_1}\oplus e_{a_2},0)$,
(b) $\psi(e_{a_1},e_{a_2})=(e_{a_1}\odot e_{a_2},1)$

Proof. (a) We define

$$\begin{aligned}
\psi^{12}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1}\oplus e_{a_2},0,e_{a_3}),\\
\psi^{23}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1},e_{a_2}\oplus e_{a_3},0).
\end{aligned}$$

For all $(e_{a_1},e_{a_2},e_{a_3})\in E^3$, using (1), (2) and (e3), we have

$$\begin{aligned}
(\psi^{12}\circ\psi^{23}\circ\psi^{12})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{12}(\psi^{23}(e_{a_1}\oplus e_{a_2},0,e_{a_3}))\\
&= \psi^{12}(e_{a_1}\oplus e_{a_2},0\oplus e_{a_3},0)\\
&= \psi^{12}(e_{a_1}\oplus e_{a_2},e_{a_3}\oplus 0,0)\\
&= \psi^{12}(e_{a_1}\oplus e_{a_2},e_{a_3},0)\\
&= ((e_{a_1}\oplus e_{a_2})\oplus e_{a_3},0,0)
\end{aligned}$$

and

$$\begin{aligned}(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1},e_{a_2},e_{a_3})))\\ &= \psi^{23}(\psi^{12}(e_{a_1},e_{a_2}\oplus e_{a_3},0))\\ &= \psi^{23}(e_{a_1}\oplus(e_{a_2}\oplus e_{a_3}),0,0)\\ &= (e_{a_1}\oplus(e_{a_2}\oplus e_{a_3}),0\oplus 0,0)\\ &= (e_{a_1}\oplus(e_{a_2}\oplus e_{a_3}),0,0)\\ &= ((e_{a_1}\oplus e_{a_2})\oplus e_{a_3},0,0).\end{aligned}$$

Then it is a solution.
(b) We define

$$\begin{aligned}\psi^{12}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1}\odot e_{a_2},1,e_{a_3}),\\ \psi^{23}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1},e_{a_2}\odot e_{a_3},1).\end{aligned}$$

For all $(e_{a_1},e_{a_2},e_{a_3})\in E^3$, we obtain

$$\begin{aligned}(\psi^{12}\circ\psi^{23}\circ\psi^{12})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1},e_{a_2},e_{a_3})))\\ &= \psi^{12}(\psi^{23}(e_{a_1}\odot e_{a_2},1,e_{a_3}))\\ &= \psi^{12}(e_{a_1}\odot e_{a_2},1\odot e_{a_3},1)\\ &= \psi^{12}((e_{a_1}\odot e_{a_2}),e_{a_3},1)\\ &= ((e_{a_1}\odot e_{a_2})\odot e_{a_3},1,1)\end{aligned}$$

and

$$\begin{aligned}(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1},e_{a_2},e_{a_3})))\\ &= \psi^{23}(\psi^{12}(e_{a_1},e_{a_2}\odot e_{a_3},1))\\ &= \psi^{23}(e_{a_1}\odot(e_{a_2}\odot e_{a_3}),1,1)\\ &= (e_{a_1}\odot(e_{a_2}\odot e_{a_3}),1\odot 1,1)\\ &= (e_{a_1}\odot(e_{a_2}\odot e_{a_3}),1,1)\\ &= ((e_{a_1}\odot e_{a_2})\odot e_{a_3},1,1).\end{aligned}$$

Therefore, it is a solution.

□

Lemma 3.5. *Let $(E;\oplus,0,1)$ be a lattice effect algebra. Then $\psi(e_{a_1},e_{a_2})=(e'_{a_2},e'_{a_1})$ is a solution of the set-theoretical Yang-Baxter equation.*

Proof. We define

$$\begin{aligned}\psi^{12}(e_{a_1},e_{a_2},e_{a_3}) &= (e'_{a_2},e'_{a_1},e_{a_3}),\\ \psi^{23}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1},e'_{a_3},e'_{a_2}).\end{aligned}$$

By using (e1), for all $(e_{a_1},e_{a_2},e_{a_3})\in E^3$, we obtain

$$\begin{aligned}(\psi^{12}\circ\psi^{23}\circ\psi^{12})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1},e_{a_2},e_{a_3})))\\ &= \psi^{12}(\psi^{23}(e'_{a_2},e'_{a_1},e_{a_3}))\\ &= \psi^{12}(e'_{a_2},e'_{a_3},e''_{a_1})\\ &= \psi^{12}(e'_{a_2},e'_{a_3},e_{a_1})\\ &= (e''_{a_3},e''_{a_2},e_{a_1})\\ &= (e_{a_3},e_{a_2},e_{a_1})\end{aligned}$$

and

$$\begin{aligned}(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1},e_{a_2},e_{a_3})))\\ &= \psi^{23}(\psi^{12}(e_{a_1},e'_{a_3},e'_{a_2}))\\ &= \psi^{23}(e''_{a_3},e'_{a_1},e'_{a_2})\\ &= \psi^{23}(e_{a_3},e'_{a_1},e'_{a_2})\\ &= (e_{a_3},e''_{a_2},e''_{a_1})\\ &= (e_{a_3},e_{a_2},e_{a_1}).\end{aligned}$$

It is a solution. □

Lemma 3.6. *Let $(E;\oplus,0,1)$ be a lattice effect algebra and $e_a \in E$ such that $e_a \leq e_{a_1}, e_{a_2}, e_{a_3}$ for all $e_{a_1}, e_{a_2}, e_{a_3} \in E$. Thus $\psi(e_{a_1}, e_{a_2}) = (e'_{a_2} \oplus e_a, e'_{a_1} \oplus e_a)$ is a solution of the set-theoretical Yang-Baxter equation.*

Proof. We define ψ^{12} and ψ^{23} as follows:

$$\begin{aligned}
\psi^{12}(e_{a_1}, e_{a_2}, e_{a_3}) &= (e'_{a_2} \oplus e_a, e'_{a_1} \oplus e_a, e_{a_3}),\\
\psi^{23}(e_{a_1}, e_{a_2}, e_{a_3}) &= (e_{a_1}, e'_{a_3} \oplus e_a, e'_{a_2} \oplus e_a).
\end{aligned}$$

By using (1) and (e5) for all $(e_{a_1}, e_{a_2}, e_{a_3}) \in E^3$, we have

$$\begin{aligned}
(\psi^{12} \circ \psi^{23} \circ \psi^{12})(e_{a_1}, e_{a_2}, e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1}, e_{a_2}, e_{a_3})))\\
&= \psi^{12}(\psi^{23}(e'_{a_2} \oplus e_a, e'_{a_1} \oplus e_a, e_{a_3}))\\
&= \psi^{12}(e'_{a_2} \oplus e_a, e'_{a_3} \oplus e_a, (e'_{a_1} \oplus e_a)' \oplus e_a)\\
&= \psi^{12}(e'_{a_2} \oplus e_a, e'_{a_3} \oplus e_a, e_a \oplus (e_a \oplus e'_{a_1})')\\
&= \psi^{12}(e'_{a_2} \oplus e_a, e'_{a_3} \oplus e_a, e_{a_1})\\
&= ((e'_{a_3} \oplus e_a)' \oplus e_a, (e'_{a_2} \oplus e_a)' \oplus e_a, e_{a_1})\\
&= (e_a \oplus (e_a \oplus e'_{a_3})', e_a \oplus (e_a \oplus e'_{a_2})', e_{a_1})\\
&= (e_{a_3}, e_{a_2}, e_{a_1})
\end{aligned}$$

and

$$\begin{aligned}
(\psi^{23} \circ \psi^{12} \circ \psi^{23})(e_{a_1}, e_{a_2}, e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1}, e_{a_2}, e_{a_3})))\\
&= \psi^{23}(\psi^{12}(e_{a_1}, e'_{a_3} \oplus e_a, e'_{a_2} \oplus e_a))\\
&= \psi^{23}((e'_{a_3} \oplus e_a)' \oplus e_a, e'_{a_1} \oplus e_a, e'_{a_2} \oplus e_a)\\
&= \psi^{23}(e_a \oplus (e_a \oplus e'_{a_3})', e'_{a_1} \oplus e_a, e'_{a_2} \oplus e_a)\\
&= \psi^{23}(e_{a_3}, e'_{a_1} \oplus e_a, e'_{a_2} \oplus e_a)\\
&= (e_{a_3}, (e'_{a_2} \oplus e_a)' \oplus e_a, (e'_{a_1} \oplus e_a)' \oplus e_a)\\
&= (e_{a_3}, e_a \oplus (e_a \oplus e'_{a_2})', e_a \oplus (e_a \oplus e'_{a_1})')\\
&= (e_{a_3}, e_{a_2}, e_{a_1}).
\end{aligned}$$

Hence it is a solution. □

Lemma 3.7. *Let $(E;\oplus,0,1)$ be a lattice effect algebra and $e_a \in E$ such that $e_a \leq e'_{a_1}, e'_{a_2}, e'_{a_3}$ for all $e_{a_1}, e_{a_2}, e_{a_3} \in E$. Then $\psi(e_{a_1}, e_{a_2}) = ((e_{a_2} \oplus e_a)', (e_{a_1} \oplus e_a)')$ is a solution of the set-theoretical Yang-Baxter equation.*

Proof. We define

$$\begin{aligned}
\psi^{12}(e_{a_1}, e_{a_2}, e_{a_3}) &= ((e_{a_2} \oplus e_a)', (e_{a_1} \oplus e_a)', e_{a_3}),\\
\psi^{23}(e_{a_1}, e_{a_2}, e_{a_3}) &= (e_{a_1}, (e_{a_3} \oplus e_a)', (e_{a_2} \oplus e_a)').
\end{aligned}$$

By using (e1), (e5), (1), for all $(e_{a_1}, e_{a_2}, e_{a_3}) \in E^3$, we obtain

$$\begin{aligned}
(\psi^{12} \circ \psi^{23} \circ \psi^{12})(e_{a_1}, e_{a_2}, e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1}, e_{a_2}, e_{a_3})))\\
&= \psi^{12}(\psi^{23}((e_{a_2} \oplus e_a)', (e_{a_1} \oplus e_a)', e_{a_3}))\\
&= \psi^{12}((e_{a_2} \oplus e_a)', (e_{a_3} \oplus e_a)', ((e_{a_1} \oplus e_a)' \oplus e_a)')\\
&= \psi^{12}((e_{a_2} \oplus e_a)', (e_{a_3} \oplus e_a)', (e_a \oplus (e_a \oplus e_{a_1})')')\\
&= \psi^{12}((e_{a_2} \oplus e_a)', (e_{a_3} \oplus e_a)', e''_{a_1})\\
&= \psi^{12}((e_{a_2} \oplus e_a)', (e_{a_3} \oplus e_a)', e_{a_1})\\
&= (((e_{a_3} \oplus e_a)' \oplus e_a)', ((e_{a_2} \oplus e_a)' \oplus e_a)', e_{a_1})\\
&= ((e_a \oplus (e_a \oplus e_{a_3})')', (e_a \oplus (e_a \oplus e_{a_2})')', e_{a_1})\\
&= (e''_{a_3}, e''_{a_2}, e_{a_1})\\
&= (e_{a_3}, e_{a_2}, e_{a_1})
\end{aligned}$$

and

$$\begin{aligned}
(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{23}(\psi^{12}(e_{a_1},(e_{a_3}\oplus e_a)',(e_{a_2}\oplus e_a)'))\\
&= \psi^{23}((((e_{a_3}\oplus e_a)'\oplus e_a)'),(e_{a_1}\oplus e_a)',(e_{a_2}\oplus e_a)')\\
&= \psi^{23}((e_a\oplus(e_a\oplus e_{a_3})')',(e_{a_1}\oplus e_a)',(e_{a_2}\oplus e_a)')\\
&= \psi^{23}(e''_{a_3},(e_{a_1}\oplus e_a)',(e_{a_2}\oplus e_a)')\\
&= \psi^{23}(e_{a_3},(e_{a_1}\oplus e_a)',(e_{a_2}\oplus e_a)')\\
&= (e_{a_3},((e_{a_2}\oplus e_a)'\oplus e_a)',((e_{a_1}\oplus e_a)'\oplus e_a)')\\
&= (e_{a_3},e''_{a_2},e''_{a_1})\\
&= (e_{a_3},e_{a_2},e_{a_1}).
\end{aligned}$$

Therefore, it is a solution. □

Lemma 3.8. *Let $(E;\odot,0,1)$ be a lattice effect algebra. If the condition*

$$e_{a_1}\odot e_{a_1} = e_{a_1} \tag{3.4}$$

holds for all $e_{a_1}\in E$, then the following are solutions of the set-theoretical Yang-Baxter equation:
(a) $\psi(e_{a_1},e_{a_2})=(e_{a_1}\odot e_{a_2},e_{a_1})$,
(b) $\psi(e_{a_1},e_{a_2})=(e_{a_2}\odot e_{a_1},e_{a_1})$,
(c) $\psi(e_{a_1},e_{a_2})=(e_{a_2}\odot e_{a_1},e_{a_2})$.

Proof. (a) We define

$$\begin{aligned}
\psi^{12}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1}\odot e_{a_2},e_{a_1},e_{a_3}),\\
\psi^{23}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1},e_{a_2}\odot e_{a_3},e_{a_2}).
\end{aligned}$$

By using (1) and (2), for all $(e_{a_1},e_{a_2},e_{a_3})\in E^3$, we obtain

$$\begin{aligned}
(\psi^{12}\circ\psi^{23}\circ\psi^{12})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{12}(\psi^{23}(e_{a_1}\odot e_{a_2},e_{a_1},e_{a_3}))\\
&= \psi^{12}(e_{a_1}\odot e_{a_2},e_{a_1}\odot e_{a_3},e_{a_1})\\
&= \psi^{12}((e_{a_1}\odot e_{a_2})\odot(e_{a_1}\odot e_{a_3}),e_{a_1}\odot e_{a_2},e_{a_1})\\
&= \psi^{12}((e_{a_1}\odot e_{a_1})\odot(e_{a_2}\odot e_{a_3}),e_{a_1}\odot e_{a_2},e_{a_1})\\
&= (e_{a_1}\odot(e_{a_2}\odot e_{a_3}),e_{a_1}\odot e_{a_2},e_{a_1})
\end{aligned}$$

and

$$\begin{aligned}
(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{23}(\psi^{12}(e_{a_1},e_{a_2}\odot e_{a_3},e_{a_2}))\\
&= \psi^{23}(e_{a_1}\odot(e_{a_2}\odot e_{a_3}),e_{a_1},e_{a_2})\\
&= (e_{a_1}\odot(e_{a_2}\odot e_{a_3}),e_{a_1}\odot e_{a_2},e_{a_1}).
\end{aligned}$$

Therefore, it is a solution.
The proofs of (b) and (c) are similiar to (a). □

Lemma 3.9. *Let $(E;\oplus,0,1)$ be a lattice effect algebra and $e_a\in E$ such that $e_a\le e_{a_1},e_{a_2},e_{a_3}$ for all $e_{a_1},e_{a_2},e_{a_3}\in E$. If the condition*

$$(e_{a_1}\oplus e_{a_2})' = e_{a_1}\oplus e'_{a_2} \tag{3.5}$$

holds for all $e_{a_1},e_{a_2}\in E$, then $\psi(e_{a_1},e_{a_2})=((e'_{a_2}\oplus e_a)'\oplus e_{a_1},e_a)$ is a solution of the set-theoretical Yang-Baxter equation.

Proof. We define

$$\begin{aligned}
\psi^{12}(e_{a_1},e_{a_2},e_{a_3}) &= ((e'_{a_2}\oplus e_a)'\oplus e_{a_1},e_a,e_{a_3}),\\
\psi^{23}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1},(e'_{a_3}\oplus e_a)'\oplus e_{a_2},e_a).
\end{aligned}$$

By using (e2), (e3),(e5), (1), (2) and (3) for all $(e_{a_1}, e_{a_2}, e_{a_3}) \in E^3$, we obtain

$$\begin{aligned}
(\psi^{12}\circ\psi^{23}\circ\psi^{12})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{12}(\psi^{23}(e'_{a_2}\oplus e_a)'\oplus e_{a_1},e_a,e_{a_3})\\
&= \psi^{12}((e'_{a_2}\oplus e_a)'\oplus e_{a_1},(e'_{a_3}\oplus e_a)'\oplus e_a,e_a)\\
&= \psi^{12}((e'_{a_2}\oplus e_a)'\oplus e_{a_1},e_a\oplus(e_a\oplus e'_{a_3})',e_a)\\
&= \psi^{12}((e'_{a_2}\oplus e_a)'\oplus e_{a_1},e_{a_3},e_a)\\
&= ((e'_{a_3}\oplus e_a)'\oplus((e'_{a_2}\oplus e_a)'\oplus e_{a_1}),e_a,e_a)\\
&= ((e'_{a_3}\oplus e'_a)\oplus((e'_{a_2}\oplus e'_a)\oplus e_{a_1}),e_a,e_a)
\end{aligned}$$

and

$$\begin{aligned}
(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{23}(\psi^{12}(e_{a_1},e'_{a_3}\oplus e_a)'\oplus e_{a_2},e_a)\\
&= \psi^{23}((((e'_{a_3}\oplus e_a)'\oplus e_{a_2})'\oplus e_a)'\oplus e_{a_1},e_a,e_a)\\
&= ((((e'_{a_3}\oplus e_a)'\oplus e_{a_2})'\oplus e_a)'\oplus e_{a_1},(e'_a\oplus e_a)'\oplus e_a,e_a)\\
&= ((((e'_{a_3}\oplus e_a)'\oplus e_{a_2})'\oplus e_a)'\oplus e_{a_1},1'\oplus e_a,e_a)\\
&= ((((e'_{a_3}\oplus e_a)'\oplus e_{a_2})'\oplus e_a)'\oplus e_{a_1},0\oplus e_a,e_a)\\
&= ((((e'_{a_3}\oplus e_a)'\oplus e_{a_2})'\oplus e_a)'\oplus e_{a_1},e_a,e_a)\\
&= ((((e'_{a_3}\oplus e'_a)\oplus e'_{a_2})\oplus e'_a)\oplus e_{a_1},e_a,e_a)\\
&= ((e'_{a_3}\oplus e'_a)\oplus((e'_{a_2}\oplus e'_a)\oplus e_{a_1}),e_a,e_a).
\end{aligned}$$

Therefore, it is a solution. □

Lemma 3.10. *Let $(E;\oplus,0,1)$ be a lattice effect algebra. If the conditions*

$$\begin{aligned}
(e_{a_1}\oplus e_{a_2})' &= e_{a_1}\oplus e'_{a_2},\\
e_{a_1}\oplus e_{a_1} &= e_{a_1}
\end{aligned}$$

hold for $\forall e_{a_1},e_{a_2},e_{a_3}\in E$, then the following are solutions to the set-theoretical Yang-Baxter equation in lattice effect algebras:
(a) $\psi(e_{a_1},e_{a_2})=(e'_{a_1}\oplus e_{a_2},e_{a_1})$,
(b) $\psi(e_{a_1},e_{a_2})=(e'_{a_2}\odot e_{a_1},e_{a_1})$,
(c) $\psi(e_{a_1},e_{a_2})=((e_{a_1}\odot e'_{a_2})',e_{a_1})$,
(d) $\psi(e_{a_1},e_{a_2})=((e'_{a_1}\oplus e_{a_2})'\oplus e_{a_2},e_{a_2})$.

Proof. (a) We define

$$\begin{aligned}
\psi^{12}(e_{a_1},e_{a_2},e_{a_3}) &= (e'_{a_1}\oplus e_{a_2},e_{a_1},e_{a_3}),\\
\psi^{23}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1},e'_{a_2}\oplus e_{a_3},e_{a_2}).
\end{aligned}$$

For all $(e_{a_1},e_{a_2},e_{a_3})\in E^3$, we obtain

$$\begin{aligned}
(\psi^{12}\circ\psi^{23}\circ\psi^{12})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{12}(\psi^{23}(e'_{a_1}\oplus e_{a_2},e_{a_1},e_{a_3}))\\
&= \psi^{12}(e'_{a_1}\oplus e_{a_2},e'_{a_1}\oplus e_{a_3},e_{a_1})\\
&= ((e'_{a_1}\oplus e_{a_2})'\oplus(e'_{a_1}\oplus e_{a_3}),e'_{a_1}\oplus e_{a_2},e_{a_1})\\
&= ((e'_{a_1}\oplus e'_{a_2})\oplus(e'_{a_1}\oplus e_{a_3}),e'_{a_1}\oplus e_{a_2},e_{a_1})
\end{aligned}$$

and

$$\begin{aligned}
(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{23}(\psi^{12}(e_{a_1},e'_{a_2}\oplus e_{a_3},e_{a_2}))\\
&= \psi^{23}(e'_{a_1}\oplus(e'_{a_2}\oplus e_{a_3}),e_{a_1},e_{a_2})\\
&= (e'_{a_1}\oplus(e'_{a_2}\oplus e_{a_3}),e'_{a_1}\oplus e_{a_2},e_{a_1})\\
&= ((e'_{a_1}\oplus e'_{a_1})\oplus(e'_{a_2}\oplus e_{a_3}),e'_{a_1}\oplus e_{a_2},e_{a_1})\\
&= ((e'_{a_1}\oplus e'_{a_2})\oplus(e'_{a_1}\oplus e_{a_3}),e'_{a_1}\oplus e_{a_2},e_{a_1}).
\end{aligned}$$

(b) We define

$$\begin{aligned}
\psi^{12}(e_{a_1},e_{a_2},e_{a_3}) &= (e'_{a_2}\odot e_{a_1},e_{a_1},e_{a_3}),\\
\psi^{23}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1},e'_{a_3}\odot e_{a_2},e_{a_2}).
\end{aligned}$$

By using (1) and (e1) for all $(e_{a_1},e_{a_2},e_{a_3})\in E^3$, we obtain

$$\begin{aligned}
(\psi^{12}\circ\psi^{23}\circ\psi^{12})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{12}(\psi^{23}(e'_{a_2}\odot e_{a_1},e_{a_1},e_{a_3}))\\
&= \psi^{12}(e'_{a_2}\odot e_{a_1},e'_{a_3}\odot e_{a_1},e_{a_1})\\
&= ((e'_{a_3}\odot e_{a_1})'\odot(e'_{a_2}\odot e_{a_1}),e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((e''_{a_3}\oplus e'_{a_1})\odot(e''_{a_2}\oplus e'_{a_1})',e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((e_{a_3}\oplus e'_{a_1})\odot(e_{a_2}\oplus e'_{a_1})',e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= (((e_{a_3}\oplus e'_{a_1})'\oplus((e_{a_2}\oplus e'_{a_1})')')',e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= (((e_{a_3}\oplus e'_{a_1})'\oplus(e_{a_2}\oplus e'_{a_1}))',e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((e_{a_3}\oplus e_{a_1})\oplus(e_{a_2}\oplus e'_{a_1})',e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((e_{a_3}\oplus e_{a_1})\oplus(e_{a_2}\oplus e_{a_1}),e'_{a_2}\odot e_{a_1},e_{a_1})
\end{aligned}$$

and

$$\begin{aligned}
(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi 23(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{23}(\psi^{12}(e_{a_1},e'_{a_3}\odot e_{a_2},e_{a_2}))\\
&= \psi^{23}((e'_{a_3}\odot e_{a_2})'\odot e_{a_1},e_{a_1},e_{a_2})\\
&= ((e'_{a_3}\odot e_{a_2})'\odot e_{a_1},e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((e''_{a_3}\oplus e'_{a_2})\odot e_{a_1},e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((e_{a_3}\oplus e'_{a_2})\odot e_{a_1},e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((((e_{a_3}\oplus e'_{a_2})'\oplus e'_{a_1})',e'_{a_2}\odot e_{a_1},e_{a_1}))\\
&= (((e_{a_3}\oplus e_{a_2})\oplus e''_{a_1}),e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((e_{a_3}\oplus e_{a_2})\oplus e_{a_1},e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((e_{a_3}\oplus e_{a_2})\oplus(e_{a_1}\oplus e_{a_1}),e'_{a_2}\odot e_{a_1},e_{a_1})\\
&= ((e_{a_3}\oplus e_{a_1})\oplus(e_{a_2}\oplus e_{a_1}),e'_{a_2}\odot e_{a_1},e_{a_1}).
\end{aligned}$$

(c) We define

$$\begin{aligned}
\psi^{12}(e_{a_1},e_{a_2},e_{a_3}) &= ((e_{a_1}\odot e'_{a_2})',e_{a_1},e_{a_3}),\\
\psi^{23}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1},(e_{a_2}\odot e'_{a_3})',e_{a_2}).
\end{aligned}$$

By using (e1) for all $(e_{a_1},e_{a_2},e_{a_3})\in E^3$, we obtain

$$\begin{aligned}
(\psi^{12}\circ\psi^{23}\circ\psi^{12})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{12}(\psi^{23}((e_{a_1}\odot e'_{a_2})',e_{a_1},e_{a_3}))\\
&= \psi^{12}((e_{a_1}\odot e'_{a_2})',(e_{a_1}\odot e'_{a_3})',e_{a_1})\\
&= (((e_{a_1}\odot e'_{a_2})'\odot((e_{a_1}\odot e'_{a_3})')')',(e_{a_1}\odot e'_{a_2})',e_{a_1})\\
&= ((e_{a_1}\odot e'_{a_2})'\odot(e_{a_1}\odot e'_{a_3})',(e_{a_1}\odot e'_{a_2})',e_{a_1})\\
&= ((((e_{a_1}\odot e'_{a_2})')'\oplus(e_{a_1}\odot e'_{a_3}))',(e_{a_1}\odot e'_{a_2})',e_{a_1})\\
&= ((e'_{a_1}\oplus e''_{a_2})'\oplus(e'_{a_1}\oplus e''_{a_3}),(e_{a_1}\odot e'_{a_2})',e_{a_1})\\
&= ((e'_{a_1}\oplus e_{a_2})'\oplus(e'_{a_1}\oplus e_{a_3}),(e_{a_1}\odot e'_{a_2})',e_{a_1})\\
&= ((e'_{a_1}\oplus e'_{a_2})\oplus(e'_{a_1}\oplus e_{a_3}),(e_{a_1}\odot e'_{a_2})',e_{a_1})
\end{aligned}$$

and

$$\begin{aligned}
(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{23}(\psi^{12}(e_{a_1},(e_{a_2}\odot e'_{a_3})',e_{a_2}))\\
&= \psi^{23}((e_{a_1}\odot(((e_{a_2}\odot e'_{a_3})')')'),e_{a_1},e_{a_2})\\
&= \psi^{23}((e_{a_1}\odot(e_{a_2}\odot e'_{a_3}))',e_{a_1},e_{a_2})\\
&= ((e_{a_1}\odot(e_{a_2}\odot e'_{a_3}))',(e_{a_1}\odot e'_{a_2})',e_{a_1})\\
&= (e'_{a_1}\oplus(e_{a_2}\odot e'_{a_3})',(e_{a_1}\odot e'_{a_2})',e_{a_1})\\
&= (e'_{a_1}\oplus(e'_{a_2}\oplus e_{a_3}),(e_{a_1}\odot e'_{a_2})',e_{a_1})\\
&= ((e'_{a_1}\oplus e'_{a_1})\oplus(e'_{a_2}\oplus e_{a_3}),(e_{a_1}\odot e'_{a_2})',e_{a_1})\\
&= ((e'_{a_1}\oplus e'_{a_2})\oplus(e'_{a_1}\oplus e_{a_3}),(e_{a_1}\odot e'_{a_2})',e_{a_1}).
\end{aligned}$$

(d) The proof (d) is clear. □

Example 3.11. *Let $(E;\oplus,0,1)$ be a lattice effect algebra. If the condition*

$$(e_{a_1}\oplus e_{a_2})' \quad = \quad e_{a_1}\oplus e'_{a_2} \tag{3.6}$$

holds for $\forall e_{a_1},e_{a_2},e_{a_3}\in E$, then $\psi(e_{a_1},e_{a_2})=((e_{a_2}\oplus e'_{a_1})',e_{a_1})$ is a solution to the set-theoretical Yang-Baxter equation in Boolean algebras while it is not a solution in lattice effect algebras.

Proof. We define

$$\begin{aligned}
\psi^{12}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_2}\oplus e'_{a_1})',e_{a_1},e_{a_3}),\\
\psi^{23}(e_{a_1},e_{a_2},e_{a_3}) &= (e_{a_1},(e_{a_3}\oplus e'_{a_2})',e_{a_2}).
\end{aligned}$$

By using (e1), for all $(e_{a_1},e_{a_2},e_{a_3})\in E^3$, we obtain

$$\begin{aligned}
(\psi^{12}\circ\psi^{23}\circ\psi^{12})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{12}(\psi^{23}(\psi^{12}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{12}(\psi^{23}((e_{a_2}\oplus e'_{a_1})',e_{a_1},e_{a_3}))\\
&= \psi^{12}((e_{a_2}\oplus e'_{a_1})',(e_{a_3}\oplus e'_{a_1})',e_{a_1})\\
&= (((e_{a_3}\oplus e'_{a_1})'\oplus((e_{a_2}\oplus e'_{a_1})')')',(e_{a_2}\oplus e'_{a_1})',e_{a_1})\\
&= (((e_{a_3}\oplus e'_{a_1})'\oplus(e_{a_2}\oplus e'_{a_1}))',(e_{a_2}\oplus e'_{a_1})',e_{a_1})\\
&= ((e_{a_3}\oplus e''_{a_1})\oplus(e_{a_2}\oplus e''_{a_1}),(e_{a_2}\oplus e'_{a_1})',e_{a_1})\\
&= (((e_{a_3}\oplus e_{a_1})\oplus(e_{a_2}\oplus e_{a_1})),(e_{a_2}\oplus e'_{a_1})',e_{a_1})
\end{aligned}$$

and

$$\begin{aligned}
(\psi^{23}\circ\psi^{12}\circ\psi^{23})(e_{a_1},e_{a_2},e_{a_3}) &= \psi^{23}(\psi^{12}(\psi^{23}(e_{a_1},e_{a_2},e_{a_3})))\\
&= \psi^{23}(\psi^{12}(e_{a_1},(e_{a_3}\oplus e'_{a_2})',e_{a_2}))\\
&= \psi^{23}(((e_{a_3}\oplus e'_{a_2})'\oplus e'_{a_1})',e_{a_1},e_{a_2})\\
&= (((e_{a_3}\oplus e'_{a_2})'\oplus e'_{a_1})',(e_{a_2}\oplus e'_{a_1})',e_{a_1})\\
&= (((e_{a_3}\oplus e''_{a_2})\oplus e''_{a_1}),(e_{a_2}\oplus e'_{a_1})',e_{a_1})\\
&= (((e_{a_3}\oplus e_{a_2})\oplus e_{a_1}),(e_{a_2}\oplus e'_{a_1})',e_{a_1}).
\end{aligned}$$

Since Boolean algebra is a bounded distributive lattice, it is a solution in Boolean algebras whereas it is not a solution in lattice effect algebra. If we define $e_{a_1}\oplus e_{a_1}=e_{a_1}$, then $\psi(e_{a_1},e_{a_2})=((e_{a_2}\oplus e'_{a_1})',e_{a_1})$ is a solution in lattice effect algebras. □

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