Konuralp Journal of Mathematics, 8 (1) (2020) 114-121

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Periodic Solutions for Some Systems of Difference Equations

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Abstract

We will show in this paper that all solutions for the systems

$$\varkappa_{n+1}^{(1)}=\frac{\varkappa_n^{(2)}}{\alpha\varkappa_n^{(2)}-1},\varkappa_{n+1}^{(2)}=\frac{\varkappa_n^{(3)}}{\alpha\varkappa_n^{(3)}-1},...,\varkappa_{n+1}^{(\kappa)}=\frac{\varkappa_n^{(1)}}{\alpha\varkappa_n^{(1)}-1},$$

and

$$\varkappa_{n+1}^{(1)}=\frac{\varkappa_n^{(\kappa)}}{\alpha\varkappa_n^{(\kappa)}-1},\varkappa_{n+1}^{(2)}=\frac{\varkappa_n^{(1)}}{\alpha\varkappa_n^{(1)}-1},...,\varkappa_{n+1}^{(\kappa)}=\frac{\varkappa_n^{(\kappa-1)}}{\alpha\varkappa_n^{(\kappa-1)}-1},$$

are periodic with period p where p is given by

$$p=\left\{\begin{array}{lll}\kappa & \text{if} & \kappa=0(mod2),\\ 2\kappa & \text{if} & \kappa\neq 0(mod2),\end{array}\right\}$$

where α and $\varkappa_0^{(1)},\varkappa_0^{(2)},...,\varkappa_0^{(\kappa)}$ are nonzero real numbers with $\varkappa_0^{(i)}\neq\frac{1}{\alpha}$, $i=1,2,...,\kappa$, for some $\kappa\in\mathbb{N}$.

Keywords: *Asymptotic behavior; difference equations; periodicity; solutions; system.*
2010 Mathematics Subject Classification: *39A10, 39A23 .*

1. Introduction

Difference equations is a very important topic in our life because they appear as mathematical models describing many real life situations for examples in probability theory, statistical problems, number theory, geometry, electrical networks, genetics, biology, physics, ecology, economics, engineering, medicine, etc [1]. Therefore, the study of difference equations has received great attention from researchers around the world. See for examples [1]-[21].

Iricanin et al. [12] investigated the positive solution for the following systems

$$\varkappa_{n+1}^{(1)}=\frac{1+\varkappa_n^{(2)}}{\varkappa_{n-1}^{(3)}},\varkappa_{n+1}^{(2)}=\frac{1+\varkappa_n^{(3)}}{\varkappa_{n-1}^{(4)}},\ldots,\varkappa_{n+1}^{(\kappa)}=\frac{1+\varkappa_n^{(1)}}{\varkappa_{n-1}^{(2)}},$$

and

$$\varkappa_{n+1}^{(1)}=\frac{1+\varkappa_n^{(2)}+\varkappa_{n-1}^{(3)}}{\varkappa_{n-2}^{(4)}},\varkappa_{n+1}^{(2)}=\frac{1+\varkappa_n^{(3)}+\varkappa_{n-1}^{(4)}}{\varkappa_{n-2}^{(5)}},\ldots,\varkappa_{n+1}^{(\kappa)}=\frac{1+\varkappa_n^{(1)}+\varkappa_{n-1}^{(2)}}{\varkappa_{n-2}^{(3)}},$$

where κ be a nonnegative integer number.

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Papaschinopoluos et al. [11] studied the behavior for the systems

$$\varkappa_1(\upsilon+1)=\frac{\alpha_\kappa\varkappa_\kappa(\upsilon)+\beta_\kappa}{\varkappa_{\kappa-1}(\upsilon-1)},$$
$$\varkappa_2(\upsilon+1)=\frac{\alpha_1\varkappa_1(\upsilon)+\beta_1}{\varkappa_\kappa(\upsilon-1)},$$
$$\varkappa_i(\upsilon+1)=\frac{\alpha_{i-1}\varkappa_{i-1}(\upsilon)+\beta_{i-1}}{\varkappa_{i-2}(\upsilon-1)},\ i=3,4,\ldots,\kappa,$$

where α_i,β_i, $i=1,2,\ldots,\kappa$, are positive constants with $\kappa\geqslant 3$ is an integer, and the values $\varkappa_i(-1),\varkappa_i(0)$, $i=1,2,\ldots,\kappa$, are positive real numbers.
In [18] the periodicity for the solutions of some systems of the form

$$\varkappa_{n+1}^{(1)}=\frac{\varkappa_n^{(2)}}{\varkappa_n^{(2)}-1},\varkappa_{n+1}^{(2)}=\frac{\varkappa_n^{(3)}}{\varkappa_n^{(3)}-1},\ldots,\varkappa_{n+1}^{(\kappa)}=\frac{\varkappa_n^{(1)}}{\varkappa_n^{(1)}-1},$$

and

$$\varkappa_{n+1}^{(1)}=\frac{\varkappa_n^{(\kappa)}}{\varkappa_n^{(\kappa)}-1},\varkappa_{n+1}^{(2)}=\frac{\varkappa_n^{(1)}}{\varkappa_n^{(1)}-1},\ldots,\varkappa_{n+1}^{(\kappa)}=\frac{\varkappa_n^{(\kappa-1)}}{\varkappa_n^{(\kappa-1)}-1},$$

was studied, where $\varkappa_0^{(i)}\neq 1,\ i=1,2,...,\kappa$.
Also, in [19] the periodic character for the positive solutions of the coming systems of difference equations

$$\varkappa_{n+1}^{(1)}=\frac{\varkappa_n^{(2)}\varkappa_{n-1}^{(3)}}{\varkappa_n^{(2)}\varkappa_{n-1}^{(3)}-\varkappa_n^{(2)}-\varkappa_{n-1}^{(3)}},$$
$$\varkappa_{n+1}^{(2)}=\frac{\varkappa_n^{(3)}\varkappa_{n-1}^{(4)}}{\varkappa_n^{(3)}\varkappa_{n-1}^{(4)}-\varkappa_n^{(3)}-\varkappa_{n-1}^{(4)}},$$
$$\vdots$$
$$\varkappa_{n+1}^{(\kappa)}=\frac{\varkappa_n^{(1)}\varkappa_{n-1}^{(2)}}{\varkappa_n^{(1)}\varkappa_{n-1}^{(2)}-\varkappa_n^{(1)}-\varkappa_{n-1}^{(2)}},$$

was studied, where $\frac{1}{\varkappa_0^{(1)}}+\frac{1}{\varkappa_{-1}^{(2)}}\neq 1,\frac{1}{\varkappa_0^{(2)}}+\frac{1}{\varkappa_{-1}^{(3)}}\neq 1,\ldots,\frac{1}{\varkappa_0^{(\kappa)}}+\frac{1}{\varkappa_{-1}^{(1)}}\neq 1$ and $\varkappa_{-1}^{(1)},\varkappa_0^{(1)},\varkappa_{-1}^{(2)},\varkappa_0^{(2)},\ldots,\varkappa_{-1}^{(\kappa)},\varkappa_0^{(\kappa)}$ are nonzero real numbers.

Throughout the present paper we will deal with the periodicity character to the solutions for some systems of the form

$$\varkappa_{n+1}^{(1)}=\frac{\varkappa_n^{(2)}}{\alpha\varkappa_n^{(2)}-1},\ \varkappa_{n+1}^{(2)}=\frac{\varkappa_n^{(3)}}{\alpha\varkappa_n^{(3)}-1},...,\ \varkappa_{n+1}^{(\kappa)}=\frac{\varkappa_n^{(1)}}{\alpha\varkappa_n^{(1)}-1},$$

and

$$\varkappa_{n+1}^{(1)}=\frac{\varkappa_n^{(\kappa)}}{\alpha\varkappa_n^{(\kappa)}-1},\ \varkappa_{n+1}^{(2)}=\frac{\varkappa_n^{(1)}}{\alpha\varkappa_n^{(1)}-1},...,\ \varkappa_{n+1}^{(\kappa)}=\frac{\varkappa_n^{(\kappa-1)}}{\alpha\varkappa_n^{(\kappa-1)}-1},$$

where α and $\varkappa_0^{(1)},\varkappa_0^{(2)},...,\varkappa_0^{(\kappa)}$ are non zero real numbers with $\varkappa_0^{(i)}\neq\frac{1}{\alpha},\ i=1,2,...,\kappa$, for some $\kappa\in\mathbb{N}$.

Now we present and prove some lemmas that will be used in proving the main results in this paper. Let $\gcd(\kappa,y)$ denotes the greatest common divisor for the integers κ and y.

Lemma 1.1. *Assume that* $\gcd(\kappa,2)=1$ *for some* $\kappa\in\mathbb{N}$, *then the numbers* $p_y=2y+1$, (*or* $p_y=-2y+1$), $y=0,1,...,\kappa-1$, *satisfy* $p_{y_1}-p_{y_2}\neq 0(mod\kappa)$, *whenever* $y_1\neq y_2$.

Proof. For the sake of a contradiction assume that $p_{y_1}-p_{y_2}=0\ (\text{mod}\kappa)$, with $y_1\neq y_2$. Then we have $2(y_1-y_2)=p_{y_1}-p_{y_2}=\kappa m$ for some $m\in\mathbb{Z}\backslash\{0\}$. Since $\gcd(\kappa,2)=1$, it is easy to see that κ is a divisor to y_1-y_2. Again, since $y_1,y_2\in\{0,1,...,\kappa-1\}$, it follows that $|y_1-y_2|<\kappa$, and this is a contradiction. □

Remark 1.2. *It follows from Lemma 1.1 that the remainders* q_y, $y=0,1,...,\kappa-1$ *for the numbers* $p_y=2y+1$, $y=0,1,...,\kappa-1$, *which were gotten from dividing the values* p_y *by* κ *are reciprocally different, they are included in a set* $S=\{0,1,...,\kappa-1\}$, *making some permutations of the ordered set* $(0,1,...,\kappa-1)$, *finally* $p_\kappa=2\kappa+1$ *is the first number having the form* $2y+1$, $y\in\mathbb{N}$, *with* $p_1-p_0\equiv 0\ (mod\kappa)$.

Lemma 1.3. *Every solution for the following equation*

$$\varkappa_{n+1} = \frac{\varkappa_n}{\alpha\varkappa_n - 1}, \qquad \textit{for all} \quad n \geq 0, \tag{1.1}$$

is periodic with period two where $\alpha,\ \varkappa_0 \in \mathbb{R}\backslash\{0\}$ *with* $\varkappa_0 \neq \frac{1}{\alpha}$.

Proof. Let $\{\varkappa_n\}_{n=0}^{\infty}$ is a solution for Eq.(1.1) such that $\alpha\varkappa_0 \neq 1$. Then it is easy, by direct substitution in Eq.(1.1), to obtain that $\{\varkappa_n\}_{n=0}^{\infty}$ is periodic with prime period two and has the formula

$$\left\{\varkappa_0,\ \frac{\varkappa_0}{\alpha\varkappa_0 - 1},\ \varkappa_0,\ \frac{\varkappa_0}{\alpha\varkappa_0 - 1},\ \varkappa_0,\ \frac{\varkappa_0}{\alpha\varkappa_0 - 1}, \ldots\right\}.$$

The proof is so complete. □

Let $f : G \to G$ be a continuous function on $G \subseteq \mathbf{R}$ and let D denotes the subset of all fixed points of $f^2 = f \circ f$. Define $F : G^\kappa \to G^\kappa$ by

$$F(\varkappa_1, \varkappa_2, \ldots, \varkappa_\kappa) = (f(\varkappa_2), f(\varkappa_3), \ldots, f(\varkappa_\kappa), f(\varkappa_1)), \tag{1.2}$$

or

$$F(\varkappa_1, \varkappa_2, \ldots, \varkappa_\kappa) = (f(\varkappa_\kappa), f(\varkappa_1), f(\varkappa_2) \ldots, f(\varkappa_{\kappa-1})), \tag{1.3}$$

where κ is a positive integer and assign a difference equation of the form

$$\chi_{n+1} = F(\chi_n), \qquad n \geq 0. \tag{1.4}$$

Therefore we get two systems of the form

$$\varkappa_{n+1}^{(1)} = f(\varkappa_n^{(2)}),\ \ \varkappa_{n+1}^{(2)} = f(\varkappa_n^{(3)}), \ldots,\ \ \varkappa_{n+1}^{(\kappa)} = f(\varkappa_n^{(1)}), \tag{1.5}$$

and

$$\varkappa_{n+1}^{(1)} = f(\varkappa_n^{(\kappa)}),\ \ \varkappa_{n+1}^{(2)} = f(\varkappa_n^{(1)}), \ldots,\ \ \varkappa_{n+1}^{(\kappa)} = f(\varkappa_n^{(\kappa-1)}). \tag{1.6}$$

Taking

$$f(\varkappa) = \frac{\varkappa}{\alpha\varkappa - 1},$$

which satisfies $f \circ f = I$, i. e. every $\varkappa \neq 1/\alpha$ is a fixed point of f^2. Thus, whenever $n \geq 0$,we obtain

$$\varkappa_{n+1}^{(1)} = \frac{\varkappa_n^{(2)}}{\alpha\varkappa_n^{(2)} - 1},\ \ \varkappa_{n+1}^{(2)} = \frac{\varkappa_n^{(3)}}{\alpha\varkappa_n^{(3)} - 1}, \ldots,\ \ \varkappa_{n+1}^{(\kappa)} = \frac{\varkappa_n^{(1)}}{\alpha\varkappa_n^{(1)} - 1}, \tag{1.7}$$

and

$$\varkappa_{n+1}^{(1)} = \frac{\varkappa_n^{(\kappa)}}{\alpha\varkappa_n^{(\kappa)} - 1},\ \ \varkappa_{n+1}^{(2)} = \frac{\varkappa_n^{(1)}}{\alpha\varkappa_n^{(1)} - 1},\ \ldots,\ \ \varkappa_{n+1}^{(\kappa)} = \frac{\varkappa_n^{(\kappa-1)}}{\alpha\varkappa_n^{(\kappa-1)} - 1}. \tag{1.8}$$

2. The Main Results

Here we prove and investigate the periodicity character for the solutions for Eq.(1.4).

Theorem 2.1. *Each one of the coming statement is true:*
(a) Assume that $\kappa = 0(mod2)$*, then any solution* $\{\chi_n\}_{n=0}^{\infty}$ *for Eq.(1.4) with the initial condition* $\chi_0 \in D^\kappa$*, is a periodic solution of period* κ.
(b) Whenever $\kappa \neq 0(mod2)$*, then any solution* $\{\chi_n\}_{n=0}^{\infty}$ *for Eq. (1.4) with the initial condition* $\chi_0 \in D^\kappa$*, is a periodic solution of period* 2κ.

Proof. (a) Assume that $\kappa = 0$ (mod2). Let $\{\chi_n\}$ be a solution of Eq.(1.4) with an initial condition $\chi_0 = (\varkappa_0^{(1)}, \varkappa_0^{(2)}, \ldots, \varkappa_0^{(\kappa)}) \in D^\kappa$. It suffices to show that $F^\kappa(\chi_0) = \chi_0$ where F is defined by (1.2). Now we will prove by induction that

$$F^\kappa(\chi_0) = F^{\kappa-2i}(\varkappa_0^{(2i+1)}, \varkappa_0^{(2i+2)}, \ldots, \varkappa_0^{(\kappa)}, \varkappa_0^{(1)}, \varkappa_0^{(2)}, \ldots, \varkappa_0^{(2i)}), i = 1, \ldots, \kappa/2. \tag{2.1}$$

Indeed, at $i = 1$, we obtain

$$\begin{aligned} F^\kappa(\chi_0) &= F^{\kappa-1}(f(\varkappa_0^{(2)}), f(\varkappa_0^{(3)}), \ldots, f(\varkappa_0^{(\kappa)}), f(\varkappa_0^{(1)})) \\ &= F^{\kappa-2}(f^2(\varkappa_0^{(3)}), f^2(\varkappa_0^{(4)}), \ldots, f^2(\varkappa_0^{(\kappa)}), f^2(\varkappa_0^{(1)}), f^2(\varkappa_0^{(2)})) \\ &= F^{\kappa-2}(\varkappa_0^{(3)}, \varkappa_0^{(4)}, \ldots, \varkappa_0^{(\kappa)}, \varkappa_0^{(1)}, \varkappa_0^{(2)}). \end{aligned}$$

Then relation (2.1) is true at $i=1$. Now assume that relation (2.1) is true for some i, then we obtain

$$\begin{aligned}F^{\kappa}(\chi_0) &= F^{\kappa-2i}(\varkappa_0^{(2i+1)},\varkappa_0^{(2i+2)},\dots,\varkappa_0^{(\kappa)},\varkappa_0^{(1)},\varkappa_0^{(2)},\dots,\varkappa_0^{(2i)})\\ &= F^{\kappa-(2i+1)}(f(\varkappa_0^{(2i+2)}),f(\varkappa_0^{(2i+3)})\dots,f(\varkappa_0^{(\kappa)}),f(\varkappa_0^{(1)}),f(\varkappa_0^{(2)}),\dots,f(\varkappa_0^{(2i+1)}))\\ &= F^{\kappa-(2i+2)}(\varkappa_0^{(2i+3)},\varkappa_0^{(2i+4)},\dots,\varkappa_0^{(\kappa)},\varkappa_0^{(1)},\dots,\varkappa_0^{(2i+2)}).\end{aligned}$$

Hence relation (2.1) is true.
Again whenever F is defined by (1.3), we will prove by induction that

$$F^{\kappa}(\chi_0) = F^{\kappa-2i}(\varkappa_0^{\kappa-(2i-1)},\varkappa_0^{\kappa-(2i-2)},\dots,\varkappa_0^{(\kappa)},\varkappa_0^{(1)},\varkappa_0^{(2)},\dots,\varkappa_0^{(\kappa-2i)}), i=1,\dots,\kappa/2. \tag{2.2}$$

Indeed, at $i=1$, we get

$$\begin{aligned}F^{\kappa}(\chi_0) &= F^{\kappa-1}(f(\varkappa_0^{(\kappa)}),f(\varkappa_0^{(1)}),f(\varkappa_0^{(2)}),\dots,f(\varkappa_0^{(\kappa-1)}))\\ &= F^{\kappa-2}(f^2(\varkappa_0^{(\kappa-1)}),f^2(\varkappa_0^{(\kappa)}),f^2(\varkappa_0^{(1)}),f^2(\varkappa_0^{(2)}),\dots,f^2(\varkappa_0^{(\kappa-2)}))\\ &= F^{\kappa-2}(\varkappa_0^{(\kappa-1)},\varkappa_0^{(\kappa)},\varkappa_0^{(1)},\varkappa_0^{(2)},\dots,\varkappa_0^{(\kappa-2)}).\end{aligned}$$

Thus relation (2.2) is true at $i=1$. Now assume that relation (2.2) is true for some i. We have

$$\begin{aligned}F^{\kappa}(\chi_0) &= F^{\kappa-2i}(\varkappa_0^{\kappa-(2i-1)},\varkappa_0^{\kappa-(2i-2)},\dots,\varkappa_0^{\kappa},\varkappa_0^{1},\varkappa_0^{2},\dots,\varkappa_0^{\kappa-(2i)})\\ &= F^{\kappa-(2i+2)}(\varkappa_0^{(\kappa-(2i+1))},\varkappa_0^{(\kappa-2i)},\dots,\varkappa_0^{(\kappa)},\varkappa_0^{(1)},\dots,\varkappa_0^{(\kappa-(2i+2))}).\end{aligned}$$

This shows that relation (2.2) is true for each $i=1,\dots,\kappa/2$.
(b) The proof of this part is similar to part (a) so will be left to the reader. □

As a direct consequence we get the following two results.

Theorem 2.2. *Each one of the coming statement is true:*
(a) Assume that $\kappa=0(mod2)$, then any solution $\{\chi_n=(\varkappa_n^{(1)},\dots,\varkappa_n^{(\kappa)})\}_{n=0}^{\infty}$ for system (1.5) (resp. system (1.6)) with $\chi_0=(\varkappa_0^{(1)},\varkappa_0^{(2)},\dots,\varkappa_0^{(\kappa)})\in D^{\kappa}$, is a periodic solution of period κ.
(b) Assume that $\kappa\neq 0(mod2)$, then any solution $\{\chi_n\}_{n=0}^{\infty}$ for system (1.5) (resp. system (1.6)) where $\chi_0\in D^{\kappa}$, is a periodic solution of period 2κ.

Theorem 2.3. *Each one of the coming statement is true:*
(a) Assume that $\kappa=0(mod2)$, then any solution $\{\chi_n\}_{n=0}^{\infty}$ for system (1.7) (resp. system (1.8)), is a periodic solution of period κ.
(b) Assume that $\kappa\neq 0(mod2)$, then any solution $\{\chi_n\}_{n=0}^{\infty}$ for system (1.7) (resp. system (1.8)), is a periodic solution of period 2κ.

Corollary 2.4. *Assume that $\alpha>0$, then every solution $\left(\varkappa_n^{(1)},\varkappa_n^{(2)},...,\varkappa_n^{(\kappa)}\right)$ of (1.7) where the conditions $\left(\varkappa_0^{(1)},\varkappa_0^{(2)},...,\varkappa_0^{(\kappa)}\right)$ to be such that*

$$\varkappa_0^{(j)} > \frac{1}{\alpha}, \quad j=1,2,...\kappa, \tag{2.3}$$

is positive.

Proof. Consider $\{\chi_n\}_{n=0}^{\infty}$ is a positive solution for system (1.7) where (2.3) is satisfied. Then if $\kappa=0(\text{mod}2)$, it follows from (1.7) and (2.3) that

$$\begin{aligned}&\varkappa_j^{(\mu)}=\begin{cases}\dfrac{\varkappa_0^{(j+\mu)}}{\alpha\varkappa_0^{(j+\mu)}-1} \text{ for } (j+\mu)\leq\kappa,\\ \dfrac{\varkappa_0^{(j+\mu-\kappa)}}{\alpha\varkappa_0^{(j+\mu-\kappa)}-1} \text{ for } (j+\mu)>\kappa,\end{cases} \quad \text{if } j \text{ is odd}\\ &\varkappa_j^{(\mu)}=\begin{cases}\varkappa_0^{(j+\mu)} \text{ for } (j+\mu)\leq\kappa,\\ \varkappa_0^{(j+\mu-\kappa)} \text{ for } (j+\mu)>\kappa,\end{cases} \quad \text{if } j \text{ is even}\end{aligned} \tag{2.4}$$

for some $j,\mu=1,2,...,\kappa$.
If $\kappa\neq 0(\text{mod}2)$, it follows from relations (1.7) and (2.3) that

$$\begin{aligned}&\varkappa_j^{(\mu)}=\begin{cases}\dfrac{\varkappa_0^{(j+\mu)}}{\alpha\varkappa_0^{(j+\mu)}-1} \text{ for } (j+\mu)\leq\kappa,\\ \dfrac{\varkappa_0^{(j+\mu-\kappa)}}{\alpha\varkappa_0^{(j+\mu-\kappa)}-1} \text{ for } \kappa<(j+\mu)\leq 2\kappa,\\ \dfrac{\varkappa_0^{(j+\mu-2\kappa)}}{\alpha\varkappa_0^{(j+\mu-2\kappa)}-1} \text{ for } 2\kappa<(j+\mu)\leq 3\kappa,\end{cases} \quad \text{if } j \text{ is odd}\\ &\varkappa_j^{(\mu)}=\begin{cases}\varkappa_0^{(j+\mu)} \text{ for } (j+\mu)\leq\kappa,\\ \varkappa_0^{(j+\mu-\kappa)} \text{ for } \kappa<(j+\mu)\leq 2\kappa,\\ \varkappa_0^{(j+\mu-2\kappa)} \text{ for } 2\kappa<(j+\mu)\leq 3\kappa,\end{cases} \quad \text{if } j \text{ is even}\end{aligned} \tag{2.5}$$

for $j=1,2,...,2\kappa$ and $\mu=1,2,...,\kappa$. Therefore it follows from (2.4) and (2.5) that every solution for system (1.7) is positive. □

Corollary 2.5. *Assume that $\alpha < 0$, then every solution $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ for (1.7) where the conditions $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ satisfy the following inequalities*

$$\varkappa_0^{(i)} < \frac{1}{\alpha}, \quad i = 1, 2, ... \kappa, \tag{2.6}$$

is negative.

Proof. The result follows consequentially from (2.4), (2.5) and (2.6). □

Corollary 2.6. *Assume that $\alpha > 0$, then every solution $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ for system (1.7) with $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ such that*

$$0 < \varkappa_0^{(i)} < \frac{1}{\alpha}, \quad i = 1, 2, ... \kappa, \tag{2.7}$$

is positive and negative successively. Moreover $\left(\varkappa_{2n}^{(1)}, \varkappa_{2n}^{(2)}, ..., \varkappa_{2n}^{(\kappa)}\right)$ is positive and $\left(\varkappa_{2n+1}^{(1)}, \varkappa_{2n+1}^{(2)}, ..., \varkappa_{2n+1}^{(\kappa)}\right)$ is negative for $n = 0, 1, 2, ...$

Proof. The result follows from (2.4), (2.5) and (2.7). □

Corollary 2.7. *Assume that $\alpha < 0$, then every solution $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ of (1.7) with the negative values $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ and*

$$\varkappa_0^{(i)} > \frac{1}{\alpha}, \quad i = 1, 2, ... \kappa, \tag{2.8}$$

is positive and negative successively. Moreover $\left(\varkappa_{2n}^{(1)}, \varkappa_{2n}^{(2)}, ..., \varkappa_{2n}^{(\kappa)}\right)$ is negative and $\left(\varkappa_{2n+1}^{(1)}, \varkappa_{2n+1}^{(2)}, ..., \varkappa_{2n+1}^{(\kappa)}\right)$ is positive for $n = 0, 1, 2, ...$

Proof. The result follows from (2.4), (2.5) and (2.8). □

Corollary 2.8. *Assume that $\alpha > 0$, then every solution $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ of system (1.7) with the negative initial values $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$; where*

$$\varkappa_0^{(i)} < 0, \quad i = 1, 2, ... \kappa, \tag{2.9}$$

is negative and positive successively. Moreover $\left(\varkappa_{2n}^{(1)}, \varkappa_{2n}^{(2)}, ..., \varkappa_{2n}^{(\kappa)}\right)$ is negative and $\left(\varkappa_{2n+1}^{(1)}, \varkappa_{2n+1}^{(2)}, ..., \varkappa_{2n+1}^{(\kappa)}\right)$ is positive for $n = 0, 1, 2, ...$

Proof. The proof is achieved by (2.4), (2.5) and (2.9). □

Corollary 2.9. *Assume that $\alpha < 0$, then every solution $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ for (1.7) where the conditions $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ to be such that*

$$\varkappa_0^{(i)} > 0, \quad i = 1, 2, ... \kappa, \tag{2.10}$$

is negative and positive successively. Moreover $\left(\varkappa_{2n}^{(1)}, \varkappa_{2n}^{(2)}, ..., \varkappa_{2n}^{(\kappa)}\right)$ is positive and $\left(\varkappa_{2n+1}^{(1)}, \varkappa_{2n+1}^{(2)}, ..., \varkappa_{2n+1}^{(\kappa)}\right)$ is negative for $n = 0, 1, 2, ...$

Proof. The proof is achieved by (2.4), (2.5) and (2.10). □

Corollary 2.10. *Assume that $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ is a solution for system (1.7) and $\alpha > 0$. Then, for $1 \leq i \leq \kappa$, and for $n = 0, 1, ...,$ the following statements hold*
(i) If $\varkappa_0^{(i)} \to \infty$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \infty$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \frac{1}{\alpha}^{+}$.
(ii) If $\varkappa_0^{(i)} \to \frac{1}{\alpha}^{+}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \frac{1}{\alpha}^{+}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \infty$.
(iii) If $\varkappa_0^{(i)} \to \frac{1}{\alpha}^{-}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \frac{1}{\alpha}^{-}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to -\infty$.
(iv) If $\varkappa_0^{(i)} \to 0^{+}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to 0^{+}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to 0^{-}$.
(v) If $\varkappa_0^{(i)} \to 0^{-}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to 0^{-}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to 0^{+}$.
(vi) If $\varkappa_0^{(i)} \to -\infty$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to -\infty$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \frac{1}{\alpha}^{-}$.

Proof. The proof is achieved by (2.4) and (2.5). □

Corollary 2.11. *Let $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ be a solution of system (1.7) and $\alpha < 0$. Then, for $1 \leq i \leq \kappa$, and for $n = 0, 1, ...,$ the following statements hold*
(i) If $\varkappa_0^{(i)} \to -\infty$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to -\infty$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \frac{1}{\alpha}^{-}$.
(ii) If $\varkappa_0^{(i)} \to \frac{1}{\alpha}^{-}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \frac{1}{\alpha}^{-}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to -\infty$.
(iii) If $\varkappa_0^{(i)} \to \frac{1}{\alpha}^{+}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \frac{1}{\alpha}^{+}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \infty$.
(iv) If $\varkappa_0^{(i)} \to 0^{+}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to 0^{+}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to 0^{-}$.
(v) If $\varkappa_0^{(i)} \to 0^{-}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to 0^{-}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to 0^{+}$.
(vi) If $\varkappa_0^{(i)} \to \infty$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \infty$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \frac{1}{\alpha}^{+}$.

Proof. The proof is achieved by (2.4) and (2.5). □

Corollary 2.12. *Assume that* $\alpha > 0$, *then every solution* $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ *of* (1.8) *with* $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ *such that*

$$\varkappa_0^{(\sigma)} > \frac{1}{\alpha}, \quad \sigma = 1,2,...\kappa, \tag{2.11}$$

is positive.

Proof. Consider $\{\chi_n\}_{n=0}^{\infty}$ is a positive solution for system (1.8) wherever $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ satisfying (2.11). Then if $\kappa = 0$ (mod2), it follows from (1.8) and the inequalities (2.11) that

$$\begin{aligned} &\varkappa_\sigma^{(\mu)} = \begin{cases} \dfrac{\varkappa_0^{(\mu-\sigma)}}{\alpha\varkappa_0^{(\mu-\sigma)}-1} \text{ for } 0 < (\mu-\sigma) < \kappa, \\ \dfrac{\varkappa_0^{(\kappa+\mu-\sigma)}}{\alpha\varkappa_0^{(\kappa+\mu-\sigma)}-1} \text{ for } (\mu-\sigma) \le 0, \end{cases} \quad \text{if } \sigma \text{ is odd} \\ &\varkappa_\sigma^{(\mu)} = \begin{cases} \varkappa_0^{(\mu-\sigma)} \text{ for } 0 < (\mu-\sigma) < \kappa, \\ \varkappa_0^{(\kappa+\mu-\sigma)} \text{ for } (\mu-\sigma) \le 0, \end{cases} \quad \text{if } \sigma \text{ is even} \end{aligned} \tag{2.12}$$

for some $\sigma, \mu = 1,2,...,\kappa$.
Assume that $\kappa \neq 0$ (mod2), it follows from (1.8) and (2.11) that

$$\begin{aligned} &\varkappa_\sigma^{(\mu)} = \begin{cases} \dfrac{\varkappa_0^{(\kappa+\mu-\sigma)}}{\alpha\varkappa_0^{(\kappa+\mu-\sigma)}-1} \text{ for } (\mu-\sigma) \le 0, \\ \dfrac{\varkappa_0^{(\mu-\sigma)}}{\alpha\varkappa_0^{(\mu-\sigma)}-1} \text{ for } 0 < (\mu-\sigma) < \kappa, \\ \dfrac{\varkappa_0^{(2\kappa+\mu-\sigma)}}{\alpha\varkappa_0^{(2\kappa+\mu-\sigma)}-1} \text{ for } -2\kappa < (\mu-\sigma) \le -\kappa, \end{cases} \quad \text{if } \sigma \text{ is odd} \\ &\varkappa_\sigma^{(\mu)} = \begin{cases} \varkappa_0^{(\kappa+\mu-\sigma)} \text{ for } (\mu-\sigma) \le 0, \\ \varkappa_0^{(\mu-\sigma)} \text{ for } 0 < (\mu-\sigma) < \kappa, \\ \varkappa_0^{(\mu-\sigma+2\kappa)} \text{ for } -2\kappa < (\mu-\sigma) \le -\kappa, \end{cases} \quad \text{if } \sigma \text{ is odd} \end{aligned} \tag{2.13}$$

where $1 \le \sigma \le 2\kappa$ and $1 \le \mu \le \kappa$. It follows from (2.12) and (2.13) that every solution for (1.8) is positive. □

Corollary 2.13. *Assume that* $\alpha < 0$, *then every solution* $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ *for* (1.8) *where the conditions* $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ *to be such that*

$$\varkappa_0^{(\sigma)} < \frac{1}{\alpha}, \quad \sigma = 1,2,...\kappa, \tag{2.14}$$

is negative.

Proof. The result follows consequentially from (2.12), (2.13) and (2.14). □

Corollary 2.14. *Assume that* $\alpha > 0$, *then every solution* $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ *for* (1.8) *where the conditions* $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ *satisfy the following inequalities*

$$0 < \varkappa_0^{(\sigma)} < \frac{1}{\alpha}, \quad \sigma = 1,2,...\kappa, \tag{2.15}$$

is positive and negative successively. Moreover $\left(\varkappa_{2n}^{(1)}, \varkappa_{2n}^{(2)}, ..., \varkappa_{2n}^{(\kappa)}\right)$ *is positive and* $\left(\varkappa_{2n+1}^{(1)}, \varkappa_{2n+1}^{(2)}, ..., \varkappa_{2n+1}^{(\kappa)}\right)$ *is negative for* $n = 0,1,2,...$.

Proof. The result follows from (2.12), (2.13) and (2.15). □

Corollary 2.15. *Assume that* $\alpha < 0$, *then every solution* $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ *for* (1.8) *where the conditions* $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ *to be such that*

$$\frac{1}{\alpha} < \varkappa_0^{(\sigma)} < 0, \quad \sigma = 1,2,...\kappa, \tag{2.16}$$

is positive and negative successively. Moreover $\left(\varkappa_{2n}^{(1)}, \varkappa_{2n}^{(2)}, ..., \varkappa_{2n}^{(\kappa)}\right)$ *is negative and* $\left(\varkappa_{2n+1}^{(1)}, \varkappa_{2n+1}^{(2)}, ..., \varkappa_{2n+1}^{(\kappa)}\right)$ *is positive for* $n = 0,1,2,...$.

Proof. The result follows from (2.12), (2.13) and (2.16). □

Corollary 2.16. *Assume that* $\alpha > 0$, *then every solution* $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ *for* (1.8) *where the conditions* $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ *satisfy the following inequalities*

$$\varkappa_0^{(\sigma)} < 0, \quad \sigma = 1,2,...\kappa, \tag{2.17}$$

is negative and positive successively. Moreover $\left(\varkappa_{2n}^{(1)}, \varkappa_{2n}^{(2)}, ..., \varkappa_{2n}^{(\kappa)}\right)$ *is negative and* $\left(\varkappa_{2n+1}^{(1)}, \varkappa_{2n+1}^{(2)}, ..., \varkappa_{2n+1}^{(\kappa)}\right)$ *is positive for* $n = 0,1,2,...$.

Proof. The proof is achieved by (2.12), (2.13) and (2.17). □

Corollary 2.17. *Assume that $\alpha < 0$, then every solution $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ for (1.8) where the conditions $\left(\varkappa_0^{(1)}, \varkappa_0^{(2)}, ..., \varkappa_0^{(\kappa)}\right)$ to be such that*

$$\varkappa_0^{(\sigma)} > 0, \quad \sigma = 1, 2, ...\kappa, \tag{2.18}$$

is negative and positive successively. Moreover $\left(\varkappa_{2n}^{(1)}, \varkappa_{2n}^{(2)}, ..., \varkappa_{2n}^{(\kappa)}\right)$ is positive and $\left(\varkappa_{2n+1}^{(1)}, \varkappa_{2n+1}^{(2)}, ..., \varkappa_{2n+1}^{(\kappa)}\right)$ is negative for $n = 0, 1, 2, ...$.

Proof. The proof is achieved by (2.12), (2.13) and (2.18). □

Corollary 2.18. *Let $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ be a solution of system (1.8) and $\alpha > 0$. Therefor each of the following statement holds, (where $n = 0, 1, ...,$ and $1 \leq i \leq \kappa$)*
(i) If $\varkappa_0^{(i)} \to \infty$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \infty$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \frac{1}{\alpha}^{+}$.
(ii) If $\varkappa_0^{(i)} \to \frac{1}{\alpha}^{+}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \frac{1}{\alpha}^{+}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \infty$.
(iii) If $\varkappa_0^{(i)} \to \frac{1}{\alpha}^{-}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \frac{1}{\alpha}^{-}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to -\infty$.
(iv) If $\varkappa_0^{(i)} \to 0^{+}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to 0^{+}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to 0^{-}$.
(v) If $\varkappa_0^{(i)} \to 0^{-}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to 0^{-}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to 0^{+}$.
(vi) If $\varkappa_0^{(i)} \to -\infty$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to -\infty$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \frac{1}{\alpha}^{-}$.

Proof. The proof follows by (2.12) and (2.13). □

Corollary 2.19. *Let $\left(\varkappa_n^{(1)}, \varkappa_n^{(2)}, ..., \varkappa_n^{(\kappa)}\right)$ be a solution of system (1.8) and $\alpha < 0$. Thus each of the following statement holds, with $n = 0, 1, ...,$ and $1 \leq i \leq \kappa$*
(i) If $\varkappa_0^{(i)} \to -\infty$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to -\infty$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \frac{1}{\alpha}^{-}$.
(ii) If $\varkappa_0^{(i)} \to \frac{1}{\alpha}^{-}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \frac{1}{\alpha}^{-}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to -\infty$.
(iii) If $\varkappa_0^{(i)} \to \frac{1}{\alpha}^{+}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \frac{1}{\alpha}^{+}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \infty$.
(iv) If $\varkappa_0^{(i)} \to 0^{+}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to 0^{+}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to 0^{-}$.
(v) If $\varkappa_0^{(i)} \to 0^{-}$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to 0^{-}$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to 0^{+}$.
(vi) If $\varkappa_0^{(i)} \to \infty$, then $\left\{\varkappa_{2n}^{(i)}\right\} \to \infty$ and $\left\{\varkappa_{2n+1}^{(i)}\right\} \to \frac{1}{\alpha}^{+}$.

Proof. The proof accomplishes by (2.12) and (2.13). □

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