Konuralp Journal of Mathematics, 8 (1) (2020) 137-143

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

The Fixed Point Theorem and Characterization of Bipolar Metric Completeness

Kübra Özkan[1*] and Utku Gürdal[2]

[1] *Manisa Celal Bayar University, Department of Mathematics, 45140, Manisa/TURKEY*
[2] *Mehmet Akif Ersoy University, Department of Mathematics, 15030, Burdur/TURKEY*
[*] *Corresponding author*

Abstract

In this article, we prove a fixed point theorem, which is generalization of Banach fixed point theorem and characterizes the metric completeness, for contravariant mapping on bipolar metric spaces. And, we give some results related to this fixed point theorem.

Keywords: *bipolar metric space; completeness; fixed point theory.*
2010 Mathematics Subject Classification: *47H10, 54H25, 54E35.*

1. Introduction

Suzuki introduced the Suzuki type fixed point theorem [9], which is known as one of useful generalizations of the Banach fixed point theorem, as follows: Let (X,d) be a complete metric space and let T be a selfmapping on X. We consider a nonincreasing function $\theta : [0,1) \to (\frac{1}{2},1]$ by

$$\theta(r) = \begin{cases} 1, & 0 \le r \le \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2}, & \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Assume that $\exists r \in [0,1)$ such that

$$\theta(r)d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le rd(x,y)$$

$\forall x,y \in X$. Then, there exists a unique fixed point u of T. Moreover, $\lim_{n\to\infty} T^n x = u$, $\forall x \in X$. Later, many researchers found this theory interesting and obtained many generalizations, see [1, 2, 3, 4, 5, 10, 11].

2. Bipolar Metric Spaces

Definition 2.1. *[6] Let $X,Y \neq \emptyset$ and $d : X \times Y \to \mathbb{R}^+$ be a function. d is called a bipolar metric on (X,Y) if the following properties are satisfied*
(B0) $x = y$ *if* $d(x,y) = 0$,
(B1) $d(x,y) = 0$ *if* $x = y$,
(B2) $d(x,y) = d(y,x)$ *if* $x,y \in X \cap Y$,
(B3) $d(x,y) \le d(x,y') + d(x',y') + d(x',y)$,
$\forall (x,y),(x',y') \in X \times Y$. *Then the triple (X,Y,d) is called a bipolar metric space.*

Definition 2.2. *[6] Let (X_1,Y_1,d_1) and (X_2,Y_2,d_2) be bipolar metric spaces. A function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a covariant map if $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$. Similarly, a function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a contravariant map if $f(X_1) \subseteq Y_2$ and $f(Y_1) \subseteq X_2$. These maps are denoted as $f : (X_1,Y_1,d_1) \rightrightarrows (X_2,Y_2,d_2)$ and $f : (X_1,Y_1,d_1) \nearrow\!\!\!\!\searrow (X_2,Y_2,d_2)$, respectively.*

Definition 2.3. *[6] In a bipolar metric space (X,Y,d);*
(1) (a) *The points of the set X are called left points,*
(b) *The points of the set Y are called right points,*

Email addresses: **kubra.ozkan@hotmail.com (Kübra Özkan), utkugurdal@gmail.com (Utku Gürdal)**

(c) *The points of the set* $X \cap Y$ *are called central points,*
(2) (a) *A sequence of left points is called a left sequence,*
(b) *A sequence of right points is called a right sequence,*
(c) *The term "sequence" is commonly used for left sequences and right sequences,*
(3) (a) *If* $\lim_{n\to\infty} d(a_n, y) = 0$ *for a left sequence* (a_n) *and a right point y, then* (a_n) *is called convergent to y,*
(b) *If* $\lim_{n\to\infty} d(x, b_n) = 0$ *for a right sequence* (b_n) *and a left point x, then* (b_n) *is called convergent to x,*
(4) *A sequence* (x_n, y_n) *on the set* $X \times Y$ *is called a bisequence on* (X, Y, d)*,*
(5) *A bisequence is called convergent, if both the left sequence* (x_n) *and the right sequence* (y_n) *converge,*
(6) *If* (x_n) *and* (y_n) *converge to a common point, then* (x_n, y_n) *is called biconvergent,*
(7) *A Cauchy bisequence is a bisequence* (x_n, y_n) *such that* $\lim_{n,m\to\infty} d(x_n, y_m) = 0$*,*
(8) *A bipolar metric space in which every Cauchy bisequence converges, is called a complete bipolar metric space.*

It is shown in [6] that convergence of Cauchy bisequences implies biconvergence.

Proposition 1. *[6] If a central point is a limit of a sequence, then this sequence has unique limit.*

Definition 2.4. *[6] (1) A covariant map* $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ *is called left-continuous at* $x_0 \in X_1 \iff \exists \delta = \delta(x_0, \varepsilon) > 0$ *such that* $d_1(x_0, y) < \delta \Rightarrow d_2(f(x_0), f(y)) < \varepsilon$ *for every* $\varepsilon > 0$ *and* $\forall y \in Y_1$*.*
(2) *A covariant map* $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ *is right-continuous at* $y_0 \in Y_1 \iff \exists \delta = \delta(y_0, \varepsilon) > 0$ *such that* $d_1(x, y_0) < \delta \Rightarrow d_2(f(x), f(y_0)) < \varepsilon$ *for every* $\varepsilon > 0$ *and* $\forall x \in X_1$*.*
(3) *If a covariant map f is left-continuous at each* $x \in X_1$ *and right-continuous at each* $y \in Y_1$*, then it is called continuous.*
(4) *A contravariant map* $f : (X_1, Y_1, d_1) \nearrow\!\!\!\!\swarrow (X_2, Y_2, d_2)$ *is called left-continuous at a point* $x_0 \in X_1$*, right-continuous at a point* $y_0 \in Y_1$ *or continuous,* $\iff$ *the corresponding covariant map* $f : (X_1, Y_1, d_1) \rightrightarrows (Y_2, X_2, d_2)$ *is left-continuous at* x_0*, right-continuous at* y_0 *or continuous, respectively.*

This definition implies that a covariant or a contravariant map f, which is defined from (X_1, Y_1, d_1) to (X_2, Y_2, d_2), is continuous, $\iff$ $(a_n) \to v$ on $(X_1, Y_1, d_1) \implies (f(a_n)) \to f(v)$ on (X_2, Y_2, d_2).

Proposition 2. *[6] If the point which a covariant or contravariant map f is left and right continuous, is central point, then the map f is continuous in this point.*

3. Main Results

Our first new result is the next:

Theorem 3.1. *Let* (X, Y, d) *be a complete bipolar metric space and* $T : (X, Y) \nearrow\!\!\!\!\swarrow (X, Y)$ *be a contravariant mapping. We define a nonincreasing function* θ *from* $[0, 1)$ *to* $\left(\frac{1}{2}, 1\right]$ *by*

$$\theta(r) = \begin{cases} 1, & 0 \le r \le \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2}, & \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Assume that $\exists r \in [0, 1)$ *such that*

$$\begin{aligned} \theta(r) d(x, Tx) \le d(x, y) \quad &\Rightarrow \quad d(Ty, Tx) \le r d(x, y) \\ \theta(r) d(Ty, y) \le d(x, y) \quad &\Rightarrow \quad d(Ty, Tx) \le r d(x, y). \end{aligned} \tag{3.1}$$

$\forall x \in X,\ y \in Y$*. Then* $T : (X, Y) \nearrow\!\!\!\!\swarrow (X, Y)$ *has a unique fixed point.*

Proof. Since $\theta(r) \le 1$, $\theta(r) d(x, Tx) \le d(x, Tx)$ for every $x \in X$. Using hypothesis, we can say that

$$\begin{aligned} d(T^2x, Tx) &\le r d(x, Tx) \\ d(Ty, T^2y) &\le r d(Ty, y) \end{aligned} \tag{3.2}$$

$\forall x \in X,\ y \in Y$. Let $x_0 \in X$, for each nonnegative integer n, we define $x_n = Ty_{n-1}$, $y_n = Tx_n$. Then (3.1) and (3.2) yield

$$\begin{aligned} d(x_1, y_1) &= d(Ty_0, Tx_1) \\ &= d(Ty_0, T^2y_0) \\ &\le r d(Ty_0, y_0) \\ &= d(x_1, y_0) \\ &= r d(T^2x_0, Tx_0) \\ &\le r^2 d(x_0, Tx_0) \\ &\le r^2 d(x_0, y_0). \end{aligned}$$

Then we have $d(x_n, y_n) \le r^{2n} d(x_0, y_0)$. From (3.1) and (3.2), we get

$$\begin{aligned} d(x_{n+1}, y_n) &= d(Ty_n, Tx_n) \\ &= d(T^2x_n, Tx_n) \\ &\le r d(x_n, Tx_n) \\ &\le r d(x_n, y_n) \\ &\le r^{2n+1} d(x_0, y_0). \end{aligned}$$

For all positive integer m and n, if $n \leq m$

$$\begin{aligned} d(x_n,y_m) &\leq d(x_n,y_n)+d(x_{n+1},y_n)+d(x_{n+1},y_{n+1})+d(x_{n+2},y_{n+1}) \\ &\quad +\ldots+d(x_m,y_{m-1})+d(x_m,y_m) \\ &\leq (r^{2n}+r^{2n+1}+r^{2n+2}+\cdots+r^{2m})d(x_0,y_0) \\ &\leq \frac{r^{2n}}{1-r}d(x_0,y_0) \end{aligned}$$

if $n > m$

$$\begin{aligned} d(x_n,y_m) &\leq d(x_n,y_{n-1})+d(x_{n-1},y_{n-1})+d(x_{n-1},y_{n-2})+d(x_{n-2},y_{n-2}) \\ &\quad +\ldots+d(x_{m+1},y_{m+1})+d(x_{m+1},y_m) \\ &\leq (r^{2n-1}+r^{2n-2}+r^{2n-3}+\cdots+r^{2m+2})d(x_0,y_0) \\ &\leq \frac{r^{2m+2}}{1-r}d(x_0,y_0). \end{aligned}$$

Then, we say that $d(x_n,y_m)\to 0$. So, (x_n,y_n) is a Cauchy bisequence. Since (X,Y,d) is a complete bipolar metric space, (x_n,y_n) is converges, in fact biconverges, to $u\in X\cap Y$. We next show

$$d(u,Tx) \leq rd(x,u) \text{ for } \forall x\in X-\{u\} \tag{3.3}$$
$$d(Ty,u) \leq rd(u,y) \text{ for } \forall y\in Y-\{u\}. \tag{3.4}$$

For $y\in Y-\{u\}$, $\forall n\in\mathbb{N}$,

$$\exists n_0\in\mathbb{N},\ \forall n\geq n_0, \qquad d(x_n,u)\leq \frac{d(u,y)}{3}$$
$$\exists n_1\in\mathbb{N},\ \forall n\geq n_1, \qquad d(u,y_n)\leq \frac{d(u,y)}{3}.$$

We take $n_2=\max\{n_0,n_1\}$. Then, we have

$$\begin{aligned} \theta(r)d(x_n,Tx_n)\leq d(x_n,Tx_n) &= d(x_n,y_n) \\ &\leq d(x_n,u)+d(u,y_n) \\ &\leq \tfrac{2}{3}d(u,y) \\ &= d(u,y)-\tfrac{d(u,y)}{3} \\ &\leq d(u,y)-d(x_n,u) \\ &\leq d(x_n,y). \end{aligned}$$

From (3.1), we get

$$\theta(r)d(x_n,Tx_n)\leq d(x_n,y)\Rightarrow d(Ty,Tx_n)\leq rd(x_n,y)\Rightarrow d(Ty,y_n)\leq rd(x_n,y)$$

for $n\geq n_2$. Letting n tend to ∞, we get

$$d(Ty,u)\leq rd(u,y).$$

That is, (3.4) is satisfied. On the other hand, for $x\in X-\{u\}$

$$\exists n_3\in\mathbb{N},\ \forall n\geq n_3, \qquad d(x_n,u)\leq \frac{d(x,u)}{3}$$
$$\exists n_4\in\mathbb{N},\ \forall n\geq n_4, \qquad d(u,y_n)\leq \frac{d(x,u)}{3}.$$

We take $n_5=\max\{n_3,n_4\}$. Then, we have

$$\begin{aligned} \theta(r)d(Ty_n,y_n)\leq d(Ty_n,y_n) &= d(x_{n+1},y_n) \\ &\leq d(x_{n+1},u)+d(u,y_n) \\ &\leq \tfrac{2}{3}d(x,u) \\ &= d(x,u)-\tfrac{d(x,u)}{3} \\ &\leq d(x,u)-d(u,y_n) \\ &\leq d(x,y_n). \end{aligned}$$

From (3.1), we get

$$\theta(r)d(Ty_n,y_n)\leq d(x,y_n)\Rightarrow d(Ty_n,Tx)\leq rd(x,y_n)\Rightarrow d(x_n,Tx)\leq rd(x,y_n)$$

for $n\geq n_5$. Letting n tend to ∞, we get

$$d(u,Tx)\leq rd(x,u).$$

That is, (3.3) is satisfied. Arguing by contradiction, we assume that $T^j u\neq u$, $\forall j\in\mathbb{N}$. Then (3.3) and (3.4) yields

$$d(T^{j+1}u,u)\leq r^j d(Tu,u) \quad \text{for} \quad j\in\mathbb{N}. \tag{3.5}$$

We consider the following three cases;

* $0\leq r\leq \frac{\sqrt{5}-1}{2}$
* $\frac{\sqrt{5}-1}{2}\leq r\leq \frac{1}{\sqrt{2}}$
* $\frac{1}{\sqrt{2}}\leq r\leq 1.$

Firstly, we consider the case where $0 \leq r \leq \frac{\sqrt{5}-1}{2}$. Then, $r^2+r \leq 1$ and $2r^2<1$. If we assume $d(T^2u,u) < d(T^2u,T^3u)$, then from (3.5), we have

$$\begin{aligned} d(u,Tu) &\leq d(u,T^2u)+d(T^2u,T^2u)+d(T^2u,Tu) \\ &< d(T^2u,T^3u)+d(Tu,T^2u) \\ &\leq r^2d(u,Tu)+rd(u,Tu) \\ &\leq d(u,Tu). \end{aligned}$$

This is a contradiction. So, we have (since $\theta(r)=1$)

$$d(T^2u,u) \geq d(T^2u,T^3u) = \theta(r)d(T^2u,T^3u).$$

By hypothesis, we get

$$d(Tu,T^3u) \leq rd(T^2u,u). \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned} d(u,Tu) &\leq du,T^3u)+d(T^3u,T^3u)+d(T^3u,Tu) \\ &\leq r^2d(u,Tu)+rd(T^2u,u) \\ &\leq r^2d(u,Tu)+r^2d(Tu,u) \\ &= 2r^2d(u,Tu) \\ &< d(u,Tu). \end{aligned}$$

This is a contradiction.

In the second case where $\frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}$, it is obvious $2r^2<1$. If we assume

$$d(T^2u,u) < \theta(r)d(T^2u,T^3u),$$

then, from (3.5), we get

$$\begin{aligned} d(u,Tu) &\leq d(u,T^2u)+d(T^2u,T^2u)+d(T^2u,Tu) \\ &< \theta(r)d(T^2u,T^3u)+d(T^2u,Tu) \\ &\leq \theta(r)r^2d(u,Tu)+rd(u,Tu) \\ &= (\theta(r)r^2+r)d(u,Tu) \\ &= d(u,Tu). \end{aligned}$$

This is a contradiction. So, we have

$$d(T^2u,u) \geq \theta(r)d(T^2u,T^3u).$$

By hypothesis, we get

$$d(Tu,T^3u) \leq rd(T^2u,u). \tag{3.7}$$

From (3.5) and (3.7), we have

$$d(u,Tu) \leq 2r^2d(u,Tu) < d(u,Tu).$$

This is a contradiction.

In the third case where $\frac{1}{\sqrt{2}} \leq r \leq 1$, we note that it is satisfied

$$\theta(r)d(x_{2n}Tx_{2n}) \leq d(x_{2n},u)$$

for every $n \in \mathbb{N}$. If $x_{2n}=u$, it is obvious. Otherwase, we assume that

$$\theta(r)d(x_{2n}Tx_{2n}) > d(x_{2n},u)$$

for $\exists n \in \mathbb{N}$. Then, from (3.3), we have

$$\begin{aligned} d(x_{2n},Tx_{2n}) &\leq d(x_{2n},u)+d(u,u)+d(u,Tx_{2n}) \\ &< d(x_{2n},u)+rd(x_{2n},u) \\ &= (1+r)d(x_{2n},u) \\ &< \theta(r)d(x_{2n},Tx_{2n})(1+r) \\ &= d(x_{2n},Tx_{2n}). \end{aligned}$$

This is a contradiction. Similarly, we show that

$$\theta(r)d(Ty_{2n},y_{2n}) \leq d(u,y_{2n})$$

for every $n \in \mathbb{N}$. If $y_{2n}=u$, it is obvious. We assume that

$$\theta(r)d(Ty_{2n},y_{2n}) > d(u,y_{2n})$$

for $\exists n \in \mathbb{N}$. Then from (3.4), we have

$$\begin{aligned} d(Ty_{2n},y_{2n}) &\leq d(Ty_{2n},u)+d(u,u)+d(u,y_{2n}) \\ &< (1+r)d(u,y_{2n}) \\ &< (1+r)d(Ty_{2n},y_{2n}) \\ &= d(Ty_{2n},y_{2n}). \end{aligned}$$

This is a contradiction. By hypothesis $d(Tu,Tx_{2n}) \leq rd(x_{2n},u)$ and $d(Ty_{2n},Tu) \leq rd(u,y_{2n})$ hold for every $n \in \mathbb{N}$. Since $\{x_n\}$ and $\{y_n\}$ converges to u, the above inequalities imply there exists a bisequence of (x_n,y_n) which biconverges to Tu. We get $Tu = u$. This is a contradiction. Then, in all the cases, $\exists j \in \mathbb{N}$ such that $T^j u = u$. Since $(T^n u, T^n u)$ is a Cauchy bisequence, we obtain $Tu = u$. That is, u is a fixed point of T. We assume that there exist another fixed point v for T, such that $u \neq v$.

$$d(u,v) = d(Tu,Tv) \leq rd(u,Tv) \leq r^2 d(u,v).$$

This is a contradiction. Then, T has a unique fixed point. □

It is obvious that the set of our contravariant contractions (3.1) in Theorem 3.1 includes that of the usual contravariant contractions in bipolar metric spaces. So, we can write the following Corollary 1 as an result of the Theorem (3.1).

Corollary 1. *[6] Let (X,Y,d) be a complete bipolar metric space and $T:(X,Y) \rightleftarrows (X,Y)$ be a contravariant mapping. Assume that $\exists r \in [0,1)$ such that*

$$d(Ty,Tx) \leq rd(x,y)$$

$\forall x \in X,\ y \in Y$. Then $T:(X,Y) \rightleftarrows (X,Y)$ has a unique fixed point.

On the otherhand, the set of our contravariant contractions (3.1) in Theorem 3.1 and Kannan contravariant mappings in bipolar metric spaces are independent. The following example show this.

Example 3.2. *Let $X = \{0,1,2\}$, $Y = \{1,2,3\}$, $d: X \times Y \to [0,\infty)$ be defined by $d(1,1) = d(2,2) = 0$, $d(1,2) = d(2,1) = 5$, $d(0,1) = 9$, $d(0,2) = 11$, $d(0,3) = 16$, $d(1,3) = 14$, $d(2,3) = 19$. Then, (X,Y,d) is a complete bipolar metric spaces. We define a contravariant mapping $T:(X,Y) \rightleftarrows (X,Y)$ such as $T(0) = 1, T(1) = T(2) = 2, T(3) = 0$. Then we get*

$$d(Ty,Tx) \leq \frac{3}{5} d(x,y)$$

if $(x,y) \neq (1,1)$, $(x,y) \neq (1,3)$. Since

$$\theta(r)d(1,T(1)) = \theta(r)d(1,2) \geq 5 > 0 = d(1,1)$$

and

$$\theta(r)d(3,T(3)) = \theta(r)d(0,3) \geq 15 > 13 = d(1,3)$$

for every $r \in [0,1)$, T satisfies the assumption in Theorem 3.1. Then from Theorem 3.1, T has a unique fixed point $2 \in X \cap Y$. On the other hand, T is not a Kannan contravariant mapping. Because, for $x = 0$, $y = 2$, $\alpha \in \left[0, \frac{1}{2}\right)$, it is obvious that

$$d(T(2),T(0)) = d(2,1) = 5 > \frac{9}{2} = \frac{1}{2}\left(d(0,T(0)) + d(T(2),2)\right).$$

Our second new result is the next:

Theorem 3.3. *Let (X,Y,d) be a bipolar metric space and define a function θ as in Theorem 3.1. For $r \in [0,1)$ and $\eta \in (0,\theta(r)]$, let $A_{r,\eta}$ be the family of contravariant mappings $T:(X,Y) \rightleftarrows (X,Y)$ satisfying the following:*

(a) For $x,y \in X$,

$$\eta d(x,Tx) \leq d(x,y) \quad \Rightarrow \quad d(Ty,Tx) \leq rd(x,y)$$
$$\eta d(Ty,y) \leq d(x,y) \quad \Rightarrow \quad d(Ty,Tx) \leq rd(x,y).$$

Let $B_{r,\eta}$ be the family of mappings $T:(X,Y) \rightleftarrows (X,Y)$ satisfying (a) and the following:

(b) $T(X)$ and $T(Y)$ are countably infinite.

(c) Every subsets of $T(X)$ and $T(Y)$ are closed.

Then the following are equivalent:

(i) $(X,Y,d)$is complete.

(ii) Every mapping $T \in A_{r,\theta(r)}$ has a fixed point $\forall r \in [0,1)$.

(iii) There exist $r \in (0,1)$ and $\eta \in (0,\theta(r)]$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.

Proof. By Theorem 3.1, $(i) \implies (ii)$. Since $B_{r,\eta} \subset A_{r,\theta(r)}$ for $r \in [0,1)$ and $\eta \in (0,\theta(r)]$, $(ii) \implies (iii)$. Let us prove $(iii) \implies (i)$. We assume that (iii) holds. Arguing by contradiction, we also assume that (X,Y,d) is not complete. That is, there exists a Cauchy bisequence (x_n,y_n) which does not biconverge. Define a function $f: X \cup Y \to [0,\infty)$ by $f(x) = \lim_n d(x,y_n)$ and $f(y) = \lim_n d(x_n,y)$ for $x \in X, y \in Y$. We note that f is well defined because $\{d(x,y_n)\}$ and $\{d(x_n,y)\}$ are two Cauchy bisequence for every $x \in X$, $y \in Y$. The following are obvious:

* $f(x) - f(y) \leq d(x,y) \leq f(x) + f(y)$ and $f(x) - f(y) \leq d(x,y) \leq f(x) + f(y)$ for $x \in X$, $y \in Y$,
* $f(x) > 0$ and $f(y) > 0$, $\forall x \in X, y \in Y$,
* $\lim_n f(x_n) = 0$ and $\lim_n f(y_n) = 0$.

Define a contravariant mapping T on (X,Y) as follows: For each $x\in X$, $y\in Y$, since $f(x)>0$ and $f(y)>0$ and $\lim_n f(x_n)=0$, $\lim_n f(y_n)=0$, $\exists v\in\mathbb{N}$ satisfying

$$f(y_v)\leq\frac{\eta_r}{3+\eta r}f(x),\quad f(x_v)\leq\frac{\eta_r}{3+\eta r}f(y).$$

We put $Tx=y_v$ and $Ty=x_v$. Then it is obvious that

$$f(Tx)\leq\frac{\eta_r}{3+\eta r}f(x),\quad \text{and}\quad Tx\in\{y_n:n\in\mathbb{N}\},$$

$$f(Ty)\leq\frac{\eta_r}{3+\eta r}f(y)\quad \text{and}\quad Ty\in\{x_n:n\in\mathbb{N}\}$$

$\forall x\in X$, $y\in Y$. Then $\forall u\in X\cap Y$, $Tu\neq u$, because $F(Tu)<f(u)$. That is, T does not have a fixed point. Since $T(X)\subset\{y_n:n\in\mathbb{N}\}$ and $T(Y)\subset\{x_n:n\in\mathbb{N}\}$, $T(X)$ and $T(Y)$ countably infinite. That is, (b) holds. Also, it is not difficult to prove (c). Let us prove (a). Fix $x,y\in X$ with

$$\eta d(x,Tx)\leq d(x,y).$$

In the case where $f(y)>2f(x)$, we have

$$\begin{array}{rcl} d(Ty,Tx)\leq f(Ty)+f(Tx) & \leq & \frac{\eta r}{3+\eta r}(f(x)+f(y)) \\ & \leq & \frac{r}{3}(f(x)+f(y)) \\ & \leq & \frac{r}{3}(f(x)+f(y))+\frac{2r}{3}(f(y)-2f(x)+) \\ & = & r(f(y)-f(x)) \\ & \leq & rd(x,y). \end{array}$$

In the case where $f(y)\leq 2f(x)$, we have

$$\begin{array}{rcl} d(x,y)\leq\eta d(x,Tx) & \geq & \eta(f(x)-f(Tx)) \\ & \geq & \eta\left(f(x)-\frac{\eta r}{3+\eta r}f(x)\right) \\ & = & \eta\left(1-\frac{\eta r}{3+\eta r}\right)f(x) \\ & = & \frac{3\eta}{3+\eta r}f(x) \end{array}$$

and hence

$$\begin{array}{rcl} d(Ty,Tx)\leq f(Ty)+f(Tx) & \leq & \frac{\eta r}{3+\eta r}(f(y)+f(x)) \\ & \leq & \frac{3\eta}{3+\eta r}f(x) \\ & \leq & rd(x,y). \end{array}$$

Similarly, fix $x,y\in X$ with $\eta d(Ty,y)\leq d(x,y)$. In the case where $f(x)>2f(y)$, we have

$$\begin{array}{rcl} d(Ty,Tx)\leq f(Ty)+f(Tx) & \leq & \frac{\eta r}{3+\eta r}(f(y)+f(x)) \\ & \leq & \frac{r}{3}(f(x)+f(y)) \\ & \leq & \frac{r}{3}(f(x)+f(y))+\frac{2r}{3}(f(x)-2f(y)+) \\ & = & r(f(x)-f(y)) \\ & \leq & rd(x,y). \end{array}$$

In the case where $f(x)\leq 2f(y)$, we have

$$\begin{array}{rcl} d(x,y)\leq\eta d(Ty,y) & \geq & \eta(f(y)-f(Ty)) \\ & \geq & \eta\left(f(y)-\frac{\eta r}{3+\eta r}f(y)\right) \\ & = & \eta\left(1-\frac{\eta r}{3+\eta r}\right)f(y) \\ & = & \frac{3\eta}{3+\eta r}f(y) \end{array}$$

and hence

$$\begin{array}{rcl} d(Ty,Tx)\leq f(Ty)+f(Tx) & \leq & \frac{\eta r}{3+\eta r}(f(y)+f(x)) \\ & \leq & \frac{3\eta}{3+\eta r}f(y) \\ & \leq & rd(x,y). \end{array}$$

Therefore, we have shown (a), that is, $T\in B_{r,\eta}$. By (iii), T has a fixed point which yields a contradiction. Hence, we obtain that X is complete. □

Corollary 2. *For a bipolar metric space (X,Y,d), the following are equivalent;*

(i) (X,Y,d) *is complete,*

(ii) There exists $r \in (0,1)$ such that every mapping T on X and Y satisfying the following has a fixed point:

$$\frac{1}{10000}d(x,Tx) \leq d(x,y) \implies d(Ty,Tx) \leq rd(x,y) \text{ and } \frac{1}{10000}d(Ty,y) \leq d(x,y) \implies d(Ty,Tx) \leq rd(x,y),\ \forall x \in X,\ y \in Y.$$

Proof. In the Theorem 3.3, if we take the η as $\frac{1}{10000}$, we obtain this corollary. □

References

[1] M. Abbas, B. Ali and C. Vetro, A Suzuki type fixed point theorem for a generalized multivalued mapping on partial Hausdorff metric spaces, Topology and Its Applications, Vol:160, No.3, (2013), 553–563.

[2] M. Abbas, H. Iqbal and A. Petrusel, Fixed points for multivalued Suzuki type $(\theta,\mathscr{R})$-contraction mapping with applications, Journal of Function Spaces, Vol: 2019, Article ID 9565804, 13 pages, 2019. https://doi.org/10.1155/2019/9565804.

[3] N. Chandraa, M.C. Aryaa and Mahesh C. Joshia, A Suzuki-Type Common Fixed Point Theorem, Filomat, Vol:31, No.10 (2017), 2951–2956.

[4] L. Ćirić, M. Abbas, M. Rajović and B. Ali, Suzuki type fixed point theorems for generalized multi-valued mappings on a set endowed with two b-metrics, Applied Mathematics and Computation, Vol:219, No.4 (2012), 1712-1723.

[5] D. Dorić, Z. Kadelburg and S. Radenović, Edelstein-Suzuki-type fixed point results in metric spaces, Nonlinear Analysis: Theory, Methods and Applications, Vol:75, (2012), 1927–1932.

[6] A. Mutlu and U. Gürdal, Bipolar metric spaces and some fixed point theorems, Journal of Nonlinear Sciences and Applications, Vol:9, No.9 (2016), 5362–5373.

[7] A. Mutlu, K. Özkan and U. Gürdal, Coupled Fixed Point Theorems on Bipolar Metric Spaces, European Journal of Pure and Applied Mathematics, Vol:10, No.4 (2017), 655–667.

[8] A. Mutlu, K. Özkan and U. Gürdal, Fixed point theorems for multivalued mappings on bipolar metric spaces, Fixed Point Theory, Vol: 21, No.1, (2020), 271–280.

[9] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proceedings of the American Mathematical Society, Vol:136, (2008), 1861-1869.

[10] T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Analysis: Theory, Methods and Applications, Vol:71, No.11 (2009), 5313–5317.

[11] D. Paesano and P. Vetro, Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces, Topology and Its Applications, Vol:159, No.3 (2012), 911–920.