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On Trans-Sasakian Manifolds with the Schouten-van Kampen Connection

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Abstract

The object of the present paper is to characterize 3-dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection.

Keywords: *Conharmonical curvature tensor; Projective curvature tensor; Schouten-van Kampen connection; Trans-Sasakian manifolds.*
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1. Introduction

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by Chinae and Gonzales [4]. These type of manifolds appear as a natural generalization of both Sasakian and Kenmotsu manifolds. In the Gray-Hervella classification of almost Hermitian manifolds [6], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [3]. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [12] if the product manifold $M\times\mathbb{R}$ belongs to the class W_4. The class $C_6\oplus C_5$ [11] coincides with the class of the trans-Sasakian structures of type (α,β). In [11], local nature of the two subclasses, namely, C_5 and C_6 structures of trans-Sasakian structures are characterized completely.
We note that trans-Sasakian structures of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosymplectic [2], β-Kenmotsu [8] and α-Sasakian [8], respectively. Also it is proved that trans-Sasakian structures are generalized quasi-Sasakian [8]. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.
On the other hand the Schouten-van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection [1, 7, 9, 14]. Solov'ev investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection [15, 16, 17, 18]. Then Olszak studied the Schouten-van Kampen connection to an almost contact metric structure and characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection and found certain curvature properties of this connection on these manifolds [13]. Also, Yildiz studied projectively flat and conharmonically flat 3-dimensional f-Kenmotsu manifolds with the Schouten-van Kampen connection [19].
The present paper is organized as follows: After preliminaries, we give some basic information about the Schouten-van Kampen connection and trans-Sasakian manifolds. Then we adapte the Schouten-van Kampen connection on 3-dimensional trans-Sasakian manifolds. In section 5, we consider projectively flat and conharmonically flat 3-dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection. In the last section, we give an example of a 3-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g), that is, ϕ is an $(1,1)$-tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(X)=-X+\eta(X)\xi,\quad \eta(\xi)=1,\quad \phi\xi=0,\quad \eta\circ\phi=0, \tag{2.1}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.2}$$

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$$g(X,\phi Y)=-g(\phi X,Y),\quad g(X,\xi)=\eta(X), \tag{2.3}$$

for all $X,Y\in TM$ [2]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X,Y)=g(X,\phi Y), \tag{2.4}$$

for $X,Y\in TM$.
An almost contact metric struce (ϕ,ξ,η,g) on a connected manifold M is called trans-Sasakian structure [12] if $(M\times\mathbb{R}, J, G)$ belongs to the class W_4 [6], where J is the almost complex structure on $M\times\mathbb{R}$ defined by

$$J(X,fd/dt)=(\phi X-f\xi,\eta(X)d/dt),$$

for all vector fields X on M and smooth function f on $M\times\mathbb{R}$, and G is the product metric on $M\times\mathbb{R}$. This may be expressed by the condition [3]

$$(\nabla_X\phi)Y=\alpha(g(X,Y)\xi-\eta(Y)X)+\beta(g(\phi X,Y)\xi-\eta(Y)\phi X), \tag{2.5}$$

for smooth functions α and β on M. Here we say that the trans-Sasakian structure is of type (α,β). From the formula (2.5) it follows that

$$\nabla_X\xi=-\alpha\phi X+\beta(X-\eta(X)\xi), \tag{2.6}$$

$$(\nabla_X\eta)Y=-\alpha g(\phi X,Y)+\beta g(\phi X,\phi Y). \tag{2.7}$$

An explicit example of 3-dimensional proper trans-Sasakian manifold was constructed in [10]. In [5], the Ricci tensor and curvature tensor for 3-dimensional trans-Sasakian manifolds were studied and their explicit formulae were given.
From [5] we know that for a 3-dimensional trans-Sasakian manifold

$$2\alpha\beta+\xi\alpha=0, \tag{2.8}$$

$$S(X,\xi)=(2(\alpha^2-\beta^2)-\xi\beta)\eta(X)-X\beta-(\phi X)\alpha, \tag{2.9}$$

$$\begin{aligned} S(X,Y) &= (\frac{\tau}{2}+\xi\beta-(\alpha^2-\beta^2))g(X,Y)\\ &-(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Y)\\ &-(Y\beta+(\phi Y)\alpha)\eta(X)-(X\beta+(\phi X)\alpha)\eta(Y), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} R(X,Y)Z &= (\frac{\tau}{2}+2\xi\beta-2(\alpha^2-\beta^2))(g(Y,Z)X-g(X,Z)Y)\\ &-g(Y,Z)[(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\xi\\ &-\eta(X)(\phi grad\alpha-grad\beta)+(X\beta+(\phi X)\alpha)\xi]\\ &+g(X,Z)[(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\xi\\ &-\eta(Y)(\phi grad\alpha-grad\beta)+(Y\beta+(\phi Y)\alpha)\xi]\\ &-[(Z\beta+(\phi Z)\alpha)\eta(Y)+(Y\beta+(\phi Y)\alpha)\eta(Z)\\ &+(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\eta(Z)]X\\ &+[(Z\beta+(\phi Z)\alpha)\eta(X)+(X\beta+(\phi X)\alpha)\eta(Z)\\ &+(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Z)]Y, \end{aligned} \tag{2.11}$$

where S is the Ricci tensor, R is the curvature tensor and τ is the scalar curvature of the manifold M, respectively.
For constants α and β are the above relations become

$$\begin{aligned} R(X,Y)Z &= (\frac{\tau}{2}-2(\alpha^2-\beta^2))(g(Y,Z)X-g(X,Z)Y)\\ &-(\frac{\tau}{2}-3(\alpha^2-\beta^2))(g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\\ &+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y), \end{aligned} \tag{2.12}$$

$$S(X,Y)=(\frac{\tau}{2}-(\alpha^2-\beta^2))g(X,Y)-(\frac{\tau}{2}-3(\alpha^2-\beta^2))\eta(X)\eta(Y), \tag{2.13}$$

$$S(X,\xi)=2(\alpha^2-\beta^2)\eta(X), \tag{2.14}$$

$$R(X,Y)\xi=(\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y), \tag{2.15}$$

$$QX=(\frac{\tau}{2}-(\alpha^2-\beta^2))X-(\frac{\tau}{2}-3(\alpha^2-\beta^2))\eta(X)\xi. \tag{2.16}$$

From (2.8) it follows that if α and β are constants, then the manifold is either α-Sasakian or β-Kenmotsu or cosymplectic.

3. The Schouten-van Kampen connection

Let M be a connected pseudo-Riemannian manifold of an arbitrary signature $(p, n-p)$, $0 \leq p \leq n$, $n = \dim M \geq 2$. By g and ∇ we denote the pseudo-Riemannian metric and Levi-Civita connection induced from the metric g on M respectively. Assume that H and V are two complementary, orthogonal distributions on M such that $dimH = n-1$, $dimV = 1$, and the distribution V is non-null. Thus $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. Assume that ξ is a unit vector field and η is a linear form such that $\eta(\xi) = 1$, $g(\xi, \xi) = \varepsilon = \pm 1$ and

$$H = \ker \eta, \quad V = \operatorname{span}\{\xi\}. \tag{3.1}$$

We can always choose such ξ and η at least locally (in a certain neighborhood of an arbitrary chosen point of M). We also have $\eta(X) = \varepsilon g(X, \xi)$. Moreover, it holds that $\nabla_X \xi \in H$.
For any $X \in TM$, by X^h and X^v we denote the projections of X onto H and V, respectively. Thus, we have $X = X^h + X^v$ with

$$X^h = X - \eta(X)\xi, \quad X^v = \eta(X)\xi. \tag{3.2}$$

The Schouten-van Kampen connection $\tilde{\nabla}$ associated to the Levi-Civita connection ∇ and adapted to the pair of the distributions (H, V) is defined by [1]

$$\tilde{\nabla}_X Y = (\nabla_X Y^h)^h + (\nabla_X Y^v)^v, \tag{3.3}$$

and the corresponding second fundamental form B is defined by $B = \nabla - \tilde{\nabla}$. Note that the condition (3.3) implies the parallelism of the distributions H and V with respect to the Schouten-van Kampen connection $\tilde{\nabla}$.
From (3.2), one can compute

$$\begin{aligned}(\nabla_X Y^h)^h &= \nabla_X Y - \eta(\nabla_X Y)\xi - \eta(Y)\nabla_X \xi, \\ (\nabla_X Y^v)^v &= (\nabla_X \eta)(Y)\xi + \eta(\nabla_X Y)\xi,\end{aligned}$$

which enables us to express the Schouten-van Kampen connection with the help of the Levi-Civita connection in the following way [15]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi. \tag{3.4}$$

Thus, the second fundamental form B and the torsion $\tilde{T}$ of $\tilde{\nabla}$ are [15, 16]

$$B(X,Y) = \eta(Y)\nabla_X \xi - (\nabla_X \eta)(Y)\xi,$$

and

$$\tilde{T}(X,Y) = \eta(X)\nabla_Y \xi - \eta(Y)\nabla_X \xi + 2d\eta(X,Y)\xi.$$

With the help of the Schouten-van Kampen connection (3.4), many properties of some geometric objects connected with the distributions H, V can be characterized [15, 16, 17]. Probably, the most spectacular is the following statement: g, ξ and η are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla}\xi = 0$, $\tilde{\nabla}g = 0$, $\tilde{\nabla}\eta = 0$.

4. Trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

Let M be a 3-dimensional trans-Sasakian manifold with α and β are constants with respect to the Schouten-van Kampen connection. Then using (2.6) and (2.7) in (3.4), we get

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)\phi X - g(\phi X, Y)\xi\} + \beta\{g(X,Y)\xi - \eta(Y)X\}. \tag{4.1}$$

Let R and $\tilde{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\tilde{\nabla}$

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \tilde{R}(X,Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}.$$

Using (4.1), by direct calculations, we obtain the following formula connecting R and $\tilde{R}$ on a 3-dimensional trans-Sasakian manifold M

$$\begin{aligned}\tilde{R}(X,Y)Z = \; & R(X,Y)Z \\ & +\alpha^2\{g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + \eta(X)\eta(Z)Y \\ & -\eta(Y)\eta(Z)X - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi\} \\ & +\beta^2\{g(Y,Z)X - g(X,Z)Y\}.\end{aligned} \tag{4.2}$$

We will also consider the Riemann curvature $(0,4)$-tensors $\tilde{R}, R$, the Ricci tensors $\tilde{S}, S$, the Ricci operators $\tilde{Q}, Q$ and the scalar curvatures $\tilde{\tau}, \tau$ of the connections $\tilde{\nabla}$ and ∇ are given by

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = \; & R(X,Y,Z,W) \\ & +\alpha^2\{g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W) \\ & +g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z) \\ & -g(Y,Z)\eta(X)\eta(W) + g(X,Z)\eta(Y)\eta(W)\} \\ & +\beta^2\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\},\end{aligned} \tag{4.3}$$

$$\tilde{S}(Y,Z) = S(Y,Z) + 2\beta^2 g(Y,Z) - 2\alpha^2\eta(Y)\eta(Z)), \tag{4.4}$$

$$\tilde{Q}X = QX + 2\beta^2 X - 2\alpha^2\eta(X)\xi, \tag{4.5}$$

$$\tilde{\tau} = \tau - 2\alpha^2 + 6\beta^2, \tag{4.6}$$

respectively, where $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W)$ and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

5. Main results

In this section, we give some geometric results on 3-dimensional trans-Sasakian manifolds with α and β are constants with respect to the Schouten-van Kampen connection.
The *Projective curvature tensor* is an important tensor from the differential geometric point of view. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the projective curvature tensor P vanishes. In fact M is projectively flat if and only if it is of constant curvature [2]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.
In a 3-dimensional trans-Sasakian manifold, the projective curvature tensor with respect to the Schouten-van Kampen connection is given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}. \tag{5.1}$$

If $\tilde{P} = 0$, then the manifold M is called *projectively flat* with respect to the Schouten-van Kampen connection.
Let M be projectively flat manifold with respect to the Schouten-van Kampen connection. From (5.1), we have

$$\tilde{R}(X,Y)Z = \frac{1}{2}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\},$$

i.e.

$$\tilde{R}(X,Y,Z,W) = \frac{1}{2}\{\tilde{S}(Y,Z)g(X,W) - \tilde{S}(X,Z)g(Y,W)\}. \tag{5.2}$$

Then using (4.3) and (4.4) in (5.2), we get

$$\begin{aligned} & R(X,Y,Z,W) + \beta^2\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ & +\alpha^2\{g(\phi Y,Z)g(\phi X,W) - g(\phi Y,W)g(\phi X,Z) + g(Y,W)\eta(X)\eta(Z) \\ & -g(X,W)\eta(Y)\eta(Z) - g(Y,Z)\eta(X)\eta(W) + g(X,Z)\eta(Y)\eta(W)\} \\ = & \frac{1}{2}\{[S(Y,Z) + 2\beta^2 g(Y,Z) - 2\alpha^2\eta(Y)\eta(Z)]g(X,W) \\ & -[S(X,Z) + 2\beta^2 g(X,Z) - 2\alpha^2\eta(X)\eta(Z)]g(Y,W)\}. \end{aligned} \tag{5.3}$$

Taking $W = \xi$ and using (2.15) in (5.3), we obtain

$$0 = S(Y,Z)\eta(X) + 2\beta^2 g(Y,Z)\eta(X) - 2\alpha^2\eta(Y)\eta(Z)\eta(X) - S(X,Z)\eta(Y) - 2\beta^2 g(X,Z)\eta(Y) + 2\alpha^2\eta(Y)\eta(Z)\eta(X),$$

i.e.

$$0 = \{S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + 2\beta^2 g(Y,Z)\eta(X) - 2\beta^2 g(X,Z)\eta(Y)\}. \tag{5.4}$$

Again taking $X = \xi$ in (5.4), we have

$$S(Y,Z) = S(\xi,Z)\eta(Y) - 2\beta^2 g(Y,Z) + 2\beta^2\eta(Y)\eta(Z). \tag{5.5}$$

Using (2.14) in (5.5), we obtain

$$S(Y,Z) = -2\beta^2 g(Y,Z) + 2\alpha^2\eta(Y)\eta(Z). \tag{5.6}$$

Now using (5.6) in (4.4), we get

$$\tilde{S}(Y,Z) = 0.$$

Thus the manifold M is the Ricci-flat with respect to the Schouten-van Kampen connection. From (5.2), we have

$$\tilde{R} = 0.$$

Now we can say the manifold M is flat with respect to the Schouten-van Kampen connection.
Conversely, if M is flat manifold with respect to the Schouten-van Kampen connection then M is the Ricci-flat with respect to the Schouten-van Kampen connection. From (5.1), M is projectively flat with respect to the Schouten-van Kampen connection.
Thus we have the following:

Theorem 5.1. *Let M be a 3-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection. Then the following statements are equivalent: i) M is projectively flat with respect to the Schouten-van Kampen connection, ii) M is the Ricci flat with respect to the Schouten-van Kampen connection, iii) M is flat with respect to the Schouten-van Kampen connection.*

In a 3-dimensional trans-Sasakian manifold *the conharmonic curvature tensor* with respect to the Schouten-van Kampen connection is given by

$$\tilde{K}(X,Y)Z = \tilde{R}(X,Y)Z - \{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y\}. \tag{5.7}$$

If $\tilde{K} = 0$, then the manifold M is called *conharmonically flat* manifold with respect to the Schouten-van Kampen connection. Then we have

$$\tilde{R}(X,Y)Z = \{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y\}. \tag{5.8}$$

Let M be conharmonically flat trans-Sasakian manifold with respect to the Schouten-van Kampen connection. Then using (4.3), (4.4) and (4.5) in (5.8), we get

$$\begin{aligned} & R(X,Y,Z,W)+\beta^2\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\ & +\alpha^2\{g(\phi Y,Z)g(\phi X,W)-g(\phi Y,W)g(\phi X,Z)+g(Y,W)\eta(X)\eta(Z) \\ & -g(X,W)\eta(Y)\eta(Z)-g(Y,Z)\eta(X)\eta(W)+g(X,Z)\eta(Y)\eta(W)\} \\ = & S(Y,Z)g(X,W)-S(X,Z)g(Y,W) \\ & +S(X,W)g(Y,Z)-S(Y,W)g(X,Z) \\ & +4\beta^2\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\ & -2\alpha^2\{g(X,W)\eta(Y)\eta(Z)-g(Y,W)\eta(X)\eta(Z) \\ & +g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)\}. \end{aligned} \tag{5.9}$$

Taking $W=\xi$ in (5.9), we obtain

$$\begin{aligned} & R(X,Y,Z,\xi)+(\beta^2-\alpha^2)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\} \\ = & S(Y,Z)\eta(X)-S(X,Z)\eta(Y)+g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi) \\ & +(4\beta^2-2\alpha^2)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}, \end{aligned} \tag{5.10}$$

i.e.

$$0=S(Y,Z)\eta(X)-S(X,Z)\eta(Y)+2\beta^2\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}. \tag{5.11}$$

Again taking $X=\xi$ and using (2.14) in (5.11), we have

$$S(Y,Z)=-4\beta^2 g(Y,Z)+2(\alpha^2+\beta^2)\eta(Y)\eta(Z). \tag{5.12}$$

Now using (5.12) in (4.4), we get

$$\tilde{S}(Y,Z)=0.$$

Thus the manifold M is the Ricci-flat with respect to the Schouten-van Kampen connection. From (5.8), we have

$$\tilde{R}=0.$$

Now we can say the manifold M is flat with respect to the Schouten-van Kampen connection.
Conversely, if M is flat manifold with respect to the Schouten-van Kampen connection then M is the Ricci-flat with respect to the Schouten-van Kampen connection. From (5.7), M is conharmonically flat with respect to the Schouten-van Kampen connection.
Thus we have the following:

Theorem 5.2. *Let M be a 3-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection. Then the following statements are equivalent: i) M is conharmonically flat with respect to the Schouten-van Kampen connection, ii) M is the Ricci flat with respect to the Schouten-van Kampen connection, iii) M is flat with respect to the Schouten-van Kampen connection.*

6. Example

We consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3, z\neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1=z\frac{\partial}{\partial x},\quad e_2=z\frac{\partial}{\partial y},\quad e_3=z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0, \\ g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$ for any $Z\in\chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1)=-e_2$, $\phi(e_2)=e_1$, $\phi(e_3)=0$. Then using linearity of ϕ and g we have

$$\eta(e_3)=1,\quad \phi^2 Z=-Z+\eta(Z)e_3,$$

$$g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W),$$

for any $Z,W\in\chi(M)$. Thus for $e_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1,e_2]=0,\quad [e_2,e_3]=-e_2,\quad [e_1,e_3]=-e_1.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula which is

$$2g(\nabla_X Y,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]). \tag{6.1}$$

Using (6.1), we have

$$\begin{aligned} 2g(\nabla_{e_1}e_3,e_1) &= 2g(-e_1,e_1), \\ 2g(\nabla_{e_1}e_3,e_2) &= 0=2g(-e_1,e_2), \\ 2g(\nabla_{e_1}e_3,e_3) &= 0=2g(-e_1,e_3). \end{aligned}$$

Hence $\nabla_{e_1}e_3=-e_1$. Similarly, $\nabla_{e_2}e_3=-e_2$ and $\nabla_{e_3}e_3=0$. (6.1) further yields

$$\begin{aligned} \nabla_{e_1}e_2 &= 0, \qquad \nabla_{e_1}e_1=e_3, \\ \nabla_{e_2}e_2 &= e_3, \qquad \nabla_{e_2}e_1=0, \\ \nabla_{e_3}e_2 &= 0, \qquad \nabla_{e_3}e_1=0. \end{aligned} \tag{6.2}$$

We see that

$$\begin{aligned} (\nabla_{e_1}\phi)e_1 &= \nabla_{e_1}\phi e_1-\phi\nabla_{e_1}e_1=-\nabla_{e_1}e_2-\phi e_3=-\nabla_{e_1}e_2=0 \\ &= 0(g(e_1,e_1)e_3-\eta(e_1)e_1)-1(g(\phi e_1,e_1)e_3-\eta(e_1)\phi e_1). \end{aligned} \tag{6.3}$$

$$\begin{aligned} (\nabla_{e_1}\phi)e_2 &= \nabla_{e_1}\phi e_2-\phi\nabla_{e_1}e_2=-\nabla_{e_1}e_1-0=e_3 \\ &= 0(g(e_1,e_2)e_3-\eta(e_2)e_1)-1(g(\phi e_1,e_2)e_3-\eta(e_2)\phi e_1). \end{aligned} \tag{6.4}$$

$$\begin{aligned} (\nabla_{e_1}\phi)e_3 &= \nabla_{e_1}\phi e_3-\phi\nabla_{e_1}e_3=0+\phi e_1=-e_2 \\ &= 0(g(e_1,e_3)e_3-\eta(e_3)e_1)-1(g(\phi e_1,e_3)e_3-\eta(e_3)\phi e_1). \end{aligned} \tag{6.5}$$

By (6.3), (6.4) and (6.5) we see that the manifold satisfies (2.5) for $X=e_1$, $\alpha=0$, $\beta=-1$, and $e_3=\xi$. Similarly, it can be shown that for $X=e_2$ and $X=e_3$ the manifold also satisfies (2.5) for $\alpha=0$, $\beta=-1$, and $e_3=\xi$. Hence the manifold is a trans-Sasakian manifold of type $(0,-1)$. Using (6.2), we get

$$\begin{aligned} R(e_1,e_2)e_1 &= e_2, \quad R(e_1,e_2)e_2=-e_1, \quad R(e_1,e_2)e_3=0, \\ R(e_1,e_3)e_1 &= e_3, \quad R(e_1,e_3)e_2=0, \quad R(e_1,e_3)e_3=-e_1, \\ R(e_2,e_3)e_1 &= 0, \quad R(e_2,e_3)e_2=e_3, \quad R(e_2,e_3)e_3=-e_2. \end{aligned} \tag{6.6}$$

Now we consider the Schouten-van Kampen connection to this example. Using (4.1) and (6.2), we calculate

$$\begin{aligned} \tilde{\nabla}_{e_1}e_1 &= (\beta+1)e_3, \quad \tilde{\nabla}_{e_1}e_2=\alpha e_3, \quad \tilde{\nabla}_{e_1}e_3=(\beta+1)e_1-\alpha e_2, \\ \tilde{\nabla}_{e_2}e_1 &= -\alpha e_3, \quad \tilde{\nabla}_{e_2}e_2=(\beta+1)e_3, \quad \tilde{\nabla}_{e_2}e_3=\alpha e_1-(\beta+1)e_2, \\ \tilde{\nabla}_{e_3}e_1 &= 0, \quad \tilde{\nabla}_{e_3}e_2=0, \quad \tilde{\nabla}_{e_3}e_3=0. \end{aligned} \tag{6.7}$$

Thus using (4.2) and (6.6), we get

$$\begin{aligned} \tilde{R}(e_1,e_2)e_1 &= (1-\alpha^2-\beta^2)e_2, \quad \tilde{R}(e_1,e_2)e_2=(-1+\alpha^2+\beta^2)e_1, \\ \tilde{R}(e_1,e_2)e_3 &= 0, \quad \tilde{R}(e_1,e_3)e_1=(1-\alpha^2-\beta^2)e_3, \\ \tilde{R}(e_1,e_3)e_2 &= 0, \quad \tilde{R}(e_1,e_3)e_3=(-1-\alpha^2+\beta^2)e_1, \\ \tilde{R}(e_2,e_3)e_1 &= 0, \quad \tilde{R}(e_2,e_3)e_2=(1-\alpha^2-\beta^2)e_3, \\ \tilde{R}(e_2,e_3)e_3 &= (-1-\alpha^2+\beta^2)e_2. \end{aligned} \tag{6.8}$$

From (6.7), we can see that $\tilde{\nabla}_{e_i}e_j=0$ $(1\leq i,j\leq 3)$ for $\xi=e_3$ and $\alpha=0$, $\beta=\mp 1$. Hence M is a 3-dimensional trans-Sasakian manifold of type $(0,-1)$ with respect to the Schouten-van Kampen connection. Also using (6.8), it can be seen that $\tilde{R}=0$. Thus the manifold M is flat manifold with respect to the Schouten-van Kampen connection. Since a flat manifold is the Ricci-flat manifold with respect to the Schouten-van Kampen connection, the manifold M is both projectively flat and conharmonically flat 3-dimensional β-Kenmotsu manifold with respect to the Schouten-van Kampen connection. So, from Theorem 5.1 and Theorem 5.2, the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection.

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