Konuralp Journal of Mathematics, 8 (1) (2020) 211-215

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Hardy Type Inequalities for Conformable Fractional Integrals

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Abstract

The main target addressed in this article are presenting Hardy type inequalities for Katugampola conformable fractional integral. In accordance with this purpose we try to use more general type of function in order to make a generalization. Thus our results cover the previous published studies for Hardy type inequalities.

Keywords: *Hardy inequality, Hölder's inequality, confromable fractional integrals.*

2010 Mathematics Subject Classification: *26D15, 26A51, 26A33, 26A42, 34A40.*

1. Introduction & Preliminaries

Classical Hardy's integral inequality, discovered in 1920 by G. H. Hardy [4]:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(s)ds\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty (f(x))^p dx \tag{1.1}$$

where $p>1$, $x>0$, f is a nonnegative measurable function on $(0,\infty)$ and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. This interesting result was later proved by Hardy himself in 1925 and 1928, respectively [5]and [6] like as the following integral inequalities:
Let f non-negative measurable function on $(0,\infty)$,
Classical Hardy's Inequality:

$$F(x)=\begin{cases} \int_0^x f(t)dt & for \quad m>1, \\ \int_x^\infty f(t)dt & for \quad m<1, \end{cases}$$

then

$$\int_0^\infty x^{-m}F^p(x)dx \le \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{-m}(xf(x))^p dx, \quad \text{for } p\ge 1. \tag{1.2}$$

Weighted Hardy's Inequality:

$$F(x)=\begin{cases} \int_0^x f(t)dt & \text{for} \quad r<p-1, \\ \int_x^\infty f(t)dt & \text{for} \quad r>p-1, \end{cases}$$

then

$$\int_0^\infty x^{r-p}F^p(x)dx \le \left(\frac{p}{|p-1-r|}\right)^p \int_0^\infty x^r f^p(x)dx, \quad \text{for } p\ge 1. \tag{1.3}$$

Hardy's type inequalities have been studied by a large number of authors during the twentieth century and has motivated some important lines of study which are currently active. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Hardy's inequality and its generalizations, see [7], [12], [13], [14]-[17], [19].

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The purpose of this paper is to establish some generalizations of Hardy type inequalities for conformable integral. The structure of this paper is as follows: Firstly, we give the definitions of the conformable derivatives and conformable integral and introduce several useful notations conformable integral used our main results. Later, the main results are presented.
In light of recent developments in mathematics, fractional calculus is becoming extremely popular in a lot of application areas such as control theory, computational analysis and engineering [11], see also [18]. Together with these developments a number of new definitions have been introduced in academia to provide the best method for fractional calculus. For instance in more recent times a new local, limit-based definition of a conformable derivative has been introduced in [1], [9], [10], with several follow-up papers [2], [3], [8]. In this study, we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$ and $t \in [0,\infty)$ given by

$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \; D_\alpha(f)(0) = \lim_{t \to 0} D_\alpha(f)(t), \tag{1.4}$$

provided the limits exist (for detail see, [9]). If f is fully differentiable at t, then

$$D_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t). \tag{1.5}$$

A function f is α-differentiable at a point $t \geq 0$ if the limit in (1.4) exists and is finite. This definition yields the following results;

Theorem 1.1. *Let $\alpha \in (0,1]$ and f,g be $\alpha-$differentiable at a point $t > 0$. Then,*
i. $D_\alpha(af+bg) = aD_\alpha(f) + bD_\alpha(g)$, for all $a,b \in \mathbb{R}$,
ii. $D_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$,
iii. $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$,
iv. $D_\alpha\left(\frac{f}{g}\right) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$
v. $D_\alpha(t^n) = nt^{n-\alpha}$ for all $n \in \mathbb{R}$
vi. $D_\alpha(f \circ g)(t) = f'(g(t))D_\alpha g(t)$ for f is differentiable at $g(t)$.

Definition 1.2 (Conformable fractional integral)**.** *Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\,d_\alpha x := \int_a^b f(x)x^{\alpha-1}dx$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L^1_\alpha([a,b])$. Also, if

$$\|f\|^p_\alpha = \int_a^b |f(x)|^p\,d_\alpha x < \infty$$

then $f \in L^p_\alpha([a,b])$.

Remark 1.3.

$$I^a_\alpha(f)(t) = I^a_1\left(t^{\alpha-1}f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

We will also use the following important results, which can be derived from the results above.

Lemma 1.4. *Let the conformable differential operator D^α be given as in (1.4), where $\alpha \in (0,1]$ and $t \geq 0$, and assume the functions f, w and g are α-differentiable as needed. Then,*
i. $D_\alpha(\ln t) = t^{-\alpha}$ for $t > 0$
ii. $D_\alpha\left[\int_a^t f(t,s)\,d_\alpha s\right] = f(t,t) + \int_a^t D_\alpha[f(t,s)]\,d_\alpha s$
iii. $\int_a^b w(x)D_\alpha g(x)\,d_\alpha x = wg|_a^b - \int_a^b g(x)D_\alpha w(x)\,d_\alpha x$.

Theorem 1.5 (Hölder's Inequality)**.** *Let $\alpha \in (0,1]$, w,g are α-fractional integrable on $[a,b]$ and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$\int_a^b |w(t)g(t)|\,d_\alpha t \leq \left(\int_a^b |w(t)|^p\,d_\alpha t\right)^{\frac{1}{p}}\left(\int_a^b |g(t)|^q\,d_\alpha t\right)^{\frac{1}{q}}.$$

Hardy type inequality can be represented for conformable fractional integral forms as follows [17]:

Theorem 1.6. *Let $\alpha \in (0,1]$, f be a nonnegative function on $(0,\infty)$, and $p > 1$. Also assume $x^{\alpha-1}f(x)$ is continuous on $[0,\infty)$. Then, we have the following inequality*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(s)d_\alpha s\right)^p d_\alpha x \leq \left(\frac{p}{p-\alpha}\right)^p \int_0^\infty (x^{\alpha-1}f(x))^p d_\alpha x.$$

Now, we present the main results:

2. Hardy type inequalities for conformable fractional integral

Now we prove the generalized Minkokski type integral inequality need in the next theorem:

Theorem 2.1. *Let $\alpha \in (0,1]$, $1 \le p < \infty$, $f : [a,b] \times [c,d] \to \mathbb{R}$ be α-fractional integrable function. Then, we have the following inequality*

$$\left[\int_a^b \left|\int_c^d f(x,y)\, d_\alpha y\right|^p d_\alpha x\right]^{\frac{1}{p}} \le \left[\int_c^d \left[\int_a^b |f(x,y)|^p\, d_\alpha x\right]^{\frac{1}{p}} d_\alpha y\right].$$

Proof. The case $p = 1$ follows from Fubini's Theorem, so we assume that $p > 1$ and we can write

$$\begin{aligned}
\int_a^b \left|\int_c^d f(x,y)\, d_\alpha y\right|^p d_\alpha x &= \int_a^b \left|\int_c^d f(x,y)\, d_\alpha y\right|^{p-1} \left|\int_c^d f(x,y)\, d_\alpha y\right| d_\alpha x \\
&\le \int_a^b \left|\int_c^d f(x,s)\, d_\alpha s\right|^{p-1} \left[\int_c^d |f(x,y)|\, d_\alpha y\right] d_\alpha x \\
&= \int_a^b \left\{\int_c^d \left|\int_c^d f(x,s)\, d_\alpha s\right|^{p-1} |f(x,y)|\, d_\alpha y\right\} d_\alpha x \\
&= \int_c^d \left\{\int_a^b \left|\int_c^d f(x,s)\, d_\alpha s\right|^{p-1} |f(x,y)|\, d_\alpha x\right\} d_\alpha y
\end{aligned}$$

the last step coming from Fubini's Theorem. By using Hölder's inequality to the inner integral with respect to x, we have

$$\int_a^b \left|\int_c^d f(x,y)\, d_\alpha y\right|^p d_\alpha x \le \int_c^d \left\{\left(\int_a^b \left|\int_c^d f(x,s)\, d_\alpha s\right|^{(p-1)q} d_\alpha x\right)^{\frac{1}{q}} \left(\int_a^b |f(x,y)|^p\, d_\alpha x\right)^{\frac{1}{p}}\right\} d_\alpha y$$

$$= \left(\int_a^b \left|\int_c^d f(x,s)\, d_\alpha s\right|^p d_\alpha x\right)^{\frac{1}{q}} \int_c^d \left\{\left(\int_a^b |f(x,y)|^p\, d_\alpha x\right)^{\frac{1}{p}}\right\} d_\alpha y.$$

Dividing both sides by the $\left(\int_a^b \left|\int_c^d f(x,s)\, d_\alpha s\right|^p d_\alpha x\right)^{\frac{1}{q}}$ and remembering that $\frac{1}{p} + \frac{1}{q} = 1$ which gives the required inequality. □

Theorem 2.2. *Let $\alpha \in (0,1]$, f be a nonnegative function on $(0,\infty)$, and $p > 1$. Also assume $\int_0^\infty (x^{\alpha-1} f(x))^p d_\alpha x$ is convergent. Then, we have the following inequality*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(s) d_\alpha s\right)^p d_\alpha x \le \left(\frac{p}{p-\alpha}\right)^p \int_0^\infty (x^{\alpha-1} f(x))^p d_\alpha x.$$

Proof. Changing the variable $(s = xt)$ we get

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(s) d_\alpha s\right)^p d_\alpha x\right)^{\frac{1}{p}} = \left(\int_0^\infty \left(\int_0^1 f(xt) x^{\alpha-1} d_\alpha t\right)^p d_\alpha x\right)^{\frac{1}{p}}.$$

Using generalized Minkowski's integral inequality and changing again the variable $u = xt$, we conclude

$$\begin{aligned}
\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(s) d_\alpha s\right)^p d_\alpha x\right)^{\frac{1}{p}} &\le \int_0^1 \left(\left(\int_0^\infty [f(xt)]^p x^{(\alpha-1)p} d_\alpha x\right)^{\frac{1}{p}} d_\alpha t\right) \\
&= \int_0^1 \left(\left(\int_0^\infty [f(u)]^p u^{(\alpha-1)p} t^{p-\alpha(p+1)} d_\alpha u\right)^{\frac{1}{p}} d_\alpha t\right) \\
&= \left(\int_0^\infty f^p(u) u^{(\alpha-1)p} d_\alpha u\right)^{\frac{1}{p}} \int_0^1 \left(t^{1-\alpha\frac{(p+1)}{p}} d_\alpha t\right) \\
&= \frac{p}{p-\alpha} \left(\int_0^\infty f^p(u) u^{(\alpha-1)p} d_\alpha u\right)^{\frac{1}{p}}.
\end{aligned}$$

This completes the proof. □

Remark 2.3. *If we put $\alpha = 1$ in Theorem 2.2, we get inequality (1.1) for $p > 1$ and $F(x) = \int_0^x f(t)dt$.*

The following theorem generalizes the Hardy's type integral inequality by introducing power weights x^r.

Theorem 2.4. *Let $\alpha \in (0,1]$, f be a nonnegative function on $(0,\infty)$, $p \ge 1$ and $r < p - \alpha$. Also assume $\int_0^\infty (x^{\alpha-1} f(x))^p x^r d_\alpha x$ is convergent. Then, we have the following inequality*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(s) d_\alpha s\right)^p x^r d_\alpha x \le \left(\frac{p}{p-r-\alpha}\right)^p \int_0^\infty (x^{\alpha-1} f(x))^p x^r d_\alpha x. \tag{2.1}$$

Proof. Changing the variable $(s = xt)$ we get

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(s)d_\alpha s\right)^p x^r d_\alpha x\right)^{\frac{1}{p}} = \left(\int_0^\infty \left(\int_0^1 f(xt)x^{\alpha-1}x^{\frac{r}{p}}d_\alpha t\right)^p d_\alpha x\right)^{\frac{1}{p}}.$$

Using generalized Minkowski's integral inequality and changing again the variable $u = xt$, we conclude

$$\begin{aligned}\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(s)d_\alpha s\right)^p x^r d_\alpha x\right)^{\frac{1}{p}} &\le \int_0^1 \left(\left(\int_0^\infty [f(xt)]^p x^{(\alpha-1)p+r} d_\alpha x\right)^{\frac{1}{p}} d_\alpha t\right)\\ &= \int_0^1 \left(\left(\int_0^\infty [f(u)]^p u^{(\alpha-1)p+r} t^{p-r-\alpha(p+1)} d_\alpha u\right)^{\frac{1}{p}} d_\alpha t\right)\\ &= \left(\int_0^\infty f^p(u)u^{(\alpha-1)p+r}d_\alpha u\right)^{\frac{1}{p}} \int_0^1 \left(t^{1-\frac{r}{p}-\alpha\frac{(p+1)}{p}} d_\alpha t\right)\\ &= \frac{p}{p-r-\alpha}\left(\int_0^\infty f^p(u)u^{(\alpha-1)p+r}d_\alpha u\right)^{\frac{1}{p}}.\end{aligned}$$

This completes the proof of the inequality (2.1). □

Remark 2.5. *If we put $\alpha = 1$ in Theorem 2.4, we get inequality (1.3) for $r < p-1$ and $F(x) = \int_0^x f(t)dt$.*

Theorem 2.6. *Let $\alpha \in (0,1]$, f be a nonnegative function on $(0,\infty)$, r be absolutely continuous function on $(0,\infty)$, and $p > 1$. Also assume $\int_0^\infty (f(x))^p d_\alpha x$ is convergent, and*

$$\frac{p-\alpha}{p} + \frac{xD_\alpha r(x)}{r(x)} \ge \frac{1}{\lambda} \tag{2.2}$$

for almost every $x > 0$ and for some $\lambda > 0$. Then, we have the following inequality

$$\int_0^\infty (H_r f(x))^p d_\alpha x \le \lambda^p \int_0^\infty (f(x))^p d_\alpha x$$

where

$$H_r f(x) = \frac{1}{xr(x)}\int_0^x f(t)r(t)d_\alpha t.$$

Proof. We consider $0 < a < b < \infty$ and

$$h_{r,a}f(x) = \frac{1}{r(x)}\int_a^x f(t)r(t)d_\alpha t.$$

Then, defining $H_{r,a}f(x) = \frac{1}{x}h_{r,a}f(x)$, and integrating by parts, see formula iii) in the Lemma 1.4 with $w = (h_{r,a}f(x))^p$ and $D_\alpha(g)(x) = x^{-p}$ note that $g(x) = \frac{x^{\alpha-p}}{\alpha-p}$, and we get

$$\begin{aligned}\int_a^b (H_{r,a}f(x))^p d_\alpha x &= (h_{r,a}f(x))^p \frac{x^{\alpha-p}}{\alpha-p}\Big|_a^b - \frac{p}{\alpha-p}\int_a^b x^{1-p}(h_{r,a}f(x))^{p-1}(h_{r,a}f(x))' d_\alpha x\\ &= -(h_{r,a}f(b))^p \frac{b^{\alpha-p}}{p-\alpha} + \frac{p}{p-\alpha}\int_a^b x^{1-p}(h_{r,a}f(x))^{p-1}(h_{r,a}f(x))' d_\alpha x.\end{aligned}$$

We notice that $-(h_{r,a}f(b))^p \frac{b^{\alpha-p}}{p-\alpha}$ is negative, since $p-\alpha > 0$, $h_{r,a}f(b) > 0$ and $b > 0$. Also, from definition of $h_{r,a}f(x)$ we have

$$(h_{r,a}f(x))' = -x^{\alpha-1}\frac{D_\alpha r(x)}{r(x)}h_{r,a}f(x) + f(x).$$

Hence,

$$\begin{aligned}\frac{p-\alpha}{p}\int_a^b (H_{r,a}f(x))^p d_\alpha x &\le \int_a^b x^{1-p}(h_{r,a}f(x))^{p-1}\left[-x^{\alpha-1}\frac{D_\alpha r(x)}{r(x)}h_{r,a}f(x) + f(x)\right]d_\alpha x\\ &= -\int_a^b x^{\alpha-p}\frac{D_\alpha r(x)}{r(x)}(h_{r,a}f(x))^p d_\alpha x + \int_a^b x^{1-p}(h_{r,a}f(x))^{p-1}f(x)d_\alpha x,\end{aligned}$$

or equivalently,

$$\int_a^b \left[\frac{p-\alpha}{p} + x^{\alpha-1}\frac{D_\alpha r(x)}{r(x)}\right](H_{r,a}f(x))^p d_\alpha x \le \int_a^b (H_{r,a}f(x))^{p-1}f(x)d_\alpha x.$$

Now, using (2.2) and Hölder's inequality, we have

$$\frac{1}{\lambda}\int_a^b (H_{r,a}f(x))^p\, d_\alpha x \le \left(\int_a^b f^p(x)\, d_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b (H_{r,a}f(x))^{(p-1)q}\, d_\alpha x\right)^{\frac{1}{q}} = \left(\int_a^b f^p(x)\, d_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b (H_{r,a}f(x))^p\, d_\alpha x\right)^{\frac{1}{q}},$$

that is,

$$\int_a^b (H_{r,a}f(x))^p\, d_\alpha x \le \lambda^p \int_0^\infty f^p(x)\, d_\alpha x.$$

If we take $c > a$, then

$$\int_c^b (H_{r,a}f(x))^p\, d_\alpha x \le \int_a^b (H_{r,a}f(x))^p\, d_\alpha x \le \lambda^p \int_0^\infty f^p(x)\, d_\alpha x,$$

and by the Dominated Convergence Theorem, making $a \to \infty$, we get

$$\int_c^b (H_r f(x))^p\, d_\alpha x \le \lambda^p \int_0^\infty f^p(x)\, d_\alpha x$$

for all $c, b > 0$. Finally, letting $b \to \infty$ and $c \to 0$,

$$\int_0^\infty (H_r f(x))^p\, d_\alpha x \le \lambda^p \int_0^\infty f^p(x)\, d_\alpha x.$$

□

Corollary 2.7. *Let $\alpha \in (0,1]$, f be a nonnegative function on $(0,\infty)$, u be absolutely continuous function on $(0,\infty)$, and $p > 1$. Also assume $\int_0^\infty (f(x))^p d_\alpha x$ is convergent, and*

$$\frac{p-\alpha}{p} - \frac{1}{p}\frac{xD_\alpha u(x)}{u(x)} \ge \frac{1}{\lambda} \tag{2.3}$$

for almost every $x > 0$ and for some $\lambda > 0$. Then, we have the following inequality

$$\int_0^\infty (Hf(x))^p u(x) d_\alpha x \le \lambda^p \int_0^\infty (f(x))^p u(x) d_\alpha x \tag{2.4}$$

where

$$Hf(x) = \frac{1}{x}\int_0^x f(t) d_\alpha t.$$

Proof. If we consider $r(x) = \left(\frac{1}{u(x)}\right)^{\frac{1}{p}}$, then (2.3) becomes

$$\frac{p-\alpha}{p} + \frac{xD_\alpha r(x)}{r(x)} \ge \frac{1}{\lambda},$$

since

$$\frac{D_\alpha r(x)}{r(x)} = -\frac{1}{p}\frac{D_\alpha u(x)}{u(x)}.$$

Now, we express $f(x) = r(x)g(x)$ for the suitable $g(x) \ge 0$ and we apply Theorem 2.6 to g, the inequality (2.4) can be obtained. □

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