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Fixed point solution of system of inclusion problems for demicontractive operators in certain Banach spaces with application

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Abstract

In this paper, we introduce a new iterative approach for finding a common element of the set of fixed points of a demicontractive mapping and the set of solutions of system of inclusion problems with an infinite family of accretive operators in certain Banach spaces. Then, strong convergence of the scheme to a common element of the two sets is proved. Moreover, application to convex minimization problems involving an infinite family of lower semi-continuous and convex functions are included.The main theorems develop and complement the recent results announced by researchers in this area.

Keywords: Proximal point algorithm; Demicontractive operators; Accretive operators; Inclusion problems.
2010 MSC: 47H10; 47J25; 49M05, 65J15.

1. Introduction

Let E be a real Banach space and K be a nonempty, closed and convex subset of E. We denote by J the normalized duality map from E to 2^{E^*} (E^* is the dual space of E) defined by:

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \forall\, x \in E.$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{ f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\}$$

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Notice that for $x \neq 0$,

$$J_q(x) = \|x\|^{q-2} J_2(x), \; q > 1.$$

Fo more details on duality maps and geometric properties of Banach spaces, the reader can consult [6, 9, 32, 10, 1, 27].

Recall that a multivalued operator $A : E \to 2^E$ with the domain $D(A)$ and the range $R(A)$ in E is accretive if, for each $x, y \in D(A)$ there exists a $j \in J(x-y)$ such that $\langle u - v, j \rangle \geq 0, \; (x, u), \; (y, v) \in G(A)$, where $G(A) := \{(x, u) : \; x \in D(A), \; u \in Ax\}$. In a Hilbert space, an accretive operator is also called monotone operator.

Many problems arising in different areas of mathematics, such as optimization, variational analysis and differential equations, can be modeled by the following inclusion problem:

$$0 \in Ax, \tag{1}$$

where A is a set-valued accretive operator. The solution set of (1) coincides to a null points set of A. In order to find a solution of problem (1), Rockafellar [24] introduced a powerful and successful algorithm which is recognized as Rockafellar proximal point algorithm: for any initial point $x_0 \in E$, a sequence $\{x_n\}$ is generated by:

$$x_{n+1} = J_{r_n}(x_n + e_n), \; \forall \; n \geq 0,$$

where $J_r = (I + rA)^{-1}$ for all $r > 0$, is the resolvent of A and $\{e_n\}$ is an error sequence in a Hilbert space. In the recent years, inclusion problem (1) in real Hilbert spaces, Banach spaces and complete CAT(0) (Hadamard) spaces have been intensively studied by many authors; see, for example, [8, 11, 16, 20, 29, 30, 3, 23, 15, 2] and the references therein.

Let K be a nonempty subset of E and we denote by $Fix(T)$ the set of fixed points of the mapping $T : K \to K$ that is, $Fix(T) := \{x \in D(T) : x = Tx\}$. Let $D(T) \subset K$, then T is said to be

(1) a contraction if there exists $b \in [0, 1)$ such that:

$$\|Tx - Ty\| \leq b\|x - y\| \;\; x, y \in D(T).$$

If $b = 1$, T is called nonexpansive;
(2) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \;\; x \in D(T), \;\; p \in Fix(T);$$

(3) k-strictly pseudo-contractive if there exists $j(x-y) \in J(x-y)$ and a constant $k \in (0, 1)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x - y\|^2 - k\|(I - T)x - (I - T)y\|^2, \;\; x, y \in D(T);$$

(4) k-demicontractive if there exists $j_q(x-y) \in J_q(x-y)$ a constant $k \in (0, 1)$ such that

$$\langle x - Tx, j_q(x-p) \rangle \geq \frac{(1-k)^{q-1}}{2^{q-1}} \|x - Tx\|^q, \;\; x \in D(T), \;\; p \in Fix(T), \;\; q > 1.$$

Clearly, the class of nonexpansive mappings (with nonempty fixed points set) is contained in the class of quasi-nonexpansive mappings, while the class of demicontractive mappings contains both the classes of nonexpansive and quasi-nonexpansive mappings. Moreover, there are several examples in the literature which, show that the above inclusions are proper,(see, [13]). Finding the fixed points of nonlinear operators is an important topic in mathematics, due to the fact that many nonlinear problems can be reformulated as fixed point equations of nonlinear mappings. One of efficient methods to solve fixed point problem involving nonlinear mappings is the iterative method. Constructed iteration approaches to find fixed points of nonlinear mappings have received vast investigation, (see, e.g., Yao et al.[33], Chidume [6], Marino et al. [19], Moudafi [21], Halpern [25], Sow et al. [28], Goebel et al. [9] and the references therein).

Recently, many authors studied the following convex feasibility problem (for short, CFP):

$$\textit{finding} \quad \textit{an } x^* \in \bigcap_{i=1}^{m} K_i, \tag{2}$$

where $m \geq 1$ is an integer and each K_i is a nonempty closed convex subset of H. There is a considerable investigation on the CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [5], computer tomography and radiation therapy treatment planning. In this paper, we shall consider the case when K_i is the solution set of an infinite family of inclusion problems and fixed point problems involving nonlinear mapping demicontractive in real Banach spaces. In 2016, J. S. Jung [26], studied the convex feasibility problem (2) (where $K = Fix(S) \cap A^{-1}(0)$) and $i = 1$). by considering a multivalued accretive operator $A : E \to 2^E$ and a strict pseudocontraction mapping S. He established a strong convergence theorem which extends the corresponding results in [3, 24, 17].

Theorem 1.1. *[26] Let E be a real uniformly convex Banach space having a weakly continuous duality mapping J_φ with gauge function ϕ, let C be a nonempty closed convex subset of E, let $A \subset E \times E$ be an accretive operator in E such that $A^{-1}(0) \neq \emptyset$ and $\overline{D(A)} \subset C \subset R(I + rA)$ for all $r > 0$, and let J_{r_n} be the resolvent of A for each $r_n > 0$. Let $r > 0$ be any given positive number, and let $S : C \to C$ be a nonexpansive mapping with $Fix(S) \cap A^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\} \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the conditions:*

$(C1)$ $\lim\limits_{n\to\infty} \alpha_n = 0$;

$(C2)$ $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;

$(C3)$ $\sum\limits_{i=0}^{\infty} |\alpha_n - \alpha_{n+1}| \leq o(\alpha_{n+1}) + \sigma_n, \ \sum\limits_{i=0}^{\infty} \sigma_n < \infty$;

$(C4)$ $\lim\limits_{n\to\infty} r_n = r$ *and* $r_n \geq \epsilon > 0$ *for* $n \geq 0$ *and* $\sum\limits_{i=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Let $f : C \to C$ be a contractive mapping with a constant $k \in (0, 1)$ and $x_0 = x \in C$ be chosen arbitrarily. Let $\{x_n\}$ be a sequence generated by:

$$x_{n+1} = J_{r_n}\big(\alpha_n f(x_n) + (1 - \alpha_n)Sx_n\big), \ n \geq 0. \tag{3}$$

Then $\{x_n\}$ converges strongly to $q \in Fix(S) \cap A^{-1}(0)$.

In this paper, motivated by above results, we construct a new algorithm for finding a common point of the set of fixed point of demicontractive operator and the set of common zero of an infinite family of accretive operators which is also a solution of some variational inequality problems in certain real Banach spaces.Applications are also investigated in Hilbert spaces.

2. Preliminairies

The demiclosedness of a nonlinear operator T usually plays an important role in dealing with the convergence of fixed point iterative algorithms.

Definition 2.1. *Let K be a nonempty, closed convex subset of a real Hilbert space H and let $T : K \to K$ be a single-valued mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\|$ converges to zero, then $p \in Fix(T)$.*

Lemma 2.2. *[9] Let E be a real Banach space satisfying Opial's property, K be a closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then $I - T$ is demiclosed*

Lemma 2.3 ([6])**.** *Let E be a smooth real Banach space E. Then, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \ \forall x, y \in E.$$

Lemma 2.4 (Xu, [32])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that (a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \leq 0$ or $\sum_{n=0}^{\infty} |\sigma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.5. *[18] Let t_n be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence t_{n_i} of t_n such that t_{n_i} such that $t_{n_i} \leq t_{n_{i+1}}$ for all $i \geq 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leq n : \ t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \ t_n\} \leq t_{\tau(n)+1}.$$

Theorem 2.6. *[6] Let $q > 1$ be a fixed real number and E be a smooth Banach space. Then the following statements are equivalent:*
(i) E is q-uniformly smooth.
(ii) There is a constant $d_q > 0$ such that for all $x, y \in E$

$$\|x + y\|^q \leq \|x\|^q + q\langle y \ , \ J_q(x)\rangle + d_q\|y\|^q.$$

(iii) There is a constant $c_1 > 0$ such that

$$\langle x - y \ , \ J_q(x) - J_q(y)\rangle \ \leq c_1\|x - y\|^q \quad \forall \ \ x, y \in E.$$

Lemma 2.7. *[19] Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ be a mapping.*
(i) If T is a k-strictly pseudo-contractive mapping, then T satisfies the Lipschitzian condition

$$\|Tx - Ty\| \leq \frac{1 + k}{1 - k}\|x - y\|.$$

(ii) If T is a k-strictly pseudo-contractive mapping, then the mapping $I - T$ is demiclosed at 0.

Lemma 2.8 (Chang et al. [4])**.** *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B(0)_r := \{x \in E : ||x|| \leq r\}$, a closed ball with center 0 and radius $r > 0$. For any given sequence $\{u_1, u_2, ..., u_n, ...\} \subset B(0)_r$ and any positive real numbers $\{\lambda_1, \lambda_2, ..., \lambda_n, ...\}$ with $\sum_{k=1}^{\infty} \lambda_k = 1$, then there exists a continuous, strictly increasing and convex function*

$$g : [0, 2r] \to \mathbb{R}^+, \ g(0) = 0,$$

such that for any integer i, j with $i < j$,

$$\|\sum_{k=1}^{\infty} \lambda_k u_k\|^2 \leq \sum_{k=1}^{\infty} \lambda_k\|u_k\|^2 - \lambda_i\lambda_j g(\|u_i - u_j\|).$$

The resolvent operator has the following properties:

Lemma 2.9. *[10] For any $r > 0$.*
(i) A is accretive if and only if the resolvent J_r^A of A is single-valued and nonexpansive;
(ii) A is m-accretive if and only if J_r^A of A is single-valued and nonexpansive and its domain is the entire E;
(iii) $0 \in A(x^*)$ if and only if $x^* \in Fix(J_r^A)$, where $Fix(J_r^A)$ denotes the fixed-point set of J_r^A.

Lemma 2.10. *(Miyadera [23]) For any $r > 0$ and $\mu > 0$, the following holds:*

$$\frac{\mu}{r}x + (1 - \frac{\mu}{r})J_r^A x \in D(J_r^A)$$

and

$$J_r^A x = J_\mu^A x(\frac{\mu}{r}x + (1 - \frac{\mu}{r})J_r^A x).$$

Lemma 2.11. *[25] Let C and D be nonempty subsets of a smooth real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D z, J(y - Q_D z)\rangle \le 0 \tag{4}$$

for all $z \in C$ and $y \in D$.

3. Main results

We are now in a position to state and prove our main result.

Theorem 3.1. *Let E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map and K be a nonempty, closed and convex subset of E. Let $B_i : E \to 2^E$, $i \in \mathbb{N}^*$ be an infinite family of multivalued accretive operators in E and $f : K \to K$ be an b-contraction mapping. Let $T : K \to K$ be a k-demicontractive operator such that $I - T$ is demiclosed at the origin, $\Gamma := \bigcap_{i=1}^{\infty} {B_i}^{-1}(0) \cap Fix(T) \neq \emptyset$ and $\bigcap_{i=1}^{\infty} \overline{D(B_i)} \subset K \subset \bigcap_{i=1}^{\infty} R(I + rB_i)$, for all $r > 0$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} z_n = (1-\theta_n)x_n + \theta_n T x_n, \\ \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big(\beta_{n,0} z_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} z_n\big), \end{cases} \tag{5}$$

$\theta_n \in [a, b] \subset (0,\ \gamma)$ where

$$\gamma := \min\Big\{1, \Big(\frac{q\omega^{q-1}}{d_q}\Big)^{\frac{1}{q-1}}\Big\}, \ \text{with } \omega = \frac{1-k}{2}.$$

Let $\{r_n\} \subset]0, \infty[$, $\{\beta_{n,i}\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{\infty} \beta_{n,i} = 1$,
(iii) $\lim_{n\to\infty} \inf r_n > 0$, and $\lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,i} > 0$, for all $i \in \mathbb{N}$.
Then, the sequence $\{x_n\}$ generated by (5) converges strongly to $\sigma \in \Gamma$, which solves the following variational inequality:

$$\langle \sigma - f(\sigma), J(\sigma - p)\rangle \le 0, \quad \forall p \in \Gamma. \tag{6}$$

Proof. For each $n \geq 0$, we put $y_n := \beta_{n,0} z_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} z_n$. Let $p \in \Gamma$. Using (5), inequality (ii) of Theorem 2.6 and property of T, we have

$$\begin{aligned} \|z_n - p\|^q &= \Big\|(1-\theta_n)(x_n - p) + \theta_n(Tx_n - p)\Big\|^q \\ &= \Big\|(1-\theta_n)(x_n - p) + \theta_n(Tx_n - x_n) + \theta_n(x_n - p)\Big\|^q \\ &= \Big\|x_n - p + \theta_n(Tx_n - x_n)\Big\|^q \\ &\leq \|x_n - p\|^q - q\theta_n\langle x_n - Tx_n, J_q(x_n - p)\rangle + d_q\Big\|\theta_n(Tx_n - x_n)\Big\|^q \\ &\leq \|x_n - p\|^q - q\theta_n\omega^{q-1}\|x_n - Tx_n\|^q + d_q\Big\|\theta_n(Tx_n - x_n)\Big\|^q. \end{aligned} \tag{7}$$

By inequality (7), it then follows that :

$$\begin{aligned} \Big\|z_n - p\Big\|^q &\leq \Big\|x_n - p\Big\|^q - q\theta_n\omega^{q-1}\Big\|x_n - Tx_n\Big\|^q + d_q\theta_n^q\Big\|Tx_n - x_n\Big\|^q. \\ &= \Big\|x_n - p\Big\|^q - \theta_n\Big[q\omega^{q-1} - d_q\theta_n^{q-1}\Big]\Big\|x_n - Tx_n\Big\|^q. \end{aligned} \tag{8}$$

Since $q\omega^{q-1} - d_q\theta_n^{q-1} > 0$, we obtain

$$\|z_n - p\Big\| \leq \|x_n - p\Big\|. \tag{9}$$

Let $p \in \Gamma$, we have

$$\begin{aligned} \|y_n - p\| &= \|\beta_{n,0} z_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} z_n - p\| \\ &\leq \beta_{n,0}\|z_n - p\| + \sum_{i=1}^{\infty} \beta_{n,i}\|J_{r_n}^{B_i} z_n - p\| \\ &\leq \|z_n - p\|. \end{aligned}$$

By using inequality (9), we obtain

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{10}$$

From (5) and inequality (10), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)\big(\beta_{n,0} z_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} z_n\big) - p\| \\ &\leq \alpha_n\|f(x_n) - f(p)\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\|f(p) - p\| \\ &\leq (1 - \alpha_n(1-b))\|x_n - p\| + \alpha_n\|\gamma f(p) - p\| \\ &\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-b}\}. \end{aligned}$$

By induction on n, we obtain

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-b}\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded. By using property of γ and inequality (8), we get

$$\begin{aligned}
\|x_{n+1}-p\|^q &= \|\alpha_n f(x_n)+(1-\alpha_n)\big(\beta_{n,0}z_n+\sum_{i=1}^{\infty}\beta_{n,i}J_{r_n}^{B_i}z_n\big)-p\|^q \\
&\leq \|y_n-p\|^q - q\alpha_n\langle y_n-f(x_n),J_q(y_n-p)\rangle + d_q\Big\|\alpha_n f(x_n)+\alpha_n y_n\Big\|^q \\
&\leq \|z_n-p\|^q + q\alpha_n\|y_n-f(x_n)\|\|y_n-p\|^{q-1} + d_q\Big\|\alpha_n f(x_n)-\alpha_n y_n\Big\|^q \\
&\leq \Big\|x_n-p\Big\|^q - \theta_n\Big[q\omega^{q-1}-d_q\theta_n^{q-1}\Big]\Big\|x_n-Tx_n\Big\|^q + q\alpha_n\|y_n-f(x_n)\|\|y_n-p\|^{q-1} \\
&\quad + d_q\Big\|\alpha_n f(x_n)-\alpha_n y_n\Big\|^q.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\theta_n\Big[q\omega^{q-1}-d_q\theta_n^{q-1}\Big]\|x_n-Tx_n\|^q &\leq \|x_n-p\|^q-\|x_{n+1}-p\|^q+q\alpha_n\|y_n-f(x_n)\|\|y_n-p\|^{q-1} \\
&\quad + d_q\|\alpha_n f(x_n)-\alpha_n y_n|^q.
\end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then there exists a constant $B>0$ such that

$$\theta_n\Big[q\omega^{q-1}-d_q\theta_n^{q-1}\Big]\|x_n-Tx_n\|^q \leq \|x_n-p\|^q-\|x_{n+1}-p\|^q+\alpha_n B. \tag{11}$$

We prove that $\{x_n\}$ converges strongly to σ. Now we divide the rest of the proof into two cases.

Case 1.Assume that there is $n_0\in\mathbb{N}$ such that $\|x_n-p\|$ is decreasing from n_0. Since $\|x_n-p\|$ is bounded, there it is convergent. Then, we have Clearly, we have

$$\lim_{n\to\infty}\Big[\|x_n-p\|-\|x_{n+1}-p\|\Big]=0. \tag{12}$$

It then implies from (11) that

$$\lim_{n\to\infty}\theta_n\Big[q\omega^{q-1}-qd_q\theta_n^{q-1}\Big]\Big\|x_n-Tx_n\Big\|^q=0. \tag{13}$$

Since $q\omega^{q-1}-d_q\theta_n^{q-1}>0$, we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{14}$$

Observing that,

$$\begin{aligned}
\|z_n-x_n\| &= \|(1-\theta_n)x_n+\theta_n Tx_n-x_n\| \\
&= \|(1-\theta_n)x_n+\theta_n Tx_n-(1-\theta_n)x_n-\theta_n\| \\
&= \theta_n\|Tx_n-x_n\|.
\end{aligned}$$

Therefore, from (14) we get that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{15}$$

Let $VI(f,\Gamma)$ be the solutions set of variational inequality problem (6). Using $(I-f)$ is strongly accretive and Γ is closed convex, then $VI(f,\Gamma)$ has only one element. In what follows, we σ to be the unique solution of $VI(f,\Gamma)$.

Next we prove that $\limsup_{n\to+\infty}\langle\sigma-f(\sigma),J(\sigma-x_n)\rangle\leq 0$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle\sigma-f(\sigma),J(\sigma-x_n)\rangle=\lim_{j\to+\infty}\langle\sigma-f(\sigma),J(\sigma-x_{n_j})\rangle.$$

From (14) and $I-T$ is demiclosed, we obtain $a \in Fix(T)$. Let $k \in \mathbb{N}^*$, from Lemma 2.8, the fact that $J_{r_n}^{B_i}$ is nonexpansive and (10), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_{n,0} z_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} z_n - p\|^2 \\
&\leq \beta_{n,0}\|z_n - p\|^2 + \sum_{i=1}^{\infty} \beta_{n,i}\|J_{r_n}^{B_i} z_n - p\|^2 - \beta_{n,0}\beta_{n,k} g(\|J_{r_n}^{B_k} z_n - z_n\|) \\
&\leq \|x_n - p\|^2 - \beta_{n,0}\beta_{n,k} g(\|J_{r_n}^{B_k} z_n - z_n\|).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n) y_n - p\|^2 \\
&\leq \|\alpha_n (f(x_n) - p) + (1-\alpha_n)(y_n - p)\|^2 \\
&\leq \alpha_n^2 \|f(x_n) - p\|^2 + (1-\alpha_n)^2 \|y_n - p\|^2 + 2\alpha_n(1-\alpha_n)\|f(x_n) - p\|\|y_n - p\| \\
&\leq \alpha_n^2 \|f(x_n) - p\|^2 + (1-\alpha_n)^2 \|x_n - p\|^2 - (1-\alpha_n)^2 \beta_{n,0}\beta_{n,k} g(\|J_{r_n}^{B_k} z_n - z_n\|) \\
&\quad +2\alpha_n(1-\alpha_n)\|f(x_n) - p\|\|x_n - p\|.
\end{aligned}$$

Thus, for every $k \in \mathbb{N}^*$, we get

$$\begin{aligned}
(1-\alpha_n)^2 \beta_{n,0}\beta_{n,k} g(\|J_{r_n}^{B_k} z_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2 \|f(x_n) - p\|^2 \\
+2\alpha_n(1-\alpha_n)\|f(x_n) - p\|\|x_n - p\|.
\end{aligned} \tag{16}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, then there exists a constant $C > 0$ sucht that

$$(1-\alpha_n)^2 \beta_{n,0}\beta_{n,k} g(\|J_{r_n}^{B_k} z_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n C. \tag{17}$$

It then implies from (17) and (12), that

$$\lim_{n\to\infty} \beta_{n,0}\beta_{n,k} g(\|J_{r_n}^{B_k} z_n - z_n\|) = 0. \tag{18}$$

Since $\lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,k} > 0$ and property of g, we have

$$\lim_{n\to\infty} \|z_n - J_{r_n}^{B_k} z_n\| = 0. \tag{19}$$

By using the resolvent identity (Lemma 2.10), for any $r > 0$, we conclude that

$$\begin{aligned}
\|z_n - J_r^{B_k} z_n\| &\leq \|z_n - J_{r_n}^{B_k} z_n\| + \|J_{r_n}^{B_k} z_n - J_r^{B_k} z_n\| \\
&\leq \|z_n - J_{r_n}^{B_k} z_n\| + \|J_r^{B_k}\left(\frac{r}{r_n} z_n + (1-\frac{r}{r_n}) J_{r_n}^{B_k} z_n\right) - J_r^{B_k} z_n\| \\
&\leq \|z_n - J_{r_n}^{B_k} z_n\| + \|\frac{r}{r_n} z_n + \left(1-\frac{r}{r_n}\right) J_{r_n}^{B_k} z_n - z_n\| \\
&\leq \|z_n - J_{r_n}^{B_k} z_n\| + |1-\frac{r}{r_n}|\, \|J_{r_n}^{B_k} z_n - z_n\| \to 0, \;\; n\to\infty, \;\; \forall k \in \mathbb{N}^*.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|z_n - J_r^{B_k} z_n\| = 0. \tag{20}$$

Since $z_{n_j} \rightharpoonup a$, it follows from (20) and Lemma 2.2, we have $a \in \bigcap_{k=1}^{\infty} {B_k}^{-1}(0)$. Therefore, $a \in \Gamma$. On other hand, the assumption that the duality mapping J is weakly continuous and $\sigma \in VI(f,\Gamma)$, we then have

$$\begin{aligned}
\limsup_{n\to+\infty} \langle \sigma - f(\sigma), J(\sigma - x_n)\rangle &= \lim_{j\to+\infty} \langle \sigma - f(\phi), J(\sigma - x_{n_j})\rangle \\
&= \langle \sigma - f(\sigma), J(\sigma - a)\rangle \leq 0.
\end{aligned}$$

Finally, we show that $x_n \to \sigma$. From (5) and Lemma 2.3, we get that

$$\begin{aligned}
\|x_{n+1}-\sigma\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)\big(\beta_{n,0}z_n + \sum_{i=1}^{\infty}\beta_{n,i}J_{r_n}^{B_i}z_n\big) - \sigma\|) \\
&\leq \|\alpha_n(f(x_n)-f(\sigma)) + (1-\alpha_n)(\beta_{n,0}z_n + \sum_{i=1}^{\infty}\beta_{n,i}J_{r_n}^{B_i}z_n - \sigma)\|^2 \\
&\quad +2\alpha_n\langle \sigma - f(\sigma), J(\sigma - x_{n+1})\rangle \\
&\leq (\alpha_n\|f(x_n)-f(\sigma)\| + \|(1-\alpha_n)(\beta_{n,0}z_n + \sum_{i=1}^{\infty}\beta_{n,i}J_{r_n}^{B_i}z_n - \sigma)\|)^2 \\
&\quad +2\alpha_n\langle \sigma - f(\sigma), J(\sigma - x_{n+1})\rangle \\
&\leq (\alpha_n b\|x_n-\sigma\| + (1-\alpha_n)\|y_n-\sigma\|)^2 + 2\alpha_n\langle \sigma - f(\sigma), J(\sigma - x_{n+1})\rangle \\
&\leq [(1-(1-b)\alpha_n)\|x_n-\sigma\|]^2 + 2\alpha_n\langle \sigma - f(\sigma), J(\sigma - x_{n+1})\rangle \\
&\leq (1-(1-b)\alpha_n)\|x_n-\sigma\|^2 + 2\alpha_n\langle \sigma - f(\sigma), J(\sigma - x_{n+1})\rangle.
\end{aligned}$$

We can check that all the assumptions of Lemma 2.4 are satisfied. Therefore, we deduce $x_n \to \sigma$.

Case 2. Assume that the sequence $\{\|x_n-\sigma\|\}$ is not monotonically decreasing. Set $\Pi_n = \|x_n - \sigma\|$ and $\chi : \mathbb{N} \to \mathbb{N}$ by $\chi(n) = \max\{k \in \mathbb{N} : k \leq n,\ \ \Pi_k \leq \Pi_{k+1}\}$.
We have χ is a non-decreasing sequence such that $\chi(n) \to \infty$ as $n \to \infty$ and $\Pi_{\chi(n)} \leq \Pi_{\chi(n)+1}$ for $n \geq n_0$. From (11), we have

$$\theta_{\chi(n)}\Big[q\omega^{q-1} - 2^{(m-1)q}d_q\theta_{\chi(n)}^{q-1}\Big]\Big\|x_{\chi(n)} - Tx_{\chi(n)}\Big\|^q \leq \alpha_{\chi(n)}C.$$

Furthermore, we have

$$\lim_{n\to\infty}\theta_{\chi(n)}\Big[q\omega^{q-1} - d_q\theta_{\chi(n)}^{q-1}\Big]\Big\|x_{\chi(n)} - Tx_{\chi(n)}\Big\|^q = 0.$$

Since $q\omega^{q-1} - 2^{(m-1)q}d_q\theta_{\chi(n)}^{q-1} > 0$, we have

$$\lim_{n\to\infty}\Big\|x_{\chi(n)} - Tx_{\chi(n)}\Big\|^q = 0. \tag{21}$$

As in the proof of Case 1, we obtain $\limsup_{\chi(n)\to+\infty}\langle \sigma - f(\sigma), J(\sigma - x_{\chi(n)})\rangle \leq 0$ and for all $n \geq n_0$, we get

$$0 \leq \|x_{\chi(n)+1}-\sigma\|^2 - \|x_{\chi(n)}-\sigma\|^2 \leq \alpha_{\chi(n)}[-(1-b)\|x_{\chi(n)}-\sigma\|^2 + 2\langle \sigma - f(\sigma), J(\sigma - x_{\chi(n)+1})\rangle],$$

which implies that

$$\|x_{\chi(n)}-\sigma\|^2 \leq \frac{2}{1-b}\langle \sigma - f(\sigma), J(\sigma - x_{\chi(n)+1})\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\chi(n)}-\sigma\| = 0.$$

Hence,

$$\lim_{n\to\infty}\Pi_{\chi(n)} = \lim_{n\to\infty}\Pi_{\chi(n)+1} = 0.$$

Thank Lemma 2.5, we have

$$0 \leq \Pi_n \leq \max\{\Pi_{\tau(n)},\ \Pi_{\tau(n)+1}\} = \Pi_{\chi(n)+1}.$$

Thus we get, $\lim_{n\to\infty}\|x_n-\sigma\| = 0$. The proof is completed. □

In the special case, where $T \equiv I$, the indentity map, then Theorem 3.1 is reduced to the following:

Theorem 3.2. *Let E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map and K be a nonempty, closed and convex subset of E. Let $B_i : E \to 2^E$, $i \in \mathbb{N}^*$ be an infinite family of multivalued accretive operators in E and $f : K \to K$ be an b-contraction mapping such that , $\bigcap_{i=1}^{\infty} B_i^{-1}(0) \neq \emptyset$ and $\bigcap_{i=1}^{\infty} \overline{D(B_i)} \subset K \subset \bigcap_{i=1}^{\infty} R(I + rB_i)$, for all $r > 0$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big(\beta_{n,0}x_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} x_n\big). \tag{22}$$

Let $\{r_n\} \subset]0, \infty[$, $\{\beta_{n,i}\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{\infty} \beta_{n,i} = 1$,

(iii) $\lim_{n\to\infty} \inf r_n > 0$, *and* $\lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,i} > 0$, *for all* $i \in \mathbb{N}$.

Then, the sequence $\{x_n\}$ generated by (22) converges strongly to $\sigma \in \bigcap_{i=1}^{\infty} B_i^{-1}(0)$, which solves the following variational inequality:

$$\langle \sigma - f(\sigma), J(\sigma - p)\rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{\infty} B_i^{-1}(0). \tag{23}$$

In the special case, where $B_i \equiv 0$, $i \in \mathbb{N}^*$, then Theorem 3.1 is reduced to the following:

Theorem 3.3. *Let E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map and K be a nonempty, closed and convex subset of E. Let $f : K \to K$ be an b-contraction mapping and, let $T : K \to K$ be a k-demicontractive operator such that $I - T$ is demiclosed at the origin and $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} z_n = (1-\theta_n)x_n + \theta_n T x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) z_n, \end{cases} \tag{24}$$

$\theta_n \in [a, b] \subset (0,\ \gamma)$ *where*

$$\gamma := \min\Big\{1, \Big(\frac{q\omega^{q-1}}{d_q}\Big)^{\frac{1}{q-1}}\Big\}, \ \ \textit{with } \omega = \frac{1-k}{2}.$$

Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequence $\{x_n\}$ generated by (24) converges strongly to $\sigma \in Fix(T)$, which solves the following variational inequality:

$$\langle \sigma - f(\sigma), J(\sigma - p)\rangle \leq 0, \quad \forall p \in Fix(T). \tag{25}$$

Finally, we consider the following unconstrained minimization problem involving an infinite family of lower semi-continuous and convex functions in a real Hilbert space, namely, find an σ with the property:

$$\sigma \in \bigcap_{i=1}^{\infty} \operatorname{argmin}_{x\in H} g_i(x), \tag{26}$$

where $\operatorname{argmin}_{x\in H} g_i(x)$ denotes the set of minimizers of g_i.

Hence, one has the following result.

Theorem 3.4. *Let H be a real Hilbert space and $f : H \to H$ be an contraction mapping with a constant $b \in [0,1)$. Let $g_i : H \to \mathbb{R} \cup \{\infty\}$, $i \in \mathbb{N}^*$ be a proper lower semi-continuous and convex function such that $\bigcap_{i=1}^{\infty} \partial {g_i}^{-1}(0) \neq \emptyset$.*
Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in H$ by:

$$\begin{cases} y_n = \beta_{n,0} x_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{\partial g_i} x_n, \\ \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n. \end{cases} \tag{27}$$

Let $\{r_n\} \subset]0,\infty[$, $\{\beta_{n,i}\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{\infty} \beta_{n,i} = 1$,
(iii) $\lim_{n\to\infty} \inf r_n > 0$, *and* $\lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,i} > 0$, *for all* $i \in \mathbb{N}$.
Then the sequence $\{x_n\}$ generated by (27) converges strongly to a solution of Problem (26).

Proof. For each $i \in \mathbb{N}^*$, we set $B_i = \partial g_i$ into Theorem 3.1. Then $\partial {g_i}^{-1}(0) = {B_i}^{-1}(0)$, for all $i \in \mathbb{N}^*$ and hence $\bigcap_{i=1}^{\infty} \partial {g_i}^{-1}(0) = \bigcap_{i=1}^{\infty} {B_i}^{-1}(0)$. Furthermore, each B_i is maximal monotone (see, e.g., Minty [22]). Therefore, the proof is complete from Theorem 3.2 with $T = I$. □

Remark 3.5. *Many already studied problems in the literature can be considered as special cases of this paper; see, for example,[20, 26, 8, 21, 25, 28, 14] and the references therein.*

References

[1] F.E. Browder, Convergenge theorem for sequence of nonlinear operator in Banach spaces, Z., 100 (1967) 201-225.

[2] R.E. Bruck Jr, A strongly convergent iterative solution of $0 \in U(x)$ for a maximal monotone operator U in Hilbert spaces, J. Math. Anal. Appl., 48,114-126. (1974).

[3] R.D. Chen, Z. C. Zhu, Viscosity approximation fixed point for nonexpansive and m-accretive operators, Fixed Point Theory Appl., 2006 (2006), 10 pages.

[4] S. Chang, J.K. Kim, X.R. Wang, Modified block iterative algorithm for solving convex feasibility problems in Banach spaces, Journal of Inequalities and Applications, vol. 2010, Article ID 869684, 14 pages.

[5] L.C. Ceng and J.C. Yao, An extragradient-like approximation method for variational inequality problems and fixed point problems, Appl. Math. Comput., 190 (2007), 205-215.

[6] C.E. Chidume, Geometric Properties of Banach spaces and Nonlinear Iterations, Springer Verlag Series: Lecture Notes in Mathematics, Vol. 1965,(2009), ISBN 978-1-84882-189.

[7] C.E. Chidume, The solution by iteration of nonlinear equations in certain Banach spaces, J. Nigerian Math. Soc., 3 (1984), 57-62.

[8] C.E. Chidume, N. Djitte, Strong convergence theorems for zeros of bounded maximal monotone nonlinear operators, J. Abstract and Applied Analysis, Volume 2012, Article ID 681348, 19 pages, doi:10.1155/2012/681348.

[9] K. Goebel and W.A. Kirk, Topics in metric fixed poit theory, Cambridge Studies, in Advanced Mathemathics, Vol. **28**, University Cambridge Press, Cambridge 1990.

[10] K. Goebel, S. Reich, (1984). Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York.

[11] O. Guler, On the convergence of the proximal point algorithm for convex minimization, SIAM J. Control Optim., 29 (1991), 403-419.

[12] B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc., 3 (1967), 957-961.

[13] Hicks, T.L. Kubicek, J.D., On the Mann iteration process in a Hilbert spaces, J. Math. Anal. Appl. 59 (1977) 498-504.

[14] S. Ishikawa, Fixed points and iteration of nonexpansive mapping in a Banach space, Proc. Amer. Math. Soc., 73 (1976), 61-71.

[15] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. of Optimization **13**(3) (5003), 938-945.

[16] N. Lehdili, A. Moudafi, Combining the proximal algorithm and Tikhonov regularization, Optimization **37**(1996), 239-252.
[17] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., **4** (1953) 506-510.
[18] P.E. Mainge, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Analysis, 16, 899-912 (2008).
[19] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Math. Appl., 329(2007), 336-346.
[20] B. Martinet, Régularisation dinéquations variationnelles par approximations successives, (French) Rev. Franaise Informat. Recherche OpÃĩrationnelle, 4 (1970), 154-158.
[21] A. Moudafi, Viscosity approximation methods for fixed-point problems, J. Math. Anal. Appl. 241 (2000), 46-55.
[22] G.J. Minty, Monotone (nonlinear) operator in Hilbert space. Duke Math. 29, 341-V346 (1962).
[23] I. Miyadera, (1992) Nonlinear semigroups, Translations of Mathematical Monographs, 109 American Mathematical Society, Providence, RI.
[24] R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optimization, 14 (1976), 877-898
[25] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, Journal of Mathematical Analysis and Applications, vol. 67, no. 2, pp. 4276, 1979.
[26] J. J. Soo, Strong convergence of an iterative algorithm for accretive operators and nonexpansive mappings, Journal of Nonlinear Sciences and Applications, (2016), Vol 9, ISSN 2008-1901
[27] T.M.M. Sow, A modified Mann iterative scheme based on the generalized explicit methods for quasi-nonexpansive mappings in Banach spaces, Advances in the Theory of Nonlinear Analysis and its Applications **3** (2019) No. 2, 90-101
[28] T.M.M. Sow, N. Djitte, and C.E. Chidume,A path convergence theorem and construction of fixed points for nonexpansive mappings in certain Banach spaces, Carpathian J.Math.,**32**(2016),No.2,217-226,2016.
[29] A. Taiwo, L.O. Jolaoso and O.T. Mewomo, General alternative regularization method for solvng Split Equality Common Fixed Point Problem for quasi-pseudocontractive mappings in Hilbert spaces, Ricerche Mat., (2019), DOI: 10.1007/s11587-019-00460-0. minimization problem and fixed point problem in uniformly convex Banach spaces, Comput. Appl. Math., 38 (2), (2019), Article 77.
[30] A. Taiwo, L.O. Jolaoso and O.T. Mewomo, Parallel hybrid algorithm for solving pseudomonotone equilibrium and Split Common Fixed point problems, Bull. Malays. Math. Sci. Soc., (2019), DOI: 10.1007/s40840-019-00781-1.
[31] H.K. Xu, A regularization method for the proximal point algorithm, J. Global. Optim., 36, 115-125. (2006).
[32] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. **66** (2002), no. 2, 240 - 256.
[33] Y. Yao, H. Zhou, Y.C. Liou, Strong convergence of modified Krasnoselskii-Mann iterative algorithm for nonexpansive mappings, J. Math. Anal. Appl. Comput. **29** (2009) 383-389.