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A Note on Gradient $*$-Ricci Solitons

Krishnendu De*

Abstract
In the offering exposition we characterize $(k, \mu)'$- almost Kenmotsu 3-manifolds admitting gradient $*$-Ricci soliton. It is shown that in a $(k, \mu)'$- almost Kenmotsu manifold with $k < -1$ admitting a gradient $*$-Ricci soliton, either the soliton is steady or the manifold is locally isometric to a rigid gradient Ricci soliton $\mathbb{H}^2(-4) \times \mathbb{R}$.

Keywords: *$(k, \mu)'$- almost Kenmotsu manifolds, $*$-Ricci solitons, gradient $*$-Ricci solitons.*

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**Corresponding author*

1. Introduction

In the present paper we study the nullity distributions which play a functional role in contemporary mathematics. In the study of Riemannian manifolds (M, g), Gray [10] and Tanno [20] introduced the concept of k**-nullity distribution** $(k \in \mathbb{R})$, which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}, \tag{1.1}$$

for any $X, Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1, 3)$. Recently, the (k, μ)**-nullity distribution** which is a generalized notion of the k-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ introduced by Blair, Koufogiorgos and Papantoniou [5] and defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$N_p(k, \mu) = \{Z \in T_pM^{2n+1} : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\},$$

for any $X, Y \in T_pM$ and $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie differentiation.

In 2009, Dileo and Pastore [7] introduced another generalized notion of the (k, μ)-nullity distribution which is named the $(k, \mu)'$**-nullity distribution** on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k, \mu)' = \{Z \in T_pM^{2n+1} : R(X, Y)Z &= k[g(Y, Z)X - g(X, Z)Y] \\ &+\mu[g(Y, Z)h'X - g(X, Z)h'Y]\}, \end{aligned} \tag{1.1}$$

for any $X, Y \in T_pM$ and $h' = h \circ \phi$.

The idea of $*$*-Ricci tensor* on almost Hermitian manifolds was introduced by Tachibana [19] in 1959. Later, in [11] Hamada studied $*$-Ricci flat real hypersurfaces in non-flat complex space forms and Blair [4] defined $*$-Ricci tensor in contact metric manifolds by

$$S^*(X, Y) = g(Q^*X, Y) = Trace\{\phi \circ R(X, \phi Y)\}, \tag{1.2}$$

where Q^* is called the $*$-*Ricci operator*.

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A Ricci soliton is nothing but a generalization of an Einstein metric. On a Riemannian manifold (M, g) [12], a Ricci soliton is defined by

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.3}$$

for a vector field V (called potential vector field) and λ a real scalar and is denoted by a triple (g, V, λ), where $\pounds$ is the Lie derivative. The Ricci soliton is said to be *shrinking*, *steady* and *expanding* according as λ is negative, zero and positive respectively.

Ricci solitons have been generalized in several ways, such as almost Ricci solitons ([8],[17]), η-Ricci solitons ([1],[2]), generalized Ricci soliton, $*$-Ricci solitons and many others.

Definition 1.1. [13] A Riemannian metric g on M is called *$*$-Ricci soliton* if

$$\pounds_V g + 2S^* + 2\lambda g = 0, \tag{1.4}$$

where λ is a constant.

Definition 1.2. [13] A Riemannian metric g on M is called *gradient $*$-Ricci soliton* if

$$\nabla\nabla f + S^* + \lambda g = 0, \tag{1.5}$$

where $\nabla\nabla f$ denotes the Hessian of the smooth function f on M with respect to g and λ is a constant.

In 2018, Ghosh and Patra [9] first undertook the study of $*$-Ricci solitons on almost contact metric manifolds. In the same year, Majhi et. al. [14] studied $*$-Ricci solitons on Sasakian 3-manifolds. Here we also mention the works of Prakasha and Veeresha [18] within the frame-work of paracontact geometry. If a $(k, \mu)'$- almost Kenmotsu manifold M satisfies the relation (1.4), then we say that M admits a $*$-Ricci soliton. In the year 2019, Dai et. al. [6] studied $*$-Ricci solitons on a $(k, \mu)'$- almost Kenmotsu manifold.
Motivated from the above studies, we make the contribution to investigate gradient $*$-Ricci soliton in a 3-dimensional $(k, \mu)'$- almost Kenmotsu manifold. More precisely, the following theorem is proved.

Theorem 1.1. *Let $(M^3, \phi, \xi, \eta, g)$ be a $(k, \mu)'$- almost Kenmotsu manifold with $k < -1$ which admits a gradient $*$-Ricci soliton. Then either, the soliton is steady or, M^3 is locally isometric to a rigid gradient Ricci soliton $\mathbb{H}^2(-4) \times \mathbb{R}$.*

2. Almost Kenmotsu manifolds

A differentiable manifold M^{2n+1} of dimension $2n + 1$ is called **almost contact metric manifold** if it admits a $(1, 1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where I denotes the identity endomorphism ([3, 4]). Then also $\phi\xi = 0$ and $\eta \circ \phi = 0$; in a straight forward calculation both can be derived from (2.1).

On an almost Kenmotsu manifold M^{2n+1}, the two symmetric tensor fields $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot, \xi)\xi$, satisfy the following relations [7]

$$h\xi = 0, \ l\xi = 0, \ tr(h) = 0, \ tr(h') = 0, \ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi^2 X + h'X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.5}$$

for any vector fields X, Y.
Now we furnish some basic results on almost Kenmotsu manifolds with ξ belongs to the $(k, \mu)'$-nullity distribution. The $(1, 1)$-type symmetric tensor field h' satisfies $h'\phi + \phi h' = 0$ and $h'\xi = 0$. Also it is clear that

$$h = 0 \Leftrightarrow h' = 0, \ \ h'^2 = (k + 1)\phi^2 (\Leftrightarrow h^2 = (k + 1)\phi^2). \tag{2.6}$$

For an almost Kenmotsu manifold, we have from (1.1)

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{2.7}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X], \tag{2.8}$$

where $k, \mu \in \mathbb{R}$. Contracting Y in (2.8) we have

$$S(X,\xi) = 2k\eta(X). \tag{2.9}$$

Suppose $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then $\lambda^2 = -(k+1)$, a constant, which follows from (2.6). Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. The non-zero eigen value λ and $-\lambda$ are respectively denoted by $[\lambda]'$ and $[-\lambda]'$, which are the corresponding eigen spaces associated with h'. We have the following lemmas.

Lemma 2.1. *(Prop. 4.1 of [7]) Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi]\oplus[\lambda]'$ and $[\xi]\oplus[-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves.*

In a 3-dimensional Riemannian manifold we have

$$\begin{aligned} R(X,Y)Z &= S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ &\quad -\frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{2.10}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ for all $X, Y \in T_pM$ and r is the scalar curvature of the manifold.
Putting $Y = Z = \xi$ in (2.10) and using Lemma 2.1 and (2.9) we obtain

$$QX = \left(\frac{r}{2} - k\right)X - \left(\frac{r}{2} - 3k\right)\eta(X)\xi - 2h'X, \tag{2.11}$$

which is equivalent to

$$S(X,Y) = \left(\frac{r}{2} - k\right)g(X,Y) - \left(\frac{r}{2} - 3k\right)\eta(X)\eta(Y) - 2g(h'X,Y), \tag{2.12}$$

for any $X, Y \in T_pM$.
With the help of (2.11) and (2.12), it follows from (2.10) that

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2} - 2k\right)[g(Y,Z)X - g(X,Z)Y] - \left(\frac{r}{2} - 3k\right)[g(Y,Z)\eta(X)\xi \\ &\quad -g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &\quad -2g(Y,Z)h'X + 2g(X,Z)h'Y - 2g(h'Y,Z)X + 2g(h'X,Z)Y, \end{aligned} \tag{2.13}$$

for any $X, Y, Z \in T_pM$.

Lemma 2.2. *In an $(k,\mu)'$- almost Kenmotsu manifold (M^3,ϕ,ξ,η,g) , we have*

$$\begin{aligned} \tilde{R}(X,Y,\phi Z,\phi W) &= \left(\frac{r}{2} - 2k\right)[g(Y,\phi Z)g(X,\phi W) - g(X,\phi Z)g(Y,\phi W)] \\ &\quad -2g(Y,\phi Z)g(h'X,\phi W) + 2g(X,\phi Z)g(h'Y,\phi W) \\ &\quad -2g(h'Y,\phi Z)g(X,\phi W) + 2g(h'X,\phi Z)g(Y,\phi W), \end{aligned} \tag{2.14}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, for $X, Y, Z, W \in \chi(M)$.

Proof. To prove the above Lemma we shall use the equation (2.13). From (2.13) one can easily write

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= \left(\frac{r}{2} - 2k\right)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad \left(\frac{r}{2} - 3k\right)[g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\ &\quad +\eta(Y)\eta(Z)g(X,W) - \eta(X)\eta(Z)g(Y,W)] \\ &\quad -2g(Y,Z)g(h'X,W) + 2g(X,Z)g(h'Y,W) \\ &\quad -2g(h'Y,Z)g(X,W) + 2g(h'X,Z)g(Y,W). \end{aligned}$$

Again replacing Z by ϕZ and W by ϕW in the foregoing equation and using $\eta.\phi = 0$, we get

$$\begin{aligned}\tilde{R}(X,Y,\phi Z,\phi W) &= \left(\frac{r}{2}-2k\right)[g(Y,\phi Z)g(X,\phi W)-g(X,\phi Z)g(Y,\phi W)]\\ &\quad -2g(Y,\phi Z)g(h'X,\phi W)+2g(X,\phi Z)g(h'Y,\phi W)\\ &\quad -2g(h'Y,\phi Z)g(X,\phi W)+2g(h'X,\phi Z)g(Y,\phi W).\end{aligned}$$

This completes the proof. □

Now we prove the following Lemma which will be used later.

Lemma 2.3. *In an $(k,\mu)'$- almost Kenmotsu manifold (M^3,ϕ,ξ,η,g) , the $*$-Ricci tensor is given by*

$$S^*(X,Y) = \left(\frac{r}{2}-2k\right)[g(X,Y)-\eta(X)\eta(Y)], \tag{2.15}$$

where S^* is the $*$-Ricci tensor of type $(0,2)$.

Proof. Let $\{e_i\}$, $i = 1,2,3$ be an orthonormal basis of the tangent space at each point of the manifold. From (1.1) and using (2.14), we infer

$$\begin{aligned}S^*(Y,Z) &= -\sum_{i=1}^{3}\tilde{R}(e_i,Y,\phi Z,\phi e_i)\\ &= \sum_{i=1}^{3}[\left(\frac{r}{2}-2k\right)[g(Y,\phi Z)g(e_i,\phi e_i)-g(e_i,\phi Z)g(Y,\phi e_i)]\\ &\quad -2g(Y,\phi Z)g(h'e_i,\phi e_i)+2g(e_i,\phi Z)g(h'Y,\phi e_i)\\ &\quad -2g(h'Y,\phi Z)g(e_i,\phi e_i)+2g(h'e_i,\phi Z)g(Y,\phi e_i)]\\ &= \left(\frac{r}{2}-2k\right)[g(Y,Z)-\eta(Y)\eta(Z)].\end{aligned}$$

Hence, the $*$-Ricci tensor is

$$S^*(Y,Z) = \left(\frac{r}{2}-2k\right)[g(Y,Z)-\eta(Y)\eta(Z)],$$

for any $Y,Z \in \chi(M)$. This completes the proof. □

From the above Lemma, the $(1,1)$ $*$-Ricci operator Q^* and the $*$-scalar curvature r^* are given by

$$Q^*Y = \left(\frac{r}{2}-2k\right)[Y-\eta(Y)\xi], \tag{2.16}$$

$$r^* = r-4k. \tag{2.17}$$

3. Proof of the main theorem

Let (M^3,ϕ,ξ,η,g) be a $(k,\mu)'$- almost Kenmotsu manifold with $k < -1$ and g as a gradient $*$-Ricci soliton. Then the equation (1.5) can be written as

$$\nabla_X Df + Q^*X + \lambda X = 0, \tag{3.1}$$

for any $X \in \chi(M)$, where D denotes the gradient operator with respect to g. From (3.1) it follows that

$$R(X,Y)Df = (\nabla_Y Q^*)X - (\nabla_X Q^*)Y, \quad X,Y \in \chi(M). \tag{3.2}$$

Using (2.7), we have

$$g(R(\xi,X)Df,\xi) = k[(Xf)-\eta(X)(\xi f)] - 2(h'Xf), \tag{3.3}$$

where we have used $\mu = -2$. With the help of (2.16), we have

$$(\nabla_X Q^*)Y = \frac{(Xr)}{2}[Y - \eta(Y)\xi] - \left(\frac{r}{2} - 2k\right)[g(X,Y)\xi + \eta(Y)X - 2\eta(X)\eta(Y)\xi + g(h'X,Y)\xi + h'X\eta(Y)]. \tag{3.4}$$

Interchanging X and Y, we have

$$(\nabla_Y Q^*)X = \frac{(Yr)}{2}[X - \eta(X)\xi] - \left(\frac{r}{2} - 2k\right)[g(X,Y)\xi + \eta(X)Y - 2\eta(X)\eta(Y)\xi + g(h'Y,X)\xi + h'Y\eta(X)]. \tag{3.5}$$

Making use of (3.4) and (3.5) we get

$$(\nabla_Y Q^*)X - (\nabla_X Q^*)Y = -\frac{(Xr)}{2}[Y - \eta(Y)\xi] + \frac{(Yr)}{2}[X - \eta(X)\xi] + \left(\frac{r}{2} - 2k\right)[\eta(Y)X - \eta(X)Y + h'X\eta(Y) - h'Y\eta(X)]. \tag{3.6}$$

Putting $X = \xi$ in (3.6) and taking inner product with ξ, we infer that

$$g((\nabla_Y Q^*)\xi - (\nabla_\xi Q^*)Y, \xi) = 0, \tag{3.7}$$

for any $Y \in \chi(M)$. From (3.3) and (3.7) we get

$$2(h'Xf) = k[(Xf) - \eta(X)(\xi f)], \tag{3.8}$$

for any $X \in \chi(M)$. Therefore,

$$2h'Df = k[Df - \xi(\xi f)]. \tag{3.9}$$

Taking into account the equation (2.6) and operating h' on (3.9) gives that

$$kh'Df = 2(k+1)[\xi(\xi f) - Df]. \tag{3.10}$$

Comparing the above relation with (3.9) gives that either $Df = (\xi f)\xi$ or $k = -2$. Next, we consider the above two cases as follows.

Case i:

$$Df = (\xi f)\xi. \tag{3.11}$$

Taking the covariant differentiation of (3.11) along any vector field $X \in \chi(M)$ and using (2.3) we get

$$\nabla_X Df = X(\xi f)\xi + (\xi f)X - (\xi f)\eta(X)\xi + (\xi f)h'X. \tag{3.12}$$

Putting the foregoing equation into (3.1) yields that

$$Q^*X = -(\lambda + (\xi f))X - X(\xi f)\xi + (\xi f)\eta(X)\xi - (\xi f)h'X. \tag{3.13}$$

Comparing (2.16) and (3.13) gives that

$$(\frac{r}{2} - 2k + \lambda + (\xi f))X - (\frac{r}{2} - 2k + (\xi f))\eta(X)\xi + X(\xi f)\xi + (\xi f)h'X = 0. \tag{3.14}$$

Now operating h' we get

$$(\frac{r}{2} - 2k + \lambda + (\xi f))h'X + (\xi f)(k+1)(X - \eta(X)\xi) = 0. \tag{3.15}$$

Contracting X in the above equation we get $2(\xi f)(k+1) = 0$ and hence by assumption $k < -1$ we obtain $(\xi f) = 0$. Using $(\xi f) = 0$ in (3.14) gives

$$(\frac{r}{2} - 2k + \lambda)X - (\frac{r}{2} - 2k)\eta(X)\xi = 0. \tag{3.16}$$

Putting $X = \xi$ in the above equation gives $\lambda = 0$. Thus we can say that the gradient $*$-Ricci soliton is steady.

Case ii: $k = -2$. In view of $k = \mu = -2$, according to Corollary 4.2 and Proposition 4.1 of Dileo and Pastore [7] we obtain that M^3 is locally isometric to the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$. In fact, from Peterson and Wylie ([15],[16]) we state that the product $\mathbb{H}^2(-4)\times\mathbb{R}$ is a rigid gradient Ricci soliton. This put an ends the proof of the Theorem 1.1. □

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Affiliations

KRISHNENDU DE
ADDRESS: Kabi Sukanta Mahavidyalaya, Dept. of Mathematics, 712221, West Bengal-India.
E-MAIL: krishnendu.de@outlook.in
ORCID ID:0000-0001-5264-5861