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# WIJSMAN ASYMPTOTICAL $\mathcal{I}_2$-STATISTICALLY EQUIVALENT DOUBLE SET SEQUENCES OF ORDER $\eta$

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ABSTRACT. In this study, we present notions of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence of order $\eta$, Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence of order $\eta$ and Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalence of order $\eta$ for double set sequences where $0 < \eta \leq 1$. Also, we investigate some properties of these notions and some relationships between them.

## 1. Introduction

Pringshiem [1] introduced the notion of convergence for double sequences. Then, Mursaleen and Edely [2] studied the notion of statistical convergence. After that, Das et al. [3] studied the notion of $\mathcal{I}$-convergence for double sequences. Recently, Bhunia et al. [4], Çolak and Altın [5], Savaş [6] and Altın et al. [7] presented various type of convergence of order $\alpha$ for double sequences.

Patterson [8] introduced the notion of asymptotical equivalence for double sequences. After that, the notions of asymptotical Cesàro equivalence, asymptotical $\mathcal{I}$-equivalence and asymptotical statistically equivalence for double sequences were studied by Kavita et al. [9], Hazarika and Kumar [10] and Esi and Açıkgöz [11], respectively.

To date, a variety of convergence types for set sequences have been studied by several authors. In this study, the notion of Wijsman convergence which is one of these types is handled (see, [12, 13, 14]). Several authors extended the notion of Wijsman convergence to the new notions for double set sequences via using the notions of statistical convergence, $\mathcal{I}$-convergence and Cesàro summability (see, [15, 16, 17, 18, 19, 20]).

The notions of asymptotical equivalence in Wijsman sense for double set sequences were presented by Nuray et al. [21]. Also, the notions of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence and Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence

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for double set sequences were introduced in [22] and [23], respectively. Lately, new notions of asymptotical equivalence of order $\alpha$ for double set sequences were studied by Gülle [24].

More study on the concepts of convergence or asymptotical equivalence for real sequences or set sequences can be found in [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

## 2. Definitions and Notations

The fundamental definitions and notations required for this study are following. (see, [1, 3, 8, 12, 13, 14, 21, 22, 23, 25]).

A double sequence $(x_{ij})$ is convergent to $L$ if for $\varepsilon > 0$, there exists a number $N_\varepsilon \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ for $i, j > N_\varepsilon$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be ideal if

1) $\emptyset \in \mathcal{I}$, 2) For $E, F \in \mathcal{I}$, $E \cup F \in \mathcal{I}$, 3) For $E \in \mathcal{I}$ and $F \subseteq E$, $F \in \mathcal{I}$.

An ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be non trivial if $\mathbb{N} \notin \mathcal{I}$ and a non trivial ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be admissible if $\{j\} \in \mathcal{I}$ for $j \in \mathbb{N}$.

A non trivial ideal $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be strongly admissible if $\{j\} \times \mathbb{N}$ and $\mathbb{N} \times \{j\}$ belong to $\mathcal{I}_2$ for $j \in \mathbb{N}$.

Obviously any strongly admissible ideal is admissible.

Throughout the study, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ will be taken as strongly admissible ideal.

Two non negative double sequences $(x_{ij})$ and $(y_{ij})$ are said to be asymptotical equivalent if

$$\lim_{i,j\to\infty} \frac{x_{ij}}{y_{ij}} = 1.$$

Let $X$ be any non empty set. A function $f : \mathbb{N} \to 2^X$ is defined by $f(n) = U_n \in 2^X$ for each $n \in \mathbb{N}$, where $2^X$ is power set of $X$. The sequence $\{U_n\} = (U_1, U_2, ...)$, which is the range's elements of $f$, is said to be set sequences.

Let $(X, \rho)$ be a metric space. For any point $x \in X$ and any non empty subset $U$ of $X$, distance from $x$ to $U$ is defined by

$$\mu(x, U) = \inf_{u\in U} \rho(x, u).$$

A double sequence $\{U_{ij}\}$ is Wijsman convergent to $U$ if for each $x \in X$,

$$\lim_{i,j\to\infty} \mu(x, U_{ij}) = \mu(x, U).$$

Throughout the study, we will take $(X, \rho)$ as metric space and $U_{ij}, V_{ij}$ as any non empty closed subsets of $X$.

The term $\mu_x(U_{ij}, V_{ij})$ is defined as follows:

$$\mu_x(U_{ij}, V_{ij}) = \begin{cases} \dfrac{\mu(x, U_{ij})}{\mu(x, V_{ij})} & , \quad x \notin U_{ij} \cup V_{ij} \\ \\ L & , \quad x \in U_{ij} \cup V_{ij}. \end{cases}$$

Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical equivalent if for each $x \in X$,

$$\lim_{i,j\to\infty} \mu_x(U_{ij}, V_{ij}) = 1.$$

Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-equivalent of multiple $L$ if for each $x \in X$ and $\varepsilon > 0$,

$$\Big\{(i,j) \in \mathbb{N}\times\mathbb{N} : |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\Big\} \in \mathcal{I}_2.$$

Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent of multiple $L$ if for each $x \in X$ and $\varepsilon$, $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\Big|\Big\{i \leq m, j \leq n : |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\Big\}\Big| \geq \delta\right\} \in \mathcal{I}_2.$$

The set of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent double sequences is denoted by $S(\mathcal{I}_{W_2}^L)$.

Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalent of multiple $L$ if for each $x \in X$ and $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n} |\mu_x(U_{ij}, V_{ij}) - L|^p \geq \varepsilon\right\} \in \mathcal{I}_2$$

where $0 < p < \infty$.

The set of Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalent double sequences is denoted by $C[\mathcal{I}_{W_2}^L]^p$.

## 3. New Notions

In this section, we present notions of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence of order $\eta$, Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence of order $\eta$ and Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalence of order $\eta$ for double set sequences.

**Definition 1.** *Let $0 < \eta \leq 1$. Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple $L$ of order $\eta$ if for each $x \in X$ and $\varepsilon$, $\delta > 0$,*

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(mn)^\eta}\Big|\Big\{(i,j) : i \leq m, j \leq n,\ |\mu_x(U_{ij}, V_{ij}) - L| \geq \varepsilon\Big\}\Big| \geq \delta\right\} \in \mathcal{I}_2$$

*and we write $U_{ij} \overset{\mathcal{I}_2^W(S_L^\eta)}{\sim} V_{ij}$, and simply Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent of order $\eta$ if $L = 1$.*

The class of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple $L$ of order $\eta$ double sequences will be denoted by $\mathcal{I}_2^W(S_L^\eta)$.

**Example 2.** *Let $X=\mathbb{R}^2$ and double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ be defined as following:*

$$U_{ij} := \begin{cases} \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+(x_2-\frac{ij}{2})^2=\frac{(ij)^2}{4}\} & , \text{ if } ij=c^2 \text{ and } c\in\mathbb{N} \\ \{(0,1)\} & , \text{ if not.} \end{cases}$$

*and*

$$V_{ij} := \begin{cases} \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+(x_2+\frac{ij}{2})^2=\frac{(ij)^2}{4}\} & , \text{ if } ij=c^2 \text{ and } c\in\mathbb{N} \\ \{(0,1)\} & , \text{ if not.} \end{cases}$$

*If we take $\mathcal{I}_2=\mathcal{I}_2^f$, ($\mathcal{I}_2^f$ is the class of finite subsets of $\mathbb{N}\times\mathbb{N}$), then the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent of order $\eta$.*

**Remark 3.** *For $\eta=1$, the notion of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence to multiple $L$ of order $\eta$ coincides with the notion of Wijsman asymptotical $\mathcal{I}_2$-statistically equivalence of multiple $L$ for double set sequences in* [22].

**Definition 4.** *Let $0<\eta\le 1$. Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalent to multiple $L$ of order $\eta$ if for each $x\in X$ and $\varepsilon>0$,*

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \left|\frac{1}{(mn)^\eta}\sum_{i,j=1,1}^{m,n}\mu_x(U_{ij},V_{ij})-L\right|\ge\varepsilon\right\}\in\mathcal{I}_2$$

*and we write $U_{ij}\overset{\mathcal{I}_2^W(C_L^\eta)}{\sim}V_{ij}$, and simply Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalent of order $\eta$ if $L=1$.*

**Definition 5.** *Let $0<\eta\le 1$ and $0<p<\infty$. Double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent to multiple $L$ of order $\eta$ if for each $x\in X$ and $\varepsilon>0$,*

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{(mn)^\eta}\sum_{i,j=1,1}^{m,n}\left|\mu_x(U_{ij},V_{ij})-L\right|^p\ge\varepsilon\right\}\in\mathcal{I}_2$$

*and we write $U_{ij}\overset{\mathcal{I}_2^W[C_L^\eta]^p}{\sim}V_{ij}$, and simply Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent of order $\eta$ if $L=1$.*

The class of Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent to multiple $L$ of order $\eta$ double sequences will be denoted by ${\mathcal{I}_2}^W[C_L^\eta]^p$.

If $p=1$, then the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalent to multiple $L$ of order $\eta$ and we write $U_{ij}\overset{\mathcal{I}_2^W[C_L^\eta]}{\sim}V_{ij}$, and simply Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalent of order $\eta$ if $L=1$.

**Example 6.** *Let $X = \mathbb{R}^2$ and double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ be defined as following:*

$$U_{ij} := \begin{cases} \{(x_1,x_2) \in \mathbb{R}^2 : (x_1+2)^2 + x_2^2 = \frac{1}{ij}\} & , \text{ if } ij = c^2 \text{ and } c \in \mathbb{N} \\ \{(-1,1)\} & , \text{ if not.} \end{cases}$$

*and*

$$V_{ij} := \begin{cases} \{(x_1,x_2) \in \mathbb{R}^2 : (x_1-2)^2 + x_2^2 = \frac{1}{ij}\} & , \text{ if } ij = c^2 \text{ and } c \in \mathbb{N} \\ \{(-1,1)\} & , \text{ if not.} \end{cases}$$

*If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, then the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalent of order $\eta$.*

**Remark 7.** *For $\eta = 1$, the notions of Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence to multiple $L$ of order $\eta$ and Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalence to multiple $L$ of order $\eta$ coincide with the notions of Wijsman asymptotical $\mathcal{I}_2$-Cesàro equivalence of multiple $L$ and Wijsman asymptotical strongly $\mathcal{I}_2$-Cesàro equivalence of multiple $L$ for double set sequences in* [23]*, respectively.*

## 4. Inclusions Theorems

In this section, we investigate some properties of the new asymptotical equivalence notions that introduced in Section 3 and some relationships between them.

**Theorem 8.** *If $0 < \eta \leq \gamma \leq 1$, then $\mathcal{I}_2^W(S_L^\eta) \subseteq \mathcal{I}_2^W(S_L^\gamma)$.*

*Proof.* Suppose that $0 < \eta \leq \gamma \leq 1$ and $U_{ij} \overset{\mathcal{I}_2^W(S_L^\eta)}{\sim} V_{ij}$. For each $x \in X$ and $\varepsilon > 0$,

$$\frac{1}{(mn)^\gamma}\Big|\big\{(i,j) : i \leq m, j \leq n, \, |\mu_x(U_{ij},V_{ij}) - L| \geq \varepsilon\big\}\Big|$$

$$\leq \frac{1}{(mn)^\eta}\Big|\big\{(i,j) : i \leq m, j \leq n, \, |\mu_x(U_{ij},V_{ij}) - L| \geq \varepsilon\big\}\Big|$$

and so for $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\gamma}\Big|\big\{(i,j) : i \leq m, j \leq n, \, |\mu_x(U_{ij},V_{ij}) - L| \geq \varepsilon\big\}\Big| \geq \delta\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\eta}\Big|\big\{(i,j) : i \leq m, j \leq n, \, |\mu_x(U_{ij},V_{ij}) - L| \geq \varepsilon\big\}\Big| \geq \delta\right\}.$$

Consequently, by our assumption, we get $\mathcal{I}_2^W(S_L^\eta) \subseteq \mathcal{I}_2^W(S_L^\gamma)$. □

If we take $\gamma = 1$ in Theorem 8, we obtain the following:

**Corollary 9.** *If double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple $L$ of order $\eta$, then the double sequences are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent of multiple $L$, i.e., $\mathcal{I}_2^W(S_L^\eta) \subseteq S(\mathcal{I}_{W_2}^L)$.*

**Theorem 10.** *If $0 < \eta \leq \gamma \leq 1$ and $0 < p < \infty$, then $\mathcal{I}_2^W[C_L^\eta]^p \subseteq \mathcal{I}_2^W[C_L^\gamma]^p$.*

*Proof.* Suppose that $0 < \eta \leq \gamma \leq 1$ and $U_{ij} \overset{\mathcal{I}_2^W[C_L^\eta]^p}{\sim} V_{ij}$. For each $x \in X$,

$$\frac{1}{(mn)^\gamma} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p \leq \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p$$

and so for $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\gamma} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p \geq \varepsilon\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p \geq \varepsilon\right\}.$$

Consequently, by our assumption, we get $\mathcal{I}_2^W[C_L^\eta]^p \subseteq \mathcal{I}_2^W[C_L^\gamma]^p$. □

If we take $\gamma = 1$ in Theorem 10, we obtain the following:

**Corollary 11.** *If double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalent to multiple $L$ of order $\eta$, then the double sequences are Wijsman asymptotical strongly $p - \mathcal{I}_2$-Cesàro equivalent of multiple $L$, i.e., $\mathcal{I}_2^W[C_L^\eta]^p \subseteq C[\mathcal{I}_{W_2}^L]^p$.*

Now, we shall give a theorem that gives a relation between $\mathcal{I}_2^W[C_L^\eta]^p$ and $\mathcal{I}_2^W[C_L^\eta]^q$ where $0 < \eta \leq 1$ and $0 < p < q < \infty$.

**Theorem 12.** *If $0 < \eta \leq 1$ and $0 < p < q < \infty$, then $\mathcal{I}_2^W[C_L^\eta]^q \subset \mathcal{I}_2^W[C_L^\eta]^p$.*

*Proof.* Assume that $0 < p < q < \infty$ and $U_{ij} \overset{\mathcal{I}_2^W[C_L^\eta]^q}{\sim} V_{ij}$. For each $x \in X$,

$$\frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p < \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^q$$

and so for $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^p \geq \varepsilon\right\}$$

$$\subset \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\eta} \sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij}, V_{ij}) - L\right|^q \geq \varepsilon\right\}.$$

Hence, by our assumption, we get $U_{ij} \overset{\mathcal{I}_2^W[C_L^\eta]^p}{\sim} V_{ij}$. Consequently, $\mathcal{I}_2^W[C_L^\eta]^q \subset \mathcal{I}_2^W[C_L^\eta]^p$. □

**Theorem 13.** *If double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent to multiple $L$ of order $\eta$, then the double sequences are Wijsman asymptotical $\mathcal{I}_2$-statistically to multiple $L$ of order $\gamma$ where $0<\eta\leq\gamma\leq 1$ and $0<p<\infty$.*

*Proof.* Assume that $0<\eta\leq\gamma\leq 1$ and the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent to multiple $L$ of order $\eta$. For each $x\in X$ and $\varepsilon>0$,

$$\sum_{i,j=1,1}^{m,n} \left|\mu_x(U_{ij},V_{ij})-L\right|^p \geq \sum_{\substack{i,j=1,1\\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon}}^{m,n} \left|\mu_x(U_{ij},V_{ij})-L\right|^p \geq \varepsilon^p \left|\left\{(i,j):\ i\leq m, j\leq n,\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon\right\}\right|$$

and so

$$\frac{1}{\varepsilon^p\,(mn)^\eta}\sum_{i,j=1,1}^{m,n}\left|\mu_x(U_{ij},V_{ij})-L\right|^p \geq \frac{1}{(mn)^\eta}\left|\left\{(i,j):\ i\leq m, j\leq n,\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon\right\}\right| \geq \frac{1}{(mn)^\gamma}\left|\left\{(i,j):\ i\leq m, j\leq n,\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon\right\}\right|.$$

Then for $\delta>0$,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{(mn)^\gamma}\left|\left\{(i,j):\ i\leq m, j\leq n,\ |\mu_x(U_{ij},V_{ij})-L|\geq\varepsilon\right\}\right|\geq\delta\right\} \subseteq \left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{(mn)^\eta}\sum_{i,j=1,1}^{m,n}|\mu_x(U_{ij},V_{ij})-L|^p\geq\varepsilon^p\,\delta\right\}.$$

Consequently, by our assumption, we get that the double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple $L$ of order $\gamma$. □

If we take $\gamma=\eta$ in Theorem 13, we obtain the following:

**Corollary 14.** *If double sequences $\{U_{ij}\}$ and $\{V_{ij}\}$ are Wijsman asymptotical strongly $p-\mathcal{I}_2$-Cesàro equivalent to multiple $L$ of order $\eta$, then the double sequences are Wijsman asymptotical $\mathcal{I}_2$-statistically equivalent to multiple $L$ of order $\eta$ where $0<\eta\leq 1$ and $0<p<\infty$.*

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