



A METRIC FORMULA ON A QUOTIENT SPACE WHICH IS RELATED TO THE SEQUENCE SPACE Σ_2

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ABSTRACT. In this paper, we first define an equivalence relation on the sequence space Σ_2 . Then we equip the quotient set Σ_2/\sim with a metric d_1 . We also determine an isometry map between the metric spaces $(\Sigma_2/\sim, d_1)$ and $([0, 1], d_{eucl})$. Finally, we investigate the symmetry conditions with respect to some points on the metric space $(\Sigma_2/\sim, d_1)$ and we compare truncation errors for the computations which is obtained by the metrics d_{eucl} and d_1 .

1. INTRODUCTION

Let Σ_n denote the set of all infinite sequences of 0's, 1's, ..., $n - 1$'s. That is,

$$\Sigma_n = \{s_1 s_2 s_3 \dots \mid s_i \in \{0, 1, \dots, n - 1\}\}$$

(see [1, 2, 8]). This set is called the sequence space on the n symbols $0, 1, \dots, n - 1$. (Σ_n, d) is the code space on n symbols with the metric

$$d(s, t) = \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{n^i} \quad \text{for all } s, t \in \Sigma_n. \quad (1)$$

A semi-metric on Σ_n is also defined as follows:

$$d'(s, t) = \left| \sum_{i=1}^{\infty} \frac{s_i - t_i}{n^i} \right| \quad \text{for all } s, t \in \Sigma_n. \quad (2)$$

It is well-known that there is the surjective mapping h between Σ_2 and $[0, 1]$ such that

$$h : \Sigma_2 \rightarrow [0, 1], \quad h(s_1 s_2 \dots s_n \dots) = \frac{s_1}{2} + \frac{s_2}{2^2} + \dots + \frac{s_n}{2^n} + \dots$$

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But, the mapping h is not injective. To exemplify, let us consider $01111\dots$ and $1000\dots$ of Σ_2 . It is clear that

$$h(01111\dots) = h(1000\dots) = \frac{1}{2}.$$

Generally, any rational number $\frac{p}{2^k} \in [0, 1]$ (p, k are integers) has exactly two preimages s', s'' under h ; they have forms $s' = s_1s_2\dots s_n\alpha\bar{\beta}$ and $s'' = s_1s_2\dots s_n\beta\bar{\alpha}$ for some $n \in \mathbb{N}, s_i, \alpha, \beta \in \{0, 1\}$ and \bar{a} stands for the sequence in Σ_2 with all the items equal to a where $a \in \{0, 1\}$. To obtain an injective mapping, an equivalence relation \sim on Σ_2 is defined by

$$s' \sim s'' \Leftrightarrow s' = s'' \text{ or there are } s_i, \alpha, \beta \in \{0, 1\} \text{ such that} \tag{3}$$

$$s' = s_1s_2\dots s_n\alpha\bar{\beta}, s'' = s_1s_2\dots s_n\beta\bar{\alpha} \text{ for an integer } n.$$

Therefore, the mapping h' defined as

$$h' : \Sigma_2/\sim \rightarrow [0, 1] \tag{4}$$

$$h'(s_1s_2\dots s_n\dots) = \frac{s_1}{2} + \frac{s_2}{2^2} + \dots + \frac{s_n}{2^n} + \dots$$

is bijective.

From now on, one can identify each element of Σ_2/\sim with an appropriate element of $[0, 1]$. That is, Σ_2/\sim can be considered as the code set of the points on $[0, 1]$. Note that, d is not well defined on Σ_2/\sim . For example, for the elements $s = 00\bar{1}$, $s' = 0\bar{1}$ and $s'' = 1\bar{0}$ ($s' \sim s''$) of Σ_2/\sim , we compute $d(s, s') \neq d(s, s'')$. Therefore, d is not a metric on Σ_2/\sim , but $(\Sigma_2/\sim, d')$ is a metric space. Obviously, the metric spaces $(\Sigma_2/\sim, d')$ and $([0, 1], d_{eucl})$ are isometric since h' is an isometry. On the other hand, different metrics can be defined on the same set X . The interesting question is whether these metrics are equivalent or not. Because equivalent metrics on the same set X generate the same topology (see [3]). It is a remarkable problem, how to define a metric d_1 , which is formulated in a different way from the metric d' given in (2) on Σ_2/\sim such that $(\Sigma_2/\sim, d_1)$ and $([0, 1], d_{eucl})$ are isometric.

Any set can be expressed as a quotient space which is related to a sequence space. Using the metrics defined on code sets, many geometric features of the corresponding space can be proved. Moreover, unusual metric formulas can be obtained on different quotient spaces. For example, the Sierpinski Gasket which is one of the fundamental models of fractals can be defined as the quotient space Σ_3/\sim where

$$c' \sim c'' \Leftrightarrow c' = c'' \text{ or there are } c_i, \alpha, \beta \in \{0, 1, 2\} \text{ such that} \tag{5}$$

$$c' = c_1c_2\dots c_n\alpha\bar{\beta}, c'' = c_1c_2\dots c_n\beta\bar{\alpha} \text{ for an integer } n.$$

(see Figure 1).

In [6], the geodesic metric on the code set of the Sierpinski Gasket is defined as follows:

Definition 1. Let $a_1a_2\dots a_{k-1}a_ka_{k+1}\dots$ and $b_1b_2\dots b_{k-1}b_kb_{k+1}\dots$ be two representations of the points a and b respectively on the Sierpinski Gasket such that $a_i = b_i$ for $i = 1, 2, \dots, k-1$ and $a_k \neq b_k$. The distance $d_{geo}(a, b)$ between a and b is determined with the following formula:

$$d_{geo}(a, b) = \min \left\{ \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i} \right\} \quad (6)$$

where

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \quad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases},$$

$$\gamma_i = \begin{cases} 0, & a_i \neq a_k \text{ and } a_i \neq b_k \\ 1, & \text{otherwise} \end{cases}, \quad \delta_i = \begin{cases} 0, & b_i \neq b_k \text{ and } b_i \neq a_k \\ 1, & \text{otherwise} \end{cases}.$$

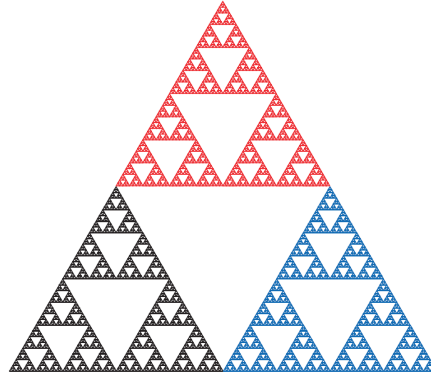


FIGURE 1. The Sierpinski Gasket

This metric formula describes the shortest distance between any two points on the code sets of the Sierpinski Gasket. Thanks to the metric formula d_{geo} , many different and interesting properties of the Sierpinski Gasket can be proved (for details see [6, 7]).

As seen in (6), this metric is actually expressed on the quotient space Σ_3/\sim' . In a similar way, a metric on the quotient space Σ_2/\sim can be defined. Note that, although relations \sim and \sim' defined in (3) and (5) are quite similar, the corresponding spaces are different.

In this paper, to define the metric d_1 we take into account the construction of the intrinsic metric on the Sierpinski Gasket (for details see proof of the Theorem 1 in the paper [5]). We use the quotient space Σ_2/\sim given in (3) instead of the quotient space Σ_3/\sim' given in (5) which is the code set of the Sierpinski Gasket. We also take the first sum, the same α_i and β_i given in the metric formula (6). Thus we obtain the metric space $(\Sigma_2/\sim, d_1)$ and show that this metric space is isometric to

the metric space $([0, 1], d_{eucl})$ in Proposition 1. Since d_1 has a different construction than the metric d_{eucl} , we compare some properties of these two metrics. So, we get some interesting results in Proposition 6 and Proposition 8.

2. A DIFFERENTLY FORMULATED METRIC ON THE QUOTIENT SET Σ_2/\sim

We first define a distance on the quotient set Σ_2/\sim and then we show that this distance function is a metric on Σ_2/\sim .

Suppose that $a = a_1a_2 \dots a_{k-1}a_k a_{k+1} \dots$ and $b = b_1b_2 \dots b_{k-1}b_k b_{k+1} \dots$ are two elements of Σ_2/\sim . If $a_i = b_i$ for all $i = 1, 2, 3, \dots$, then $d_1(a, b) = 0$. If a_i does not equal to b_i for any $i = 1, 2, 3, \dots$, then there is at least one k such that $a_k \neq b_k$. k is chosen as the smallest index $i = 1, 2, \dots$ such that $a_i \neq b_i$. That is, let $a_i = b_i$ for $i = 1, 2, \dots, k - 1$ and $a_k \neq b_k$. We define the distance $d_1(a, b)$ between a and b as

$$d_1(a, b) = \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}$$

where

$$\alpha_i = \begin{cases} 0 & \text{if } a_i = b_k \\ 1 & \text{if } a_i \neq b_k \end{cases}, \quad \beta_i = \begin{cases} 0 & \text{if } b_i = a_k \\ 1 & \text{if } b_i \neq a_k \end{cases}. \tag{7}$$

Since there are many stages to prove that $(\Sigma_2/\sim, d_1)$ is a metric space, we give it as a result of the following proposition:

Proposition 2. $d_{eucl}(h'(a), h'(b)) = d_1(a, b)$ for all $a, b \in \Sigma_2/\sim$. Thus, $(\Sigma_2/\sim, d_1)$ is a metric space.

Proof. Let us first consider the bijective mapping h' defined in (4) as follows:

$$h' : \Sigma_2/\sim \rightarrow [0, 1]$$

$$h'(s_1s_2 \dots s_n \dots) = \frac{s_1}{2} + \frac{s_2}{2^2} + \dots + \frac{s_n}{2^n} + \dots$$

We now show that

$$d_{eucl}(h'(a), h'(b)) = d_1(a, b)$$

for all $a, b \in \Sigma_2/\sim$. Suppose that

$$\begin{array}{cccccccc} a & = & a_1 & a_2 & \dots & a_{k-1} & a_k & a_{k+1} & a_{k+2} & \dots \\ b & = & b_1 & b_2 & \dots & b_{k-1} & b_k & b_{k+1} & b_{k+2} & \dots \end{array}$$

where $a_i = b_i$ for $i = 1, 2, \dots, k - 1$ and $a_k \neq b_k$. So, we have either $a_k > b_k$ or $a_k < b_k$. Assume that $a_k < b_k$ (that is, chosen as $a_k = 0$ and $b_k = 1$). Then,

$|a_{i+1} - b_k| = b_k - a_{i+1}$ and $|b_{i+1} - a_k| = b_{i+1} - a_k$ for $i = k, k+1, k+2, \dots$. The other case is done similarly. Observe that

$$\begin{aligned}
d_1(a, b) &= \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i} \\
&= \frac{|a_{k+1} - b_k| + |b_{k+1} - a_k|}{2^{k+1}} + \frac{|a_{k+2} - b_k| + |b_{k+2} - a_k|}{2^{k+2}} + \dots \\
&= \frac{(b_k - a_{k+1}) + (b_{k+1} - a_k)}{2^{k+1}} + \frac{(b_k - a_{k+2}) + (b_{k+2} - a_k)}{2^{k+2}} + \dots \\
&= \frac{(b_k - a_k) + (b_{k+1} - a_{k+1})}{2^{k+1}} + \frac{(b_k - a_k) + (b_{k+2} - a_{k+2})}{2^{k+2}} + \dots \\
&= (b_k - a_k) \left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \dots \right) + \frac{(b_{k+1} - a_{k+1})}{2^{k+1}} + \frac{(b_{k+2} - a_{k+2})}{2^{k+2}} + \dots \\
&= \frac{(b_k - a_k)}{2^k} + \frac{(b_{k+1} - a_{k+1})}{2^{k+1}} + \frac{(b_{k+2} - a_{k+2})}{2^{k+2}} + \dots \\
&= \left(\frac{b_k}{2^k} + \frac{b_{k+1}}{2^{k+1}} + \frac{b_{k+2}}{2^{k+2}} + \dots \right) - \left(\frac{a_k}{2^k} + \frac{a_{k+1}}{2^{k+1}} + \frac{a_{k+2}}{2^{k+2}} + \dots \right) \\
&= h'(b) - h'(a) \\
&= |h'(a) - h'(b)| \\
&= d_{eucl}(h'(a), h'(b))
\end{aligned}$$

So, $(\Sigma_2/\sim, d_1)$ is a metric space and it is also isometric to the metric space $([0, 1], d_{eucl})$. This concludes the proof. \square

Observe that if we define the mapping h'' as below, we have the following result:

$$\begin{aligned}
h'' : (\Sigma_2/\sim, d_2) &\rightarrow ([m, n], d_{eucl}) \\
h''(s_1 s_2 \dots s_n \dots) &= m + (n - m) \left(\frac{s_1}{2} + \frac{s_2}{2^2} + \dots + \frac{s_n}{2^n} + \dots \right)
\end{aligned}$$

where

$$d_2(a, b) = \sum_{i=k+1}^{\infty} \frac{\alpha'_i + \beta'_i}{2^i}$$

such that

$$\alpha'_i = \begin{cases} 0 & \text{if } a_i = b_k \\ n - m & \text{if } a_i \neq b_k \end{cases}, \quad \beta'_i = \begin{cases} 0 & \text{if } b_i = a_k \\ n - m & \text{if } b_i \neq a_k \end{cases} \quad \text{where } i \geq k + 1.$$

Corollary 3. *The metric space $(\Sigma_2/\sim, d_2)$ is isometric to the metric space $([m, n], d_{eucl})$ for any $m, n \in \mathbb{R}$ and $n > m$.*

Remark 4. One can also use the metric d_2 defined on Σ_2/\sim instead of the metric d_{eucl} defined on $[m, n]$ because of Corollary (3).

Using the metric d_1 , we can make some calculations easier than using Euclidean metric. To better understand the importance of this metric, we will give the following example:

Example 5. Consider the points $a = \overline{010100}$ and $b = \overline{101011}$ of Σ_2/\sim . Let us first compute $d_1(a, b)$ and then let us find the point c of Σ_2/\sim such that $d_1(a, b) = d_1(\overline{0}, c)$.

$$\begin{aligned}
 d_1(a, b) &= \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i} \\
 &= \left(\frac{0+0}{2^2} + \frac{1+1}{2^3} + \frac{0+0}{2^4} + \frac{1+1}{2^5} + \frac{1+1}{2^6} + \frac{1+1}{2^7} \right) + \\
 &\quad \left(\frac{0+0}{2^8} + \frac{1+1}{2^9} + \frac{0+0}{2^{10}} + \frac{1+1}{2^{11}} + \frac{1+1}{2^{12}} + \frac{1+1}{2^{13}} \right) + \dots \\
 &= \left(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} \right) + \left(\frac{1}{2^8} + \frac{1}{2^{10}} + \frac{1}{2^{11}} + \frac{1}{2^{12}} \right) + \dots \\
 &= \frac{1}{2^2} \left(1 + \frac{1}{2^6} + \frac{1}{2^{12}} + \dots \right) + \frac{1}{2^4} \left(1 + \frac{1}{2^6} + \frac{1}{2^{12}} + \dots \right) \\
 &\quad + \frac{1}{2^5} \left(1 + \frac{1}{2^6} + \frac{1}{2^{12}} + \dots \right) + \frac{1}{2^6} \left(1 + \frac{1}{2^6} + \frac{1}{2^{12}} + \dots \right) \\
 &= \frac{64}{63} \left(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} \right) \\
 &= \frac{64}{63} \cdot \frac{23}{64} = \frac{23}{63}
 \end{aligned}$$

Note that the calculation above gives us the point c of Σ_2/\sim such that $d_1(a, b) = d_1(\overline{0}, c)$. That is, the point c equals to

$$\left(\frac{0}{2^1} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} \right) + \left(\frac{0}{2^7} + \frac{1}{2^8} + \frac{0}{2^9} + \frac{1}{2^{10}} + \frac{1}{2^{11}} + \frac{1}{2^{12}} \right) + \dots$$

or equals to $\overline{010111}$ as an element of Σ_2/\sim . If we try to calculate this distance and to find the point c with the Euclidean metric, then we will have to deal with a bit more processing. Because we do not deal with subtraction and absolute value in calculations made with the metric d_1 . Moreover, we directly obtain this distance as a binary number.

In the following proposition, we emphasize a geometrical property of the metric d_1 different from the metric d_{eucl} .

Proposition 6. *Let $a = a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots$ and $b = b_1 b_2 \dots b_{k-1} b_k b_{k+1} \dots$ be arbitrary elements of Σ_2 / \sim . If $a_1 \neq b_1$ and $\alpha_i = \beta_i$ for all $i = 2, 3, \dots$, then these points are symmetric with respect to the point $0\bar{1}$ (or equivalently $1\bar{0}$).*

Proof. We must show that

$$d_1(a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots, 0\bar{1}) = d_1(b_1 b_2 \dots b_{k-1} b_k b_{k+1} \dots, 0\bar{1}).$$

Then

$$d_1(a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots, 0\bar{1}) = \sum_{i=k+1}^{\infty} \frac{\alpha'_i + \beta'_i}{2^i}$$

and

$$d_1(b_1 b_2 \dots b_{k-1} b_k b_{k+1} \dots, 0\bar{1}) = \sum_{i=k+1}^{\infty} \frac{\alpha''_i + \beta''_i}{2^i}$$

where $\alpha'_i, \alpha''_i, \beta'_i$ and β''_i are determined in accordance with (7). Since $a_1 \neq b_1$, we have

$$\alpha_i = \begin{cases} 0 & \text{if } a_i = b_1 \\ 1 & \text{if } a_i \neq b_1 \end{cases}, \quad \beta_i = \begin{cases} 0 & \text{if } b_i = a_1 \\ 1 & \text{if } b_i \neq a_1 \end{cases}$$

and $\alpha_i = \beta_i$ for all $i = 2, 3, \dots$. In both cases, we get $b_i \neq a_i$ for all $i = 2, 3, \dots$ because $a_1 = 0, b_1 = 1$ or $a_1 = 1, b_1 = 0$. Let us compute and compare the two distances between $a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots$ and $0\bar{1}$ and the distance between $b_1 b_2 \dots b_{k-1} b_k b_{k+1} \dots$ and $0\bar{1}$. Without loss of generality, suppose that $a_1 = 0$ and $b_1 = 1$. For all $i = 2, 3, \dots$ if $a_i = 1$ then $b_i = 0$ and $a_i = 1$ for all $i > 1$ yields

$$d_1(a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots, 0\bar{1}) = d_1(b_1 b_2 \dots b_{k-1} b_k b_{k+1} \dots, 0\bar{1}) = 0.$$

Otherwise, there is at least one s such that $a_s = 0$ and $a_i = 1$ for $i = 2, 3, \dots, s-1$ and then we have $b_s = 1$ and $b_i = 0$ for $i = 2, 3, \dots, s-1$. We obtain $\alpha''_i = 0$ for all $i = 2, 3, \dots, s-1$ and $\alpha''_s = 1$, and also $\beta'_i = 1$ for all $i = s+1, s+2, s+3, \dots$ and $\beta'_i = 0$ for all $i = 2, 3, 4, \dots$. Furthermore, for any $t \in \{s+1, s+2, s+3, \dots\}$, if $a_t = 1$, then we get $b_t = 0$ and thus $\alpha'_t = 0$ and $\alpha''_t = 0$. Similarly, for any $t \in \{s+1, s+2, s+3, \dots\}$, if $a_t = 0$, then we obtain $b_t = 1$ and thus $\alpha'_t = 1$ and $\alpha''_t = 1$. Therefore

$$\begin{aligned} d_1(a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots, 0\bar{1}) &= \sum_{i=s+1}^{\infty} \frac{\alpha'_i + \beta'_i}{2^i} \\ &= \frac{x_{s+1} + 1}{2^{s+1}} + \frac{x_{s+2} + 1}{2^{s+2}} + \frac{x_{s+3} + 1}{2^{s+3}} + \dots \\ &= \frac{1}{2^s} + \frac{x_{s+1}}{2^{s+1}} + \frac{x_{s+2}}{2^{s+2}} + \frac{x_{s+3}}{2^{s+3}} + \dots \end{aligned}$$

$$\begin{aligned}
 d_1(b_1 b_2 \dots b_{k-1} b_k b_{k+1} \dots, 0\bar{1}) &= \sum_{i=2}^{\infty} \frac{\alpha''_i + \beta''_i}{2^i} \\
 &= \frac{0+0}{2^2} + \frac{0+0}{2^3} + \dots + \frac{0+0}{2^{s-1}} + \frac{1+0}{2^s} \\
 &\quad + \frac{x_{s+1}+0}{2^{s+1}} + \frac{x_{s+2}+0}{2^{s+2}} + \dots \\
 &= \frac{1}{2^s} + \frac{x_{s+1}}{2^{s+1}} + \frac{x_{s+2}}{2^{s+2}} + \frac{x_{s+3}}{2^{s+3}} + \dots
 \end{aligned}$$

where $x_i \in \{0, 1\}$ for $i = s + 1, s + 2, s + 3, \dots$. This completes the proof. □

In a similar way, we can generalize Proposition 6 as follows:

Corollary 7. *Let $a = a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots$ and $b = b_1 b_2 \dots b_{k-1} b_k b_{k+1} \dots$ be arbitrary elements of Σ_2 / \sim . If $a_i = b_i = \sigma_i$ for all $i = 1, 2, \dots, k - 1$, $a_k \neq b_k$ and $\alpha_i = \beta_i$ for all $i = k + 1, k + 2, \dots$, then these points are symmetric with respect to the point $\sigma_1 \sigma_2 \dots \sigma_{k-1} 0\bar{1}$ (or equivalently $\sigma_1 \sigma_2 \dots \sigma_{k-1} 1\bar{0}$) where $x_i \in \{0, 1\}$ for $i = 1, 2, \dots, k - 1$.*

Proof. The proof will be omitted. □

The following proposition gives us the relationship between truncation errors for the computations which is obtained by the metrics d_{eucl} and d_1 . In Proposition 8, we choose a and b as

$$a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots a_n a_{n+1} a_{n+1} \dots$$

and

$$b_1 b_2 \dots b_{k-1} b_k b_{k+1} \dots b_n b_{n+1} b_{n+1} \dots$$

respectively where $a_i = b_i$ for $i = 1, 2, \dots, k - 1$ and $a_k \neq b_k$.

Proposition 8. *The truncation errors E_n and E'_{n+1} are determined such that*

$$d_1(a, b) = \sum_{i=k+1}^n \frac{\alpha_i + \beta_i}{2^i} + E_n$$

and

$$d_{eucl}(h'(a), h'(b)) = \left| \sum_{i=k}^n \frac{a_i - b_i}{2^i} + E'_{n+1} \right|.$$

Then we have

$$E_n = 2^{-n} + E'_{n+1}. \tag{8}$$

Proof. It must be either $b_k > a_k$ or $a_k > b_k$ since $a_k \neq b_k$. Without loss of generality, let us take $a_k > b_k$ and then we have $a_k = 1$ and $b_k = 0$. With simple calculations, we obtain

$$\begin{aligned}
d_1(a, b) &= \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i} \\
&= \sum_{i=k+1}^n \frac{\alpha_i + \beta_i}{2^i} + E_n \\
&= \frac{|a_{k+1} - b_k| + |b_{k+1} - a_k|}{2^{k+1}} + \frac{|a_{k+2} - b_k| + |b_{k+2} - a_k|}{2^{k+2}} \\
&\quad + \cdots + \frac{|a_n - b_k| + |b_n - a_k|}{2^n} + E_n \\
&= \frac{(a_{k+1} - b_k) + (a_k - b_{k+1})}{2^{k+1}} + \frac{(a_{k+2} - b_k) + (a_k - b_{k+2})}{2^{k+2}} \\
&\quad + \cdots + \frac{(a_n - b_k) + (a_k - b_n)}{2^n} + E_n \\
&= \frac{(a_k - b_k) + (a_{k+1} - b_{k+1})}{2^{k+1}} + \frac{(a_k - b_k) + (a_{k+2} - b_{k+2})}{2^{k+2}} \\
&\quad + \cdots + \frac{(a_k - b_k) + (a_n - b_n)}{2^n} + E_n \\
&= \left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots + \frac{1}{2^n} \right) + \frac{(a_{k+1} - b_{k+1})}{2^{k+1}} + \frac{(a_{k+2} - b_{k+2})}{2^{k+2}} \\
&\quad + \cdots + \frac{(a_n - b_n)}{2^n} + E_n \\
&= \frac{1 - 2^{k-n}}{2^k} + \frac{(a_{k+1} - b_{k+1})}{2^{k+1}} + \frac{(a_{k+2} - b_{k+2})}{2^{k+2}} + \cdots + \frac{(a_n - b_n)}{2^n} + E_n
\end{aligned}$$

Moreover, we compute

$$\begin{aligned}
d_{eucl}(h'(a), h'(b)) &= \left| \sum_{i=k}^n \frac{a_i - b_i}{2^i} + E'_{n+1} \right| \\
&= \sum_{i=k}^n \frac{a_i - b_i}{2^i} + E'_{n+1} \\
&= \frac{1}{2^k} + \frac{(a_{k+1} - b_{k+1})}{2^{k+1}} + \frac{(a_{k+2} - b_{k+2})}{2^{k+2}} + \cdots + \frac{(a_n - b_n)}{2^n} \\
&\quad + E'_{n+1}
\end{aligned}$$

We know that $d_1(a, b) = d_{eucl}(h'(a), h'(b))$ because of Proposition 2. Thus,

$$E_n = 2^{-n} + E'_{n+1}$$

is obtained if the required simplifications are made. Observe that, this equation does not depend on the choice of k and the error E_n is always positive while the error E'_{n+1} can be positive or negative. \square

As seen in the formula (8), the metric d_1 works better in some computations. An example which shows that the absolute value of truncation error E_n is less than the absolute value of truncation error E'_{n+1} is given as follows:

Example 9. Consider the points $a_1a_2 \dots a_{k-1}0111 \dots$ and $a_1a_2 \dots a_{k-1}1000 \dots$. It is obvious that truncation error E_n is always zero for every natural number n . But, truncation error E'_{n+1} equals to -2^{-n} .

3. CONCLUSION

We give a metric formula on the quotient space Σ_2/\sim . Using a similar method, metric formulas on quotient spaces, which are related to the sequence spaces Σ_n for every natural n , can be defined (for an example see ([6]). Moreover, the metric spaces which are isometric to these quotient spaces equipped with these metrics can be investigated. In this paper, we show the metric space $(\Sigma_2/\sim, d_1)$ is isometric to the $([0, 1], d_{eucl})$. The metric d_1 provides some facilities since there is no absolute value and subtraction in the metric formula (for details see Example 5, 9 and Proposition 8). The distance is also obtained as a binary number. As the metric formula d_1 is simple and understandable, it can also be used in the computer science applications (that is, the metric formula (7) is quite suitable for the basic coding commands).

The metric formula d_1 also has a different interpretation from the Euclidean metric defined on $[0, 1]$: The distance between two points in the Euclidean metric is formulated as the absolute value of the difference of the distances of these two points to the point 0 while the distance between two points in the metric d_1 is formulated as the sum of the distances of these points to the point $\frac{1}{2^k}$ where k is the smallest index such that $a_i \neq b_i$ for $i = 1, 2, 3, \dots$ (for details see Proposition 6 and Corollary 7).

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