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Research Article

Structure of rings with commutative factor rings for some ideals contained in their centers

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Abstract

This article concerns commutative factor rings for ideals contained in the center. A ring R is called *CIFC* if R/I is commutative for some proper ideal I of R with $I \subseteq Z(R)$, where $Z(R)$ is the center of R. We prove that (i) for a CIFC ring R, $W(R)$ contains all nilpotent elements in R (hence Köthe's conjecture holds for R) and $R/W(R)$ is a commutative reduced ring; (ii) R is strongly bounded if $R/N_*(R)$ is commutative and $0 \neq N_*(R) \subseteq Z(R)$, where $W(R)$ (resp., $N_*(R)$) is the Wedderburn (resp., prime) radical of R. We provide plenty of interesting examples that answer the questions raised in relation to the condition that R/I is commutative and $I \subseteq Z(R)$. In addition, we study the structure of rings whose factor rings modulo nonzero proper ideals are commutative; such rings are called *FC*. We prove that if a non-prime FC ring is noncommutative then it is subdirectly irreducible.

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Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We use $N(R)$, $J(R)$, $N_*(R)$, $N^*(R)$, and $W(R)$ to denote the set of all nilpotent elements, Jacobson radical, lower nilradical (i.e., prime radical), upper nilradical (i.e., the sum of all nil ideals), and the Wedderburn radical (i.e., the sum of all nilpotent ideals) of R, respectively. The center of R is denoted by $Z(R)$. It is well-known that $W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i,j)-entry

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1 and zeros elsewhere. I_n denotes the identity matrix in $Mat_n(R)$. $\prod$ means the direct product. Use $|S|$ to denote the cardinality of a given set S. The characteristic of R is written by $ch(R)$. An element u of R is called right (resp., left) regular if $ur = 0$ (resp., $ru = 0$) for $r \in R$ implies $r = 0$. An element is regular if it is both left and right regular. The monoid of all regular elements in R is denoted by $C(R)$.

Due to Jacobson [7], a nonzero right ideal of a ring R is called *bounded* if it contains a nonzero ideal of R. This concept has been extended in several ways. Following Faith [2], a ring is called *strongly right* (resp., *left*) *bounded* if every nonzero right (resp., left) ideal is bounded. A ring is called *strongly bounded* if it is both strongly right and left bounded. It is well-known that the class of strongly right bounded rings contains right duo rings, right subdirectly irreducible rings, right valuation rings which are not subdirectly irreducible, and bounded principal ideal domains.

In Section 1, we study the structure of rings for which factor rings are commutative by some ideals contained in centers, such rings are called CIFC; and provide a method of constructing a kind of noncommutative strongly bounded ring. Indeed we prove that that for a CIFC ring R, $W(R)$ contains all nilpotent elements in R and $R/W(R)$ is a commutative reduced ring, and that R is strongly bounded if $R/N_*(R)$ is commutative and $0 \neq N_*(R) \subseteq Z(R)$, where $W(R)$ (resp., $N_*(R)$) is the Wedderburn (resp., prime) radical of R. We provide a kind of interesting examples that answer the questions raised in relation to the condition that R/I is commutative and $I \subseteq Z(R)$. It is observed that the CIFC property goes up to polynomial rings. In Section 2 we study the structure of FC rings, focusing on the relation among FC rings, commutative rings and simple rings. We investigate that in several kinds of ring extensions that play important roles in ring theory.

A ring is usually called *reduced* if it has no nonzero nilpotents. It is easily checked that a ring R is reduced if and only if $a^2 = 0$ for $a \in R$ implies $a = 0$. A ring is called *Abelian* if every idempotent is central. Reduced rings are clearly Abelian, but not conversely by [5, Lemma 2]. Following Feller [3], a ring is called *right duo* if every right ideal is two-sided. Left duo rings are defined similarly. A ring is called *duo* if it is both left and right duo. Following [9], a ring R is called *right π-duo* provided that for any $a \in R$ there is an integer $n \geq 1$ such that $Ra^n \subseteq aR$. Left π-duo rings are defined similarly. A ring is called *π-duo* if it is both left and right π-duo. Right duo rings are clearly right π-duo but not conversely by [9, Theorem 1.7]. Right or left π-duo rings are Abelian by [9, Proposition 1.9(4)].

1. When R/I is commutative for some specific ideal I

In this section we study the structure of rings R for which R/I is commutative for some proper ideal I of R with a specific condition. We first study the structure of such R when $I \subseteq Z(R)$. A ring R will be called *CIFC* if R/I is commutative for some proper ideal I of R with $I \subseteq Z(R)$.

Lemma 1.1. *CIFC rings are π-duo.*

Proof. (1) Let R be a CIFC ring. Then R/I is commutative for some proper ideal I of R with $I \subseteq Z(R)$. If $I = 0$ then R is commutative. Assume that $I \neq 0$. Let $a \in R$. Then for any $r \in R$ we get $ar - ra \in I \subseteq Z(R)$. So $a(ar - ra) = (ar - ra)a$ and $ra^2 = 2ara - a^2r = a(2ra - ar)$ follows. This implies $Ra^2 \subseteq aR$, and hence R is right π-duo. Similarly R can be shown to be left π-duo. □

Every CIFC ring is Abelian by Lemma 1.1 and [9, Proposition 1.9(4)]. In the following we see a CIFC ring that is noncommutative.

Example 1.2. Let K be a field and $A = K\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x, y over K. Consider the ideal I of A generated by abc and set $R = A/I$, where $a, b, c \in \{f \in A \mid$ the constant term of f is zero$\}$, say B. We identify elements in A with their images in R for simplicity. Note $B^3 = 0$.

Next let J be the ideal of R generated by xy and yx. Then $J \subseteq Z(R)$ since $Js = 0 = sJ$ for all $s \in B$. Write $\bar{r} = r + J$ for all $r \in R$. Every element in R/J is of the form $\bar{k}_0 + \bar{k}_1\bar{x} + \bar{k}_2\bar{y} + \bar{k}_3\bar{x}^2 + \bar{k}_4\bar{y}^2$, where $k_i \in K$. So R/J is commutative since R/J is isomorphic to $K[x, y]/(xy, x^3, y^3)$, the factor ring of the polynomial ring $K[x, y]$ modulo the ideal (xy, x^3, y^3) of $K[x, y]$ generated by xy, x^3, y^3. Therefore R is CIFC. But $xy \neq yx$ in R, so that R is noncommutative.

Next we see some conditions under which CIFC rings are commutative. Recall that *Köthe's conjecture* means "*the sum of two nil left ideals is nil*".

Theorem 1.3. (1) *Let R be a CIFC ring. Then $W(R) = N(R)$ and $R/W(R)$ is a commutative reduced ring. Especially Köthe's conjecture holds for CIFC rings.*

(2) *Let R be a CIFC ring. Then each of $R/N_*(R)$, $R/N^*(R)$ and $R/J(R)$ is a commutative ring.*

(3) *Noncommutative CIFC rings have nonzero Wedderburn radicals, and semiprime CIFC rings are commutative.*

(4) *Let R be a CIFC ring such that R/I is commutative for some proper ideal I of R with $I \subseteq Z(R)$. If $I \cap C(R) \neq \emptyset$ then R is commutative.*

Proof. (1) Since R is CIFC, R/I is commutative for some proper ideal I of R with $I \subseteq Z(R)$. Let $a \in N(R)$ with $a^n = 0$ for $n \geq 2$. We proceed with the proof based on the fact that $(ra - ar)s \in I$ for all $r, s \in R$. Let $r_i \in R$ for $i = 1, \ldots, n-1$.

Suppose $n = 2$. Then

$$0 = a^2(ra - ar)r_1 = a(ra - ar)r_1a = arar_1a, \text{ so that } aRaRa = 0.$$

This implies $(RaR)^3 = 0$.

Suppose $n = 3$. Then

$$0 = a^3(ra - ar)r_1 = a^2(ra - ar)r_1a = a^2rar_1a, \text{ so that } a^2RaRa = 0.$$

From this we can obtain

$$0 = a^3(ra - ar)r_1r_2 = a^2(ra - ar)r_1ar_2 = a(ra - ar)r_1ar_2a = arar_1ar_2a, \text{ so that}$$

$aRaRaRa = 0$. This implies $(RaR)^4 = 0$.

Suppose that $a^n = 0$ for $n \geq 2$. Then, by applying the method above, we get

$$0 = a^n(ra - ar)r_1 = a^{n-1}(ra - ar)r_1a = a^{n-1}rar_1a, \text{ so that } a^{n-1}RaRa = 0.$$

From this we can obtain

$$\begin{aligned} 0 &= a^n(ra - ar)r_1r_2 = a^{n-1}(ra - ar)r_1ar_2 = a^{n-2}(ra - ar)r_1ar_2a \\ &= a^{n-2}rar_1ar_2a - a^{n-1}rr_1ar_2a = a^{n-2}rar_1ar_2a, \end{aligned}$$

so that $a^{n-2}RaRaRa = 0$ (hence $a^{n-2}(Ra)^3 = 0$).

Now suppose by induction that

$$a^{n-k}(Ra)^{k+1} = 0 \text{ for } k < n-1.$$

Then we obtain

$$0 = a^{n-k}(ra - ar)r_1ar_2a \cdots r_kar_{k+1} = a^{n-k-1}(ra - ar)r_1ar_2 \cdots r_kar_{k+1}a =$$

$$a^{n-k-1}rar_1ar_2 \cdots r_kar_{k+1}a,$$

so that $a^{n-(k+1)}(Ra)^{k+2} = 0$. Therefore $a^{n-(n-1)}(Ra)^{(n-1)+1} = 0$ and $(RaR)^{n+1} = 0$. So $a \in W(R)$ and $N(R) = W(R)$ follows. From this we now conclude that $R/W(R)$ is reduced and Köthe's conjecture holds for R.

Next we claim that $R/W(R)$ is commutative. Assume on the contrary that $ab - ba \notin W(R)$ for some $a, b \in R$. But $ab - ba \in I$ since R/I is commutative. Moreover since $ab - ba \in I \subseteq Z(R)$, we have the following computation.

First we get

$$ba^2b - baba = ba(ab - ba) = (ab - ba)ba = ab^2a - baba,$$

entailing $ba^2b = ab^2a$. Moreover since $a(ab - ba) \in I$, we also get

$$ba^2b - baba = ba(ab - ba) = a(ab - ba)b = (ab - ba)ab = abab - ba^2b,$$

entailing $2ba^2b = abab + baba$. It then follows that $ab^2a + ba^2b = 2ba^2b = abab + baba$, and this yields

$$\begin{aligned}(ab - ba)^2 &= abab - ab^2a - ba^2b + baba \\ &= abab - (ab^2a + ba^2b) + baba = abab - (abab + baba) + baba = 0,\end{aligned}$$

so that $ab - ba \in N(R) = W(R)$, contrary to $ab - ba \notin W(R)$. This gives the desired result.

(2) This is clear from (1) since $W(R) = N_*(R) = N^*(R) \subseteq J(R)$.

(3) This is an immediate consequences of (1) and (2).

(4) Let $a, b \in R$ and take $g \in I \cap C(R)$. By hypothesis, we get $g, ag, bg \in I \subseteq Z(R)$ and hence

$$(ab - ba)g = abg - bag = bga - bga = 0.$$

But $g \in C(R)$ and so we have $ab - ba = 0$, entailing $ab = ba$. Thus R is commutative. □

We can see noncommutative CIFC rings R such that $W(R) \neq 0$ in Examples 1.2 and 1.6 to follow.

We see next other information about CIFC rings in relation to powers of elements.

Proposition 1.4. (1) *Let R be a ring and suppose that $ab - ba \in Z(R)$ for $a, b \in R$. Then*

$$a^nb - ba^n = na^{n-1}(ab - ba)$$

for any $n \geq 2$.

(2) *Let R ba a ring of $ch(R) = n \geq 2$. If $ab - ba \in Z(R)$ for $a, b \in R$ then $a^nb = ba^n$.*

(3) *Let R be a noncommutative CIFC ring of $ch(R) = n \geq 2$. Then $a^n \in Z(R)$ for all $a \in R$. Especially $a^nR = Ra^nR = Ra^n$ for all $a \in R$.*

Proof. (1) From $ab - ba \in Z(R)$, we get $a(ab - ba) = (ab - ba)a$ and $a^2b - aba = aba - ba^2$ follows; hence $a^2b - ba^2 = 2aba - 2ba^2 = 2(ab - ba)a = 2a(ab - ba)$. We proceed by induction on n. Assume that $a^kb - ba^k = ka^{k-1}(ab - ba)$ for $k \geq 2$. Note $ka^{k-1}(ab - ba) = k(ab - ba)a^{k-1} = kaba^{k-1} - kba^k$.

From $a^k(ab - ba) = (ab - ba)a^k$, we get $a^{k+1}b - a^kba = aba^k - ba^{k+1}$. This yields

$$\begin{aligned}a^{k+1}b - ba^{k+1} &= a^kba + aba^k - 2ba^{k+1} = (a^kb + aba^{k-1} - 2ba^k)a \\ &= (\underline{a^kb - ba^k} + aba^{k-1} - ba^k)a = (kaba^{k-1} - kba^k + aba^{k-1} - ba^k)a \\ &= [(k + 1)aba^{k-1} - (k + 1)ba^k]a = (k + 1)(ab - ba)a^k \\ &= (k + 1)a^k(ab - ba).\end{aligned}$$

This completes the proof.

(2) This is obtained from (1).

(3) Let $a \in R$. Then by the proof of Theorem 1.3(1), $ar - ra \in Z(R)$ for all $r \in R$. This implies $a^n r = ra^n$ by (2), so that $a^n \in Z(R)$. It then follows that $a^n R = Ra^n R = Ra^n$. $\square$

The ring R in Example 1.6 to follow is an example of Proposition 1.4(3). Indeed, $Z(R) = \mathbb{Z}_2 + B^2$ where $B = \{f \in R \mid$ the constant term of f is zero$\}$. So $f^2 \in Z(R)$ for all $f \in R$ since $ch(R) = 2$. Moreover this ring R is a strongly bounded ring that is neither right nor left duo.

We see a condition under which a kind of CIFC rings are strongly bounded in the result below.

Theorem 1.5. *Let R be a ring such that $0 \neq N_*(R) \subseteq Z(R)$ and $R/N_*(R)$ is commutative. Then R is strongly bounded.*

Proof. Let K be a nonzero right ideal of R. Suppose $K \cap N_*(R) \neq 0$. Then, for every $0 \neq a \in K \cap N_*(R)$, we have $RaR = aR \subseteq K$ since $a \in Z(R)$.

Suppose $K \cap N_*(R) = 0$. Write $\bar{R} = R/N_*(R)$. Since $\bar{R}$ is commutative, $(K + N_*(R))/N_*(R)$ is an ideal of $\bar{R}$. This yields $RK \subseteq K + N_*(R)$, and furthermore we have

$$RK^2 = RKK \subseteq (K + N_*(R))K = K^2 + N_*(R)K = K^2 + KN_*(R) \subseteq K^2 + K = K$$

because $N_*(R) \subseteq Z(R)$. But $K^2 \neq 0$ because $K \cap N_*(R) = 0$ and $(K + N_*(R))/N_*(R)$ is a nonzero right ideal of the semiprime ring $\bar{R}$. In fact, if $K^2 = 0$ then $[(K+N_*(R))/N_*(R)]^2 = (K^2 + N_*(R))/N_*(R) = 0$ in $\bar{R}$; hence $K^2 \subseteq N_*(R)$ and $K \subseteq N_*(R)$ follows, contrary to $K \cap N_*(R) = 0$. Thus RK^2 is a nonzero ideal of R. Therefore R is strongly right bounded. R being strongly left bounded can be proved similarly. $\square$

It is easily checked that Theorem 1.5 also holds for $N^*(R)$ in place of $N_*(R)$. For Theorem 1.5, it is natural to ask whether the CIFC ring R is commutative when $0 \neq N_*(R) \subseteq Z(R)$ and $R/N_*(R)$ is commutative. But the answer is negative as follows. For an element $\alpha = \sum_{g \in G} a_g g$ in a monoid ring, we write $supp(\alpha) = \{g \in G \mid a_g \neq 0\}$, the support of α. Moreover we provide a method of constructing a kind of noncommutative strongly bounded ring through the ring below.

Example 1.6. Let $A = \mathbb{Z}_2\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x, y over $\mathbb{Z}_2$. Set $R = A/I$, where I is the ideal of A generated by the following subset:

$$\{a_1 a_2 \cdots a_n - a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} \mid n \geq 3,\ a_i \in \{x, y\} \text{ and } \sigma \in S_n\},$$

where S_n is the symmetric group on n letters.

We identify x and y with their images in R. As relations are homogeneous, R is graded and we can consider the degree of monomials in R.

For a monomial $m = a_1 a_2 \cdots a_k \in R$ (where $k \geq 1$) and $a_1, a_2, \ldots, a_k \in \{x, y\}$, we denote by $m(x)$ the number of x's appearing in m and by $m(y)$ the number of y's appearing in m.

Moreover, for a fixed presentation of an element $\alpha \in R$ and nonnegative integers s, t, we denote by $\alpha_{s,t}$ the number of monomials m in α such that $m(x) = s$ and $m(y) = t$.

`Remark 1`. Obviously if $s + t \geq 3$ then $\alpha_{s,t} \in \{0, 1\}$. But for example $\alpha = xy + yx$ is nonzero and $\alpha_{1,1} = 2$.

`Claim 1`. An element $\alpha \in R$ belongs to $N_*(R)$ if and only if for any presentation of α and any positive integers s, t we have $\alpha_{s,t}$ is an even number and $1 \notin supp(\alpha)$.

Proof. Assume that $1 \notin supp(\alpha)$ and for any presentation of α and any positive integers s, t we have $\alpha_{s,t}$ is an even number. Then by Remark 1, $\alpha = xy + yx$ or $\alpha = 0$, so that $(R\alpha R)^2 = 0$.

Now, suppose $0 \neq \alpha \in N_*(R)$. It should be obvious that $1 \notin supp(\alpha)$. Notice that either $\alpha = \beta + (xy + yx)$ or $\alpha = \beta$ where $\beta_{s,t}$ is an odd number for all nonnegative integers s, t. We will show that $\beta = 0$. Obviously, in any case $\alpha^2 = \beta^2$. Suppose for a contradiction that $\beta \neq 0$. Let s be the biggest number such that $\beta_{s,t} \neq 0$ for some nonnegative integer t. Between all such t's we choose one which is the biggest and we call it q. It is not difficult to see that $(\beta^2)_{2s,2q} = 1$ and for any $w \geq 2$, $(\beta^w)_{ws,wq} = 1$ which gives β is not nilpotent. Therefore $\alpha = xy + yx$ or $\alpha = 0$.

Remark 2. In fact, by the above consideration, $N_*(R) = \{0, xy+yx\} = N^*(R) = W(R)$.

Now, by the relations defining R, $N_*(R)$ belongs to $Z(R)$ since $(xy+yx)y = y(xy+yx) = (xy + yx)x = x(xy + yx) = 0$.

Moreover, $R/N_*(R)$ is commutative because $R/N_*(R)$ is isomorphic to the polynomial ring $\mathbb{Z}_2[x, y]$. Finally $xy \neq yx$ in R, so that R is noncommutative; and R is strongly bounded by Theorem 1.5.

Right duo rings are seated between commutative rings and strongly right bounded rings. So one may ask whether R is right duo in Theorem 1.5. But the answer in negative by Example 1.6. Indeed, $Rx \nsubseteq xR$ and $Rx \nsupseteq xR$.

The affirmative result below is compared with the negative one in Example 2.9 (i.e., the FC property is not preserved by polynomial (power series) rings).

Proposition 1.7. *If a ring R is CIFC then so is $R[x]$.*

Proof. Let R be a CIFC ring. Then R/I is commutative for some proper ideal I of R with $I \subseteq Z(R)$. So $R[x]/I[x]$ is commutative through $\frac{R}{I}[x] \cong \frac{R[x]}{I[x]}$. Since $I \subseteq Z(R)$ and $I \subsetneq R$, $I[x]$ is a proper ideal of $R[x]$ such that $I[x] \subseteq Z(R[x])$. Thus $R[x]$ is CIFC. □

We next consider the condition that for a ring R, R/I is commutative for some proper ideal I of R such that I is a commutative subring of R without identity; and ask whether R is commutative in this situation. We answer this question negatively in the following which shows that such noncommutative rings are of various kinds. $\lfloor - \rfloor$ denotes the greatest integer function (i.e., floor function).

Example 1.8. (1) Let S be a commutative ring and $R = T_2(S)$. Let $I = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$. Then I is a commutative ring because $I^2 = 0$, and moreover $R/I \cong S \times S$. But R is noncommutative.

(2) Consider a subring of $T_n(\mathbb{Z}_{m^k})$, where $k, m, n \geq 2$. Let $l = \lfloor \frac{k+1}{2} \rfloor$. Define

$$R = \{(a_{ij}) \in T_n(\mathbb{Z}_{m^k}) \mid a_{ij} \in m^l\mathbb{Z}_{m^k} \text{ for all } i, j \text{ with } i < j\}.$$

Let $I = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}$. Since $I^2 = 0$, I is a commutative ring. Furthermore R/I is isomorphic to $\prod_{i=1}^{n} R_i$ with $R_i = \mathbb{Z}_{m^k}$ for all i.

(3) Let $A = \mathbb{Z}_4\langle x, y\rangle$ be the free algebra in x, y over $\mathbb{Z}_4$. Then $A \cong \mathbb{Z}_4[x] *_{\mathbb{Z}_4} \mathbb{Z}_4[y]$, the ring coproduct of $\mathbb{Z}_4[x]$ and $\mathbb{Z}_4[y]$ over $\mathbb{Z}_4$. Let

$$B = \{f \in A \mid \text{the constant term of } f \text{ is zero}\}$$

and define

$$R = \{(a_{ij}) \in D_n(A) \mid a_{ii} \in \mathbb{Z}_4[x] \text{ and } a_{ij} \in 2B \text{ for all } i, j \text{ with } i < j\},$$

where $n \geq 2$. Let $I = \{(a_{ij}) \in R \mid a_{ii} = 0\}$. Then $I^2 = 0$ and so I is a commutative ring. Moreover R/I is isomorphic to $\mathbb{Z}_4[x]$. But R is noncommutative as can be seen by

$$(xI_n)((2y)E_{12}) = (2xy)E_{12} \neq (2yx)E_{12} = ((2y)E_{12})(xI_n).$$

(4) The argument in (3) can be extended to the case of $T_n(A)$. We write this for completeness. Define

$$R = \{(a_{ij}) \in T_n(A) \mid a_{ii} \in \mathbb{Z}_4[x] \text{ for all } i \text{ and } a_{ij} \in 2B \text{ for all } i, j \text{ with } i < j\},$$

where $n \geq 2$. Let $I = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}$. Then $I^2 = 0$ and so I is a commutative ring. Moreover R/I is isomorphic to $\prod_{i=1}^n R_i$ with $R_i = \mathbb{Z}_4[x]$ for all i. But R is noncommutative by the same computation as in (3).

(5) Let A be the same free algebra as in (3) and I be the ideal of A generated by x^2. Set $A_1 = A/I$ and identify x, y with their images in A_1 for simplicity. Then $A_1 \cong \frac{\mathbb{Z}_4[x]}{\langle x^2\rangle} *_{\mathbb{Z}_4} \mathbb{Z}_4[y]$, the ring coproduct of $\frac{\mathbb{Z}_4[x]}{\langle x^2\rangle}$ and $\mathbb{Z}_4[y]$ over $\mathbb{Z}_4$, where $\langle x^2\rangle$ is the ideal of $\mathbb{Z}_4[x]$ generated by x^2. Next let $B_1 = \{f \in A_1 \mid \text{the constant term of } f \text{ is zero}\}$. Consider the subring

$$R = \{(a_{ij}) \in T_n(A_1) \mid a_{ii} \in \frac{\mathbb{Z}_4[x]}{\langle x^2\rangle} \text{ for all } i \text{ and } a_{ij} \in 2B_1 \text{ for all } i, j \text{ with } i < j\}$$

of $T_n(A_1)$, where $n \geq 2$. Let $I = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}$. Then $I^2 = 0$ and so I is a commutative ring. Moreover R/I is isomorphic to $\prod_{i=1}^n R_i$ with $R_i = \frac{\mathbb{Z}_4[x]}{\langle x^2\rangle}$ for all i. But R is noncommutative as can be seen by

$$(xI_n)((2y)E_{1n}) = (2xy)E_{1n} \neq (2yx)E_{1n} = ((2y)E_{1n})(xI_n).$$

2. When factor rings are commutative

In this section we are concerned with the class of rings whose factor rings modulo nonzero proper ideals are commutative. We study the structure of such rings in relation to several ring extensions which play an important role in ring theory.

Example 2.1. (1) Let F be a field and $R = T_2(F)$. A nonzero proper ideal of R is one of the following: $I_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, and $I_3 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then $R/I_1 \cong F \times F$, $R/I_2 \cong F \cong R/I_3$. But R is noncommutative.

(2) Let A be any commutative ring and $R = T_n(A)$ for $n \geq 3$. Consider the nonzero ideal $I = AE_{1n}$. Then R/I is isomorphic to the ring $R_1 = \{(a_{ij}) \in R \mid a_{1n} = 0\}$, with usual addition and multiplication with $a_{1i}b_{in} = 0$ for all $i = 1, 2, \ldots, n-1$, where $(a_{ij}), (b_{ij}) \in R_1$. Thus R_1 is noncommutative as can be seen by $E_{11}E_{12} = E_{12} \neq 0 = E_{12}E_{11}$.

(3) Let A be any commutative ring and $R = D_n(A)$ for $n \geq 4$. Consider the nonzero ideal $I = AE_{1n}$. Then R/I is isomorphic to the ring $R_2 = \{(a_{ij}) \in R \mid a_{1n} = 0\}$, with usual addition and multiplication with $a_{1i}b_{in} = 0$ for all $i = 1, 2, \ldots, n-1$, where $(a_{ij}), (b_{ij}) \in R_2$. R_2 is noncommutative as can be seen by $E_{12}E_{23} = E_{13} \neq 0 = E_{23}E_{12}$.

We consider the following notion, based on Example 2.1.

Definition 2.2. A ring R is called *FC* if R is simple, or else R/I is a commutative ring for every nonzero proper ideal I of R.

Commutative rings and simple rings are FC, but the converses are not true in general by Example 2.1(1). $T_n(A)$ (resp., $D_n(A)$) is not FC over any ring A by Example 2.1(2) (resp., Example 2.1(3)) when $n \geq 3$ (resp., $n \geq 4$). Moreover we will see that the concepts of FC and CIFC are independent of each other by Remark 2.5 to follow.

Following Birkhoff [1], a ring R is called *subdirectly irreducible* if the intersection of all nonzero ideals in R is nonzero. It is obvious that a ring R is subdirectly irreducible if and only if for every set of nonzero proper ideals of R, $\{K_l \mid l \in L\}$ say, we have $\cap_{l \in L} K_l \neq 0$. We will use this fact freely. It is proved in [1] that any ring is isomorphic to a subdirect product of subdirectly irreducible rings.

Lemma 2.3. *Let R be a non-prime FC ring.*

(1) *If R is not subdirectly irreducible, then R is commutative. Equivalently, if R is noncommutative then R is subdirectly irreducible.*

(2) *$R/N_*(R)$ is a subdirect product of commutative domains, and $R/J(R)$ is a subdirect product of fields.*

(3) *If R is semiprime then R is a commutative reduced ring.*

Proof. (1) Let R be not subdirectly irreducible. Then there exist nonzero proper ideals J_i $(i \in I)$ such that $\cap_{i\in I} J_i = 0$. So R is a subdirect product of R/J_i's. Since R is FC, every R/J_i is commutative and hence the direct product of R/J_i's is commutative. Therefore R is commutative.

(2) Let P_i $(i \in I)$ be all prime ideals of R. Every P_i is nonzero since R is not prime. Since R is FC, every R/P_i is a commutative prime ring (hence a domain). Thus the result follows. The remainder follows immediately since commutative primitive rings are fields.

(3) is an immediate consequence of (2). □

There exists a non-prime commutative ring (hence FC) that is not subdirectly irreducible. In fact, the ring $\mathbb{Z}_{pq}$ is not subdirectly irreducible because $p\mathbb{Z}_{pq} \cap q\mathbb{Z}_{pq} = 0$, where p and q are distinct prime numbers. This elaborates on Lemma 2.3(1).

Following [11], a ring R is called *right* (resp., *left*) *quasi-duo* if every maximal right (resp., left) ideal of R is two-sided. A ring is called *quasi-duo* if it is both right and left quasi-duo. It is obvious that a ring R is right quasi-duo if and only if $R/J(R)$ is right quasi-duo. Right π-duo rings are right quasi-duo by [9, Proposition 1.9(1)], entailing that right duo rings are right quasi-duo. It is proved by [4, Proposition 1] that a ring R is right quasi-duo if and only if every right primitive factor ring of R is a division ring.

Theorem 2.4. *A non-prime FC ring is either commutative or its right (left) primitive factor rings are fields. In particular, every non-prime FC ring is quasi-duo.*

Proof. Let R be a non-prime FC ring. Suppose $J(R) = 0$. Then R is commutative by Lemma 2.3(3). Suppose that $J(R) \neq 0$ and R is noncommutative. Note that every right (left) primitive ideal of R is nonzero, say P. Thus R/P is commutative because R is FC, so that R/P is a field.

It then follows from the preceding result and [4, Proposition 1] that every non-prime FC ring is quasi-duo. □

The following elaborates on Theorem 2.4.

Remark 2.5. (1) By Theorem 2.4, if a non-prime FC ring R is noncommutative then $R/J(R)$ is a commutative reduced ring.

(2) Simple (hence FC) rings need not be quasi-duo by the existence of simple domains which are not division rings (e.g., the first Weyl algebra over a field of characteristic zero), which is related to the second statement of Theorem 2.4. Indeed this domain is neither right nor left quasi-duo.

(3) There exist non-prime noncommutative FC rings as can be seen by $T_2(K)$ over a field K (see Example 2.1(1)). This provides examples to Theorem 2.4.

(4) Based on Theorem 2.4, one may ask whether a non-prime quasi-duo ring is FC. But the answer is negative. Let A be a right quasi-duo ring and $R = T_n(A)$ for $n \geq 3$. Then R is right quasi-duo by [11, Proposition 2.1]. As in Example 2.1(2), let $I = AE_{1n}$. Then R/I is noncommutative by the argument in Example 2.1(2), and so R is not FC. Furthermore, there exists a non-prime duo ring which is not FC. It is easily checked that $D[[x]] \times D[[x]]$ is a non-prime duo ring over any noncommutative division ring D. But $(D[[x]] \times D[[x]])/(D[[x]] \times xD[[x]]) \cong D$ is noncommutative, hence $D[[x]] \times D[[x]]$ is not FC.

(5) Noncommutative FC rings need not be CIFC. Let R be a noncommutative simple (hence FC) ring. Then since $R/0 \cong R$ is noncommutative, R cannot be CIFC. Next there exists a noncommutative non-simple FC ring that is not CIFC. Consider $R = T_2(F)$ over a field F. Then R is noncommutative non-simple ring that is FC by Example 2.1(1). It is well-known that $Z(R) = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in F\}$. So the only proper ideal of R that is contained in $Z(R)$ is the zero ideal. But $R/0 \cong R$ is noncommutative, and hence R is not CIFC.

(6) Noncommutative CIFC rings need not be FC. Consider the non-simple CIFC ring R in Example 1.2 to follow. Let K be the ideal of R generated by x^2. Then R/K is noncommutative since $\bar{x}\bar{y} \neq \bar{y}\bar{x}$; hence R is not FC.

The following provides useful information about semiprime rings and FC rings.

Proposition 2.6. *Let R be a non-prime semiprime ring. Then the following are equivalent:*

(1) *R is FC;*
(2) *Every prime factor ring is commutative;*
(3) *R is a commutative reduced ring;*
(4) *R is commutative.*

Proof. (1) $\Rightarrow$ (3) is proved by Lemma 2.3(3). (2) $\Rightarrow$ (4) is obtained from the fact that R is a subdirect product of prime factor rings. (3) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are obvious. □

The condition "semiprime" in Proposition 2.6 is not superfluous by Example 2.1(1). One may ask whether the condition, that every primitive factor ring is commutative, is also equivalent to commutativity in Proposition 2.6. But the following answers this negatively.

Example 2.7. We apply the construction and argument in [6, Example 1.2] and [8, Theorem 2.2(2)]. Let K be a field and $R_n = D_{2^n}(K)$ for $n \geq 1$ with the function $\sigma : R_n \to R_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Set $R = \bigcup_{n=1}^{\infty} R_n$, noting that R_n can be considered as a subring of R_{n+1} via σ. Then R is a semiprime ring by [8, Theorem 2.2(2)]. But

$$J(R) = N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\} \text{ and } R/J(R) \cong K.$$

This implies that $J(R)$ is maximal (hence primitive), entailing that every primitive factor ring of R is commutative. But R is not commutative.

Next we study the relation between FC and commutativity in several kinds of ring extensions.

Theorem 2.8. (1) *A ring R is commutative if and only if $R[x]$ is FC if and only if $R[[x]]$ is FC.*

(2) *Let R_i be rings for all $i \in I$, and $R = \prod_{i \in I} R_i$, where $|I| \geq 2$. The following conditions are equivalent:*

(i) *R is FC;*
(ii) *R_i is commutative for all $i \in I$;*
(iii) *R is commutative.*

Proof. (1) Suppose $R[x]$ is FC and consider the nonzero proper ideal $R[x]x$. Then $R[x]/R[x]x$ is commutative. But $R[x]/R[x]x$ is isomorphic to R, hence R is commutative. The proof for $R[[x]]$ is almost the same as in the case of $R[x]$. The remainder of the proof is obvious.

(2) (i) $\Rightarrow$ (ii). Suppose R is FC. Let $j \in I$ and $I_j = \{(a_i)_{i\in I} \in R \mid a_j = 0\}$. Then I_j is a nonzero proper ideal of R such that R/I_j is isomorphic to R_j. Since R is FC, R_j is commutative. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are obvious. □

By help of Theorem 2.8, the FC property is not preserved by polynomial (power series) rings by the existence of noncommutative non-simple FC rings (e.g., see Example 2.1(1)), which is compared with Proposition 1.7 (i.e., if a ring R is CIFC then so is $R[x]$). Furthermore, also over simple rings, the FC property is not preserved by polynomial (power series) rings as in the following.

Example 2.9. Let R be any noncommutative simple ring (e.g., the first Weyl algebra over a field of characteristic zero). Consider $R[x]$ and the nonzero maximal ideal $R[x]x$. Then $R[x]/R[x]x$ is isomorphic to R, and is noncommutative. So $R[x]$ is not FC. For the case of $R[[x]]$, we use the maximal ideal $R[[x]]x$ to obtain $\frac{R[[x]]}{R[[x]]x} \cong R$.

One can compare this result with the fact that if $R[x]$ is right quasi-duo over a domain R then R is commutative [10, Theorem 3.3].

In the following we argue about the FC property of $T_n(R)$ for $n = 2$ and $D_n(R)$ for $n \leq 3$, based on Example 2.1(2, 3).

Theorem 2.10. *Let R be a ring and $n \geq 2$.*

(1) *R is simple if and only if $Mat_n(R)$ is FC if and only if $Mat_n(R)$ is simple.*
(2) *R is commutative if and only if $D_2(R)$ is FC.*
(3) *The following conditions are equivalent:*
  (i) *R is a field;*
  (ii) *$T_2(R)$ is an FC ring;*
  (iii) *$D_3(R)$ is an FC ring.*

Proof. (1) It suffices to show that if $Mat_n(R)$ is FC then R is simple. Let R be non-simple. Consider a nonzero proper ideal I of R. Then $Mat_n(R)/Mat_n(I)$ is isomorphic to $Mat_n(R/I)$ that is noncommutative. So $Mat_n(R)$ is not FC.

(2) Suppose that $D_2(R)$ is FC. Then, letting $I = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$, $R \cong D_2(R)/I$ is commutative. The converse is obvious.

(3) (i) $\Rightarrow$ (ii). If R is a field then $T_2(R)$ is FC by Example 2.1(1).

(ii) $\Rightarrow$ (i). Suppose that $T_2(R)$ is FC. If R is not simple then $T_2(R)/T_2(M)$ is isomorphic to the noncommutative ring $T_2(R/M)$ for each maximal ideal M of R, entailing that $T_2(R)$ is not FC. Thus R must be simple. Next consider the proper ideal $I = \begin{pmatrix} 0 & R \\ 0 & R \end{pmatrix}$ of $T_2(R)$. Then $T_2(R)/I$ is isomorphic to R, and hence commutative because $T_2(R)$ is FC. Summarizing, R is a field.

(i) $\Rightarrow$ (iii). Let R be a field. Then the proper ideals of $D_3(R)$ are one of the following: $\begin{pmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & R & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & R \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & R & R \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}$. So the factor rings modulo by these ideals are commutative. Hence $D_3(R)$ is FC.

(iii) $\Rightarrow$ (i). Suppose that $D_3(R)$ is FC. If R is not simple then $D_3(R)/D_3(M)$ is isomorphic to the noncommutative ring $D_3(R/M)$ for a maximal ideal M of R. So $D_3(R)$ is not FC. Thus R must be simple. Next consider the proper ideal $I = \begin{pmatrix} 0 & R & R \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}$ of $D_3(R)$. Then $D_3(R)/I$ is isomorphic to R, and hence commutative. Consequently, R is a field. □

The condition "R is FC" cannot be equivalent to the conditions in Theorem 2.10(1) because $Mat_n(R)$ $(n \geq 2)$ cannot be FC when an FC ring R is non-simple.

Recall that right π-duo rings are quasi-duo. So it is natural to consider the implications between FC rings and (right) π-duo rings. $Mat_n(A)$ is simple (hence FC) over any simple ring A for all $n \geq 2$, but this FC ring is not right π-duo since one-sided π-duo rings are Abelian by [9, Proposition 1.9(4)]. Right π-duo rings are also need not be FC as can be seen by the duo ring $D[[x]]$ over a noncommutative division ring D.

The FC property is not closed under neither direct products nor subrings as follows.

Example 2.11. (1) Let K be a field. Then $T_2(K)$ is FC by Example 2.1. But $R = T_2(K) \times T_2(K)$ is not FC by Proposition 2.8(2) because $T_2(K)$ is noncommutative.

(2) Let R be the first Weyl algebra over a field of characteristic zero. Consider $R[x]$. Then $R[x]$ is not FC by Example 2.9. But since $R[x]$ is a right Noetherian domain, it is contained in a quotient division ring Q which is clearly FC.

In the following we find a kind of subring which inherits the FC property.

Theorem 2.12. *Let R be a ring and $0 \neq e^2 = e \in R$. If R is FC then eRe is FC.*

Proof. Suppose that R is simple. Let J be a nonzero ideal of eRe. Then $J = eJe = eReJeRe$ implies $ReJeR \neq 0$. Since R is simple, $ReJeR = R$ and so $J = eReJeRe = eRe$. Thus eRe is simple (hence FC).

Suppose that R is FC and eRe is non-simple. Then R is non-simple by the preceding argument. In fact, we can construct a nonzero proper ideal of R from a given nonzero proper ideal of eRe as follows. Let J be a nonzero proper ideal of eRe. As in the case of R being simple, let $I = ReJeR$. Assume $I = R$. Then $J = eReJeRe = eRe$, contrary to $J \subsetneq eRe$. So I is a nonzero proper ideal of R such that $eIe = J$. Since R is FC, R/I is commutative.

Write $\bar{R} = R/I$ and $\bar{r} = r + I$ for $r \in R$. Assume $e \in I$. Then $e \in eIe = J$ and $eRe = J$ follows, contrary to $J \subsetneq eRe$. So $e \notin I$ and $\bar{e} \neq 0$ in $\bar{R}$. Next consider the epimorphism $f : eRe \to \bar{e}\bar{R}\bar{e}$ defined by $f(ere) = \bar{e}\bar{r}\bar{e}$. Since $\bar{R}$ is commutative, the subring $\bar{e}\bar{R}\bar{e}$ of $\bar{R}$ is also commutative. So $\frac{eRe}{Ker(f)} (\cong \bar{e}\bar{R}\bar{e})$ is commutative, where $Ker(f)$ is the kernel of f. Letting $f(ere) = 0$, $\bar{e}\bar{r}\bar{e} = 0$ and $ere \in I$. This implies $ere = e(ere)e \in eIe = J$, entailing $ere \in J$. So $Ker(f) \subseteq J$. Moreover $J = eJe = eIe \subseteq I$ and $J \subseteq Ker(f)$ follows. Consequently we have $Ker(f) = J$, and hence $(eRe)/J$ is commutative. Therefore eRe is FC. □

The converse of the first part of the proof of Theorem 2.12 is not true in general. Let $R = A \times A$ with A a simple ring, and $e = (1, 0) \in R$. Then $eRe \cong A$ is simple, but R is not simple.

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