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A CONSTRUCTION OF A CONGRUENCE IN A UP-ALGEBRA BY A PSEUDO-VALUATION

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Abstract. In our recently published paper, we study pseudo-valuations on UP-algebras and obtain some related results. In this article, we use a pseudo-metric induced by a pseudo-valuation to introduce a congruence relation on a UP-algebra. In addition, we construct the quotient algebra induced by this relation and prove that it is also a UP-algebra.

1. INTRODUCTION

The idea that universal algebra should be analyzed by means of pseudo-valuation was first developed by D. Busneag in 1996 [1]. This author has expanded the perception of pseudo-valuation on Hilbert's algebras [2]. Logical algebras and pseudo-valuations on them have become an object of interest for researchers in recent years. For example, Doh and Kang [3, 4] introduced in the concept of pseudo-valuation on BCK/BCI - algebras. Ghorbani in 2010 [5] determined a congruence on BCI-algebras based on pseudo-valuation and describe the obtained factorial structure generated by this congruence. Song, Roh and Jun described pseudo-valuation on BCK/BCI - algebras [15] and Song, Bordbar and Jun have described the quotient structure on such algebras generated by pseudo-valuation [16]. Jun, Lee and Song analyzed in article [8] several types of quasi-valuation maps on BCK-algebra and their interactions. Also, Mehrshad and Kouhestani were interested in pseudo-valuations on BCK-algebra [10]. Jun, Ahn and Roh. in [7] described pseudo-valuation on the BCC-algebras. Koam, Haider and Ansari described in 2019 pseudo-valuations on KU-algebras [9].

The concept of UP-algebras is introduced and analyzed by Iampan in 2017 [6] as a generalization of the concept of KU-algebras. This author has participated in the analysis of the properties of UP-algebras, also (See, for example: [11, 12, 13]).

In recently published article [14], he offered one way of determining of pseudo-valuation on PU-algebras. Apart from showing he demonstrated how to construct a pseudo-metric space by such mapping.

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In this article, using the pseudo-metric induced by a pseudo-valuation the author construct the quotient algebra. In addition, it has been shown that the algebra constructed in this way is also UP-algebra.

2. PRELIMINARIES

Here we give the definition of UP-algebra and some of its substructures necessary for further work.

Definition 2.1 ([6]). *An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP- algebra if it satisfies the following axioms:*
(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, *and*
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

In A we can define a binary relation $' \leqslant '$ by

$$(\forall x, y \in A)(x \leqslant y \Longleftrightarrow x \cdot y = 0).$$

Definition 2.2 ([6]). *A non-empty subset J of a UP-algebra A is called a UP-ideal of A if it satisfies the following conditions:*
(1) $0 \in J$, *and*
(2) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \wedge y \in J) \Longrightarrow x \cdot z \in J)$.

Definition 2.3 ([11]). *Let A be a UP-algebra. A subset G of A is called a* proper UP-filter *of A if it satisfies the following properties:*
(3) $\neg(0 \in G)$, *and*
(4) $(\forall x, y, z \in A)((\neg(x \cdot (y \cdot z) \in G) \wedge x \cdot z \in G) \Longrightarrow y \in G)$.

In this section, we introduce the concept of pseudo-valuations on UP-algebras, describe the basics properties of such pseudo-valuation and construct a pseudo-metric space based on this mapping.

Definition 2.4 ([14], Definition 3.1). *A real-valued function v on a UP-algebra A is called a pseudo-valuation on A if it satisfies the following two conditions:*
(5) $v(0) = 0$, *and*
(6) $(\forall x, y, z \in A)(v(x \cdot z) \leqslant v(x \cdot (y \cdot z)) + v(y))$.
A pseudo-valuation v on a UP-algebra A satisfying the following condition:
(7) $(\forall x \in A)(v(x) = 0 \Longrightarrow x = 0)$
is called a valuation on X.

Theorem 2.1 ([14], Theorem 3.16). *Let A be a UP-algebra and v be a pseudo-valuation on A. Then the mapping $d_v : A \times A \ni (x, y) \longmapsto v(x \cdot y) + v(y \cdot x) \in \mathbb{R}$ is a pseudo-metric on A.*

3. THE MAIN RESULTS

3.1. Some important properties of pseudo-metric on UP-algebras.

Proposition 3.1. *Let v be pseudo-valuation on a UP-algebra A. Then*
(8) $(\forall x, y, z \in A)(d_v(x \cdot z, y \cdot z) \leqslant d_v(x, y))$;
(9) $(\forall x, y, z \in A)(d_v(z \cdot x, z \cdot y) \leqslant d_v(x, y))$.

Proof. Let x, y, z be arbitrary elements of A. Then the following holds

$$\begin{aligned} d_v(x \cdot z, y \cdot z) &= v((x \cdot z) \cdot (y \cdot z)) + v((y \cdot z) \cdot (x \cdot z)) \\ &\leqslant v((x \cdot z) \cdot ((y \cdot x) \cdot (y \cdot z))) + v(y \cdot x) \\ &+ v((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z))) + v(x \cdot y) \\ &= (0 + v(y \cdot x)) + (0 + v(x \cdot y)) \\ &= d_v(x, y) \end{aligned}$$

since it is $v((x \cdot z) \cdot ((y \cdot x) \cdot (y \cdot z))) = v(0) = 0$ and $v((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z))) = v(0) = 0$.

On the other hand, relying on valid inequality (4) in the article [14], we have

$$\begin{aligned} d_v(z \cdot x, z \cdot y) &= v((z \cdot x) \cdot (z \cdot y)) + v((z \cdot y) \cdot (z \cdot x)) \\ &\leqslant v((x \cdot y) \cdot ((z \cdot x) \cdot (z \cdot y)) + v(x \cdot y) \\ &+ v((y \cdot x) \cdot ((z \cdot y) \cdot (z \cdot x))) + v(y \cdot x) \\ &= (0 + v(x \cdot y)) + (0 + v(y \cdot x) = v(x \cdot y) + v(y \cdot x) \\ &= d_v(x, y). \end{aligned}$$

□

3.2. **A construction of a congruence on UP-algebra.**

Definition 3.1. *Let u be a pseudo valuation on a UP-algebra A. Define the relation $\theta_v \subseteq A \times A$ by:*

$$(\forall x, y \in A)((x, y) \in \theta_v \iff d_v(x, y) = 0)$$

Theorem 3.2. *Let v be a pseudo-valuation on a UP-algebra A. Then θ_v is a congruence relation on A.*

Proof. Since θ_v induced by a pseudo-metric d_v, it is an equivalence relation on A.

To prove that θ compatible with the internal operation in A, we assume that $x, y, z \in A$ are such that $(x, y) \in \theta$. Then $d_v(x, y) = 0$. Thus

$$0 \leqslant d_v(x \cdot z, y \cdot z) \leqslant d_v(x, y) = 0 \text{ and } 0 \leqslant d_v(z \cdot x, z \cdot y) \leqslant d_v(x, y) = 0.$$

by Proposition 3.1. Hence $d_v(x \cdot z, y \cdot z) = 0$ and $d_v(z \cdot x, z \cdot y) = 0$, which means $(x \cdot z, y \cdot z) \in \theta$ and $(z \cdot x, z \cdot y) \in \theta$.

So, θ_v is a congruence relation on A. □

For the congruence relation θ_v on UP-algebra A, constructed in this way, we say that it is induced by a pseudo-valuation v.

Proposition 3.3. *Let v be a pseudo-valuation on a UP-algebra A and θ_v be the congruence relation induced by v. Then the class $[0]_v$ in $A/\theta_v = \{[x]_v : x \in A\}$, generated by the element 0 in A, is an ideal in A.*

Proof. Obviously the following applies: $x \in [0]_v$ if and only if $(x, 0) \in \theta_v$. Then $d_v(x, 0) = 0$. This means $0 = v(x \cdot 0) + v(0 \cdot x) = v(0) + v(x) = v(x)$. Therefore, $[0]_v$ is an ideal in A, according to Theorem 3.6 in article [14]. □

Corollary 3.4. *Let v be a pseudo-valuation on a UP-algebra A and θ_v be the congruence relation induced by v. Then the set $\bigcup\{[x]_v : x \in A \wedge \neg(x \in [0]_v)\}$ is a proper filter in A.*

Proof. The proof of this corollary follows from Theorem 3.7 in article [11]. □

3.3. **The quotient $A/[0]_v$ is a UP-algebra.**

Theorem 3.5. *Let v be a pseudo-valuation on a UP-algebra A, θ_v be the congruence induced by v and $[0]_v$ be the class in A/θ_v. Then the factor-set $A/[0]_v$ is a UP-algebra.*

Proof. Let v be a pseudo-valuation on a UP-algebra A and let θ_v be the congruence on A induced by v. According to the previous proposition, class $[0]_v$ is an ideal in A. We can construct a congruence relation $\sim_v$ on A using this ideal, by Theorem 3.5 in article [6], as follows

$$(\forall x, y \in A)(x \sim_v y \iff (x \cdot y \in [0]_v \ \wedge \ y \cdot x \in [0]_v)).$$

On the other hand, this pseudo-valuation v allows us to determine the ideal $J_v = \{x \in A : v(x) = 0\}$ in A, by Theorem 3.6 in article [14]. Now, we have if $x \sim_v y$, then $x \cdot y \in [0]_v$ and $y \cdot x \in [0]_v$. This means $d_v(x \cdot y, 0) = 0$ and $d_v(y \cdot x, 0) = 0$. Thus $v((x \cdot y) \cdot 0) + v(0 \cdot (x \cdot y)) = 0$ and $v((y \cdot x) \cdot 0) + v(0 \cdot (y \cdot x)) = 0$. From here, considering (UP-2), (UP-3) and (5), we have $v(x \cdot y) = 0$ and $v(y \cdot x) = 0$. Therefore, $x \cdot y \in J_v$ and $y \cdot x \in J_v$. Without much difficulty it can be checked that the reverse deduction is true, too.

On the set $A/\theta_v = \{[x]_v : x \in A\}$, we define

$$(\forall x, y \in A)([x]_v * [y] = [x \cdot y]_v).$$

According to claim (4) of Theorem 3.7 in article [6], factor-set $A/[0]_v$ is a UP-algebra. □

4. CONCLUSION

In 2010, Ghorbani presented the idea of constructing a congruence on BCI-algebras in [5] by using the pseudo-valuation given on that algebra. That idea, 2018, was discussed by S.-Z. Song, H. Bordbar and Y. B. Jun. in [16]. This author introduced the concept of pseudo-valuations on UP-algebras in [14]. Looking at the texts [5, 16], in this article, as a continuation of [14], we introduced a congruence relation θ_v generated by a given pseudo-valuation v on the UP-algebra A.

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