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Research Article

Some bounds for the $\mathbb{A}$-numerical radius of certain 2×2 operator matrices

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Abstract

For a given bounded positive (semidefinite) linear operator A on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, we consider the semi-Hilbertian space $(\mathcal{H}, \langle\cdot,\cdot\rangle_A)$ where $\langle x, y\rangle_A := \langle Ax, y\rangle$ for every $x, y \in \mathcal{H}$. The A-numerical radius of an A-bounded operator T on $\mathcal{H}$ is given by

$$\omega_A(T) = \sup\Big\{\big|\langle Tx, x\rangle_A\big| \,;\ x \in \mathcal{H},\ \langle x, x\rangle_A = 1\Big\}.$$

Our aim in this paper is to derive several $\mathbb{A}$-numerical radius inequalities for 2×2 operator matrices whose entries are A-bounded operators, where $\mathbb{A} = \operatorname{diag}(A, A)$.

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1. Introduction and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. Let $\mathbb{B}(\mathcal{H})$ stand for the C^*-algebra of all bounded linear operators on $\mathcal{H}$. The symbol I denotes the identity operator on $\mathcal{H}$. Let $\mathbb{B}(\mathcal{H})^+$ be the cone of all positive (semi-definite) operators in $\mathbb{B}(\mathcal{H})$, i.e.,

$$\mathbb{B}(\mathcal{H})^+ = \{A \in \mathbb{B}(\mathcal{H})\,;\ \langle Ax, x\rangle \geq 0,\ \forall\, x \in \mathcal{H}\}\,.$$

In all what follows, by an operator we mean a bounded linear operator. Moreover, for $T \in \mathbb{B}(\mathcal{H})$, we denote by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ the kernel and the range of T, respectively. Furthermore, T^* is the adjoint of T. For a given linear subspace $\mathcal{M}$ of $\mathcal{H}$, its closure in the norm topology of $\mathcal{H}$ will be denoted by $\overline{\mathcal{M}}$. In addition, let $P_{\mathcal{S}}$ stand for the orthogonal projection onto a closed subspace $\mathcal{S}$ of $\mathcal{H}$.

Let $A \in \mathbb{B}(\mathcal{H})^+$. Then, A induces the following semi-inner product

$$\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C},\ (x, y) \longmapsto \langle x, y\rangle_A := \langle Ax, y\rangle = \langle A^{1/2}x, A^{1/2}y\rangle.$$

Here $A^{1/2}$ stands for the square root of A. The seminorm induced by $\langle\cdot,\cdot\rangle_A$ is given by $\|x\|_A = \|A^{1/2}x\|$ for all $x \in \mathcal{H}$. One can verify that $\|\cdot\|_A$ is a norm if and only if A is one-to-one, and that the seminormed space $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$.

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The semi-inner product $\langle\cdot,\cdot\rangle_A$ induces on the quotient $\mathcal{H}/\mathcal{N}(A)$ an inner product which is not complete unless $\mathcal{R}(A)$ is closed. However, a canonical construction due to de Branges and Rovnyak [10] (see also [15]) shows that the completion of $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathcal{R}(A^{1/2})$ endowed with the following inner product

$$\langle A^{1/2}x, A^{1/2}y\rangle_{\mathbf{R}(A^{1/2})} := \langle P_{\overline{\mathcal{R}(A)}}x, P_{\overline{\mathcal{R}(A)}}y\rangle, \quad \forall\, x, y \in \mathcal{H}. \tag{1.1}$$

For the sequel, the Hilbert space $\big(\mathcal{R}(A^{1/2}), \langle\cdot,\cdot\rangle_{\mathbf{R}(A^{1/2})}\big)$ will be denoted by $\mathbf{R}(A^{1/2})$. It is worth noting that $\mathcal{R}(A)$ is dense in $\mathbf{R}(A^{1/2})$ (see [3]). For an account of results related to the Hilbert space $\mathbf{R}(A^{1/2})$, the reader is invited to consult [3] and the references therein. By using (1.1), it can be checked that

$$\langle Ax, Ay\rangle_{\mathbf{R}(A^{1/2})} = \langle x, y\rangle_A, \quad \forall\, x, y \in \mathcal{H}. \tag{1.2}$$

Let $T \in \mathbb{B}(\mathcal{H})$. An operator $S \in \mathbb{B}(\mathcal{H})$ is said to be an A-adjoint of T if for all $x, y \in \mathcal{H}$, the identity $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ holds (see [1]). So, the existence of an A-adjoint of T is equivalent to the existence of a solution of the equation $AX = T^*A$. Notice that this kind of equations can be investigated by using a well-known theorem due to Douglas [11] which briefly says that the operator equation $TX = S$ has a bounded linear solution X if and only if $\mathcal{R}(S) \subseteq \mathcal{R}(T)$ which, in turn, equivalent to the existence of a positive number λ such that $\|S^*x\| \leq \lambda\|T^*x\|$ for all $x \in \mathcal{H}$. Furthermore, among its many solutions it has only one, denoted by Q, which satisfies $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(T^*)}$. Such Q is called the Douglas solution or the reduced solution of the equation $TX = S$. Clearly, the existence of an A-adjoint operator is not guaranteed. If we denote by $\mathbb{B}_A(\mathcal{H})$ the subspace of all operators admitting A-adjoints, then by Douglas theorem, we have

$$\mathbb{B}_A(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H})\,;\ \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}\,.$$

If $T \in \mathbb{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by $T^{\sharp_A}$. Note that, $T^{\sharp_A} = A^{\dagger}T^*A$ in which $A^{\dagger}$ is the Moore-Penrose inverse of A (see [2]). Notice that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathbb{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$ and $((T^{\sharp_A})^{\sharp_A})^{\sharp_A} = T$. Moreover, if $S \in \mathbb{B}_A(\mathcal{H})$ then $TS \in \mathbb{B}_A(\mathcal{H})$ and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$. For results concerning $T^{\sharp_A}$, we refer the reader to [1,2]. An operator $U \in \mathbb{B}_A(\mathcal{H})$ is called A-unitary if $\|Ux\|_A = \|U^{\sharp_A}x\|_A = \|x\|_A$ for all $x \in \mathcal{H}$.

An operator T is called A-bounded if there exists $\lambda > 0$ such that $\|Tx\|_A \leq \lambda\|x\|_A$, $\forall\, x \in \mathcal{H}$. An application of Douglas theorem shows that the subspace of all operators admitting $A^{1/2}$-adjoints, denoted by $\mathbb{B}_{A^{1/2}}(\mathcal{H})$, is equal the collection of all A-bounded operators, i.e.,

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H})\,;\ \exists\,\lambda > 0\,;\ \|Tx\|_A \leq \lambda\|x\|_A,\ \forall\, x \in \mathcal{H}\}\,.$$

Notice that $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathbb{B}(\mathcal{H})$ which are, in general, neither closed nor dense in $\mathbb{B}(\mathcal{H})$. Moreover, we have $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$ (see [1, 3]). Clearly, $\langle\cdot,\cdot\rangle_A$ induces a seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. Indeed, if $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then it holds that

$$\|T\|_A := \sup_{\substack{x\in\overline{\mathcal{R}(A)},\\ x\neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup\left\{\|Tx\|_A\,;\ x \in \mathcal{H},\ \|x\|_A = 1\right\} < \infty.$$

Notice that it was proved in [12] that for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ we have

$$\|T\|_A = \sup\left\{|\langle Tx, y\rangle_A|\,;\ x, y \in \mathcal{H},\ \|x\|_A = \|y\|_A = 1\right\}. \tag{1.3}$$

An important observation is that for every $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\|TS\|_A \leq \|T\|_A\|S\|_A. \tag{1.4}$$

Furthermore, the A-numerical radius of an operator $T\in\mathbb{B}(\mathcal{H})$ was firstly defined by Saddi in [20] as

$$\omega_A(T) := \sup\left\{|\langle Tx, x\rangle_A|\,;\ x\in\mathcal{H}, \|x\|_A = 1\right\}.$$

It should be emphasized that it may happen that $\|T\|_A$ and $\omega_A(T)$ are equal to $+\infty$ for some $T\in\mathbb{B}(\mathcal{H})\setminus\mathbb{B}_{A^{1/2}}(\mathcal{H})$ (see [13]). However, these quantities are equivalent seminorms on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. More precisely, it was shown in [6] that for every $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\frac{1}{2}\|T\|_A \le \omega_A(T) \le \|T\|_A. \tag{1.5}$$

Notice that if $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and satisfies $AT^2=0$, then by [13, Corollary 2] we have

$$\omega_A(T) = \frac{1}{2}\|T\|_A. \tag{1.6}$$

In addition, the A-numerical radius of semi-Hilbertian space operators satisfies the weak A-unitary invariance property which asserts that

$$\omega_A(U^\sharp TU) = \omega_A(T), \tag{1.7}$$

for every A-unitary operator $U\in\mathbb{B}_A(\mathcal{H})$ and $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$ (see [5, Lemma 3.8]). For the sequel, for any arbitrary operator $T\in\mathbb{B}_A(\mathcal{H})$, we denote

$$\Re_A(T) := \frac{T+T^{\sharp_A}}{2} \quad\text{and}\quad \Im_A(T) := \frac{T-T^{\sharp_A}}{2i}.$$

For simplicity, we will write $\Re_A^2(T)$ and $\Im_A^2(T)$ instead of $[\Re_A(T)]^2$ and $[\Im_A(T)]^2$, respectively. Also, $\omega_A^2(T)$ means $[\omega_A(T)]^2$. It has been proved in [23] that for $T\in\mathbb{B}_A(\mathcal{H})$, it holds

$$\omega_A(T) = \sup_{\theta\in\mathbb{R}}\left\|\Re_A(e^{i\theta}T)\right\|_A = \sup_{\theta\in\mathbb{R}}\left\|\Im_A(e^{i\theta}T)\right\|_A. \tag{1.8}$$

Let $T\in\mathbb{B}(\mathcal{H})$. Then, it was shown in [3, Proposition 3.6.] that $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exists a unique $\widetilde{T}\in\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_AT=\widetilde{T}Z_A$. Here, $Z_A:\mathcal{H}\to\mathbf{R}(A^{1/2})$ is defined by $Z_Ax=Ax$. It has been shown in [3,13] that for every $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$ we have

$$\|T\|_A = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \quad\text{and}\quad \omega_A(T)=\omega(\widetilde{T}). \tag{1.9}$$

Recently, the concept of the A-spectral radius of A-bounded operators has been introduced by the present author in [13] as follows:

$$r_A(T) := \inf_{n\ge 1}\|T^n\|_A^{\frac{1}{n}} = \lim_{n\to\infty}\|T^n\|_A^{\frac{1}{n}}. \tag{1.10}$$

We note here that the second equality in (1.10) is also proved in [13, Theorem 1]. Moreover, like the classical spectral radius of Hilbert space operators, it was shown in [13] that $r_A(\cdot)$ satisfies the commutativity property, which asserts that

$$r_A(TS) = r_A(ST), \tag{1.11}$$

for all $T,S\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$. In addition, the following relation between A-spectral and A-numerical radii of A-bounded operators is also proved in [13]:

$$r_A(T)\le\omega_A(T),\quad \forall\, T\in\mathbb{B}_{A^{1/2}}(\mathcal{H}). \tag{1.12}$$

An operator $T\in\mathbb{B}(\mathcal{H})$ is said to be A-selfadjoint if AT is selfadjoint, that is, $AT=T^*A$. Moreover, it was shown in [13] that if T is A-self-adjoint, then

$$\|T\|_A = \omega_A(T) = r_A(T). \tag{1.13}$$

In addition, an operator T is called A-positive if $AT \geq 0$ and we write $T \geq_A 0$. Obviously, an A-positive operator is always A-selfadjoint since $\mathcal{H}$ is a complex Hilbert space. Clearly, if $T \in \mathbb{B}_A(\mathcal{H})$ then $TT^{\sharp_A} \geq_A 0$, $T^{\sharp_A}T \geq_A 0$ and

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2. \tag{1.14}$$

If $T, S \in \mathbb{B}(\mathcal{H})$ and satisfies $T - S \geq_A 0$, then we will write $T \geq_A S$. For the sequel, if $A = I$ then $\|T\|$, $r(T)$ and $\omega(T)$ denote respectively the classical operator norm, the spectral radius and the numerical radius of an operator T. In recent years, several results covering some classes of operators on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ were extended to $(\mathcal{H}, \langle\cdot,\cdot\rangle_A)$. Of course, the extension is not trivial since many difficulties arise. For instance, as it is mentioned above, it may happen that $\|T\|_A = \infty$ for some $T \in \mathbb{B}(\mathcal{H})$. Moreover, not any operator admits an adjoint operator for the semi-inner product $\langle\cdot,\cdot\rangle_A$. In addition, for $T \in \mathbb{B}_A(\mathcal{H})$ we have $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$. So, in general $(T^{\sharp_A})^{\sharp_A} \neq T$. The reader is invited to see [5–7, 14–16, 18, 22, 23] and the references therein.

In this paper, we consider the 2×2 operator diagonal matrix $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Clearly, $\mathbb{A} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})^+$. So, $\mathbb{A}$ induces the following semi-inner product

$$\langle x, y\rangle_{\mathbb{A}} = \langle \mathbb{A}x, y\rangle = \langle x_1, y_1\rangle_A + \langle x_2, y_2\rangle_A,$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{H} \oplus \mathcal{H}$. Notice that if T_{ij} are operators in $\mathbb{B}_A(\mathcal{H})$ for all $i, j \in \{1, 2\}$. Then, it was shown in [5, Lemma 3.1] that the 2×2 operator matrix $(T_{ij})_{2\times 2} \in \mathbb{B}_{\mathbb{A}}(\mathcal{H} \oplus \mathcal{H})$ and

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} T_{11}^{\sharp_A} & T_{21}^{\sharp_A} \\ T_{12}^{\sharp_A} & T_{22}^{\sharp_A} \end{pmatrix}. \tag{1.15}$$

Very recently, several inequalities for the $\mathbb{A}$-numerical radius of 2×2 operator matrices have been established by Bhunia et al. (see [8]). This paper is devoted also to prove several new $\mathbb{A}$-numerical radius inequalities of certain 2×2 operator matrices. Some of the obtained results cover and extend the following works [9, 19, 21].

2. Results

In this section, we present our results. Throughout this section $\mathbb{A}$ is denoted to be the 2×2 operator diagonal matrix whose each diagonal entry is the positive operator A. To prove our two next results, the following lemma concerning $\mathbb{A}$-numerical radius inequalities is required. Notice that the first assertion is proved in [8] for operators in $\mathbb{B}_A(\mathcal{H})$.

Lemma 2.1. *Let $P, Q, R, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then, the following assertions hold:*

(a) $\omega_{\mathbb{A}}\left[\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\right] = \max\{\omega_A(P), \omega_A(S)\}$.

(b) $\omega_{\mathbb{A}}\left[\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\right] \leq \omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\right]$.

(c) $\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] \leq \omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\right]$.

Proof. (a) Follows by proceeding as in the proof of [8, Lemma 2.4.].

(b) Clearly we have

$$\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} = \frac{1}{2}\begin{pmatrix} P & Q \\ R & S \end{pmatrix} + \frac{1}{2}\begin{pmatrix} P & -Q \\ -R & S \end{pmatrix}. \tag{2.1}$$

Let $\mathbb{U} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$. In view of (1.15) we have $\mathbb{U}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} -P_{\overline{\mathcal{R}(A)}} & 0 \\ 0 & P_{\overline{\mathcal{R}(A)}} \end{pmatrix}$. So, since $P_{\overline{\mathcal{R}(A)}} A = A P_{\overline{\mathcal{R}(A)}} = A$, then it can be verified that $\|\mathbb{U}x\|_{\mathbb{A}} = \|\mathbb{U}^{\sharp_{\mathbb{A}}}x\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$ for all $x = (x_1, x_2) \in$

$\mathcal{H}\oplus\mathcal{H}$. Hence, $\mathbb{U}$ is $\mathbb{A}$-unitary. Thus, by (1.7) we have

$$\begin{aligned}\omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q\\ R & S\end{pmatrix}\right] &= \omega_{\mathbb{A}}\left[\mathbb{U}^{\sharp_A}\begin{pmatrix} P & Q\\ R & S\end{pmatrix}\mathbb{U}\right]\\ &= \omega_{\mathbb{A}}\left[\begin{pmatrix} P_{\overline{\mathcal{R}(A)}}P & -P_{\overline{\mathcal{R}(A)}}Q\\ -P_{\overline{\mathcal{R}(A)}}R & P_{\overline{\mathcal{R}(A)}}S\end{pmatrix}\right]\\ &= \omega_{\mathbb{A}}\left[\begin{pmatrix} P_{\overline{\mathcal{R}(A)}} & 0\\ 0 & P_{\overline{\mathcal{R}(A)}}\end{pmatrix}\begin{pmatrix} P & -Q\\ -R & S\end{pmatrix}\right]\\ &= \omega_{\mathbb{A}}\left[\begin{pmatrix} P & -Q\\ -R & S\end{pmatrix}\right],\end{aligned}$$

where the last equality follows from the definition of $\omega_{\mathbb{A}}(\cdot)$ together with the fact that $P_{\overline{\mathcal{R}(A)}}A = AP_{\overline{\mathcal{R}(A)}} = A$. So, by taking into consideration (2.1) and the triangle inequality we prove the desired result.

(b) Since $\omega_{\mathbb{A}}\left[\begin{pmatrix} -P & Q\\ R & -S\end{pmatrix}\right] = \omega_{\mathbb{A}}\left[\begin{pmatrix} P & -Q\\ -R & S\end{pmatrix}\right]$, then by the proof of the assertion (a) we deduce that

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q\\ R & S\end{pmatrix}\right] = \omega_{\mathbb{A}}\left[\begin{pmatrix} -P & Q\\ R & -S\end{pmatrix}\right].$$

Moreover, by using the fact that

$$\begin{pmatrix} 0 & Q\\ R & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix} P & Q\\ R & S\end{pmatrix} + \frac{1}{2}\begin{pmatrix} -P & Q\\ R & -S\end{pmatrix},$$

and the subadditivity of the $\mathbb{A}$-numerical radius $\omega_{\mathbb{A}}(\cdot)$, we get the required result. □

Also, we need the following lemma.

Lemma 2.2 ([16]). *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then,*

$$\left\|\begin{pmatrix} 0 & T\\ S & 0\end{pmatrix}\right\|_{\mathbb{A}} = \left\|\begin{pmatrix} T & 0\\ 0 & S\end{pmatrix}\right\|_{\mathbb{A}} = \max\{\|T\|_A, \|S\|_A\}.$$

Now, we are in a position to prove our first result in this paper.

Theorem 2.3. *Let $\mathbb{T} = \begin{pmatrix} P & Q\\ R & S\end{pmatrix}$ be such that $P, Q, R, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then,*

$$\lambda_1 \leq \omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q\\ R & S\end{pmatrix}\right] \leq \lambda_2, \tag{2.2}$$

where

$$\lambda_1 = \max\left\{\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & Q\\ R & 0\end{pmatrix}\right], \max\{\omega_A(P), \omega_A(S)\}\right\}$$

and

$$\lambda_2 = \frac{\|Q\|_A + \|R\|_A}{2} + \max\{\omega_A(P), \omega_A(S)\}.$$

Proof. Clearly we have

$$\begin{pmatrix} P & Q\\ R & S\end{pmatrix} = \begin{pmatrix} P & 0\\ 0 & S\end{pmatrix} + \begin{pmatrix} 0 & Q\\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0\\ R & 0\end{pmatrix}. \tag{2.3}$$

On the other hand, it is not difficult to see that $\mathbb{A}\begin{pmatrix} 0 & Q\\ 0 & 0\end{pmatrix}^2 = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$ and $\mathbb{A}\begin{pmatrix} 0 & 0\\ R & 0\end{pmatrix}^2 = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$. So, by (1.6) and Lemma 2.2 we have

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & Q\\ 0 & 0\end{pmatrix}\right] = \frac{1}{2}\left\|\begin{pmatrix} 0 & Q\\ 0 & 0\end{pmatrix}\right\|_{\mathbb{A}} = \frac{1}{2}\|Q\|_A.$$

Similarly, we have $\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix}\right] = \frac{1}{2}\|R\|_A$. So, by using the trivial observation (2.3) and the subadditivity of the $\mathbb{A}$-numerical radius $\omega_{\mathbb{A}}(\cdot)$ together with Lemma 2.1 (a), we get

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\right] \leq \max\{\omega_A(P), \omega_A(S)\} + \frac{\|Q\|_A + \|R\|_A}{2}. \tag{2.4}$$

Furthermore, by Lemmas 2.1 and 2.2 we have

$$\begin{aligned}\omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\right] &\geq \max\left\{\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right], \omega_{\mathbb{A}}\left[\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\right]\right\} \\ &= \max\left\{\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right], \max\{\omega_A(P), \omega_A(S)\}\right\}. \end{aligned} \tag{2.5}$$

By combining (2.4) together with (2.5), we reach the desired result. ☐

In order to prove our next result, we need the following lemma.

Lemma 2.4. *Let $T, S \in \mathbb{B}(\mathcal{H})$ be two A-positive operators. Then,*

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}\right] = \frac{1}{2}\|T+S\|_A. \tag{2.6}$$

Proof. Since T and S are A-positive, then $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. So, by [3, Proposition 3.6.] there exist two unique operators $\widetilde{T}, \widetilde{S} \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T = \widetilde{T} Z_A$ and $Z_A S = \widetilde{S} Z_A$. Moreover, since $T \geq_A 0$, then for all $x \in \mathcal{H}$ we have

$$\langle ATx, x\rangle \geq 0.$$

This implies, through (1.2), that

$$\langle Tx, x\rangle_A = \langle ATx, Ax\rangle_{\mathbf{R}(A^{1/2})} = \langle \widetilde{T}Ax, Ax\rangle_{\mathbf{R}(A^{1/2})} \geq 0$$

for all $x \in \mathcal{H}$. Further, by using the density of $\mathcal{R}(A)$ in $\mathbf{R}(A^{1/2})$, we obtain

$$\langle \widetilde{T}A^{1/2}x, A^{1/2}x\rangle_{\mathbf{R}(A^{1/2})} \geq 0, \quad \forall x \in \mathcal{H}.$$

So, $\widetilde{T}$ is a positive operator on the Hilbert space $\mathbf{R}(A^{1/2})$. Similarly, we prove that $\widetilde{S} \geq 0$. Therefore, in view of [4, Corollary 3] we have

$$\omega\left[\begin{pmatrix} 0 & \widetilde{T} \\ \widetilde{S} & 0 \end{pmatrix}\right] = \frac{1}{2}\left\|\widetilde{T} + \widetilde{S}\right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = \frac{1}{2}\left\|\widetilde{T+S}\right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}, \tag{2.7}$$

where the last equality follows since $\widetilde{T+S} = \widetilde{T} + \widetilde{S}$ (see [15]). Moreover, by [5, Lemma 3.2], we have $\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix} \in \mathbb{B}_{\mathbb{A}^{1/2}}(\mathcal{H} \oplus \mathcal{H})$ and

$$\widetilde{\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}} = \begin{pmatrix} 0 & \widetilde{T} \\ \widetilde{S} & 0 \end{pmatrix}.$$

This proves the desired result by applying (2.7) together with (1.9). ☐

We are now in a position to state the following theorem.

Theorem 2.5. *Let $\mathbb{T} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_{\mathbb{A}}(\mathbb{T}) \leq \frac{1}{2}\Big(\omega_A(P) + \omega_A(Q)\Big) + \frac{1}{4}\Big(\|I + PP^{\sharp_A} + QQ^{\sharp_A}\|_A + \|I + RR^{\sharp_A} + SS^{\sharp_A}\|_A\Big).$$

Proof. We first prove that

$$\omega_{\mathbb{A}}(\mathbb{S}) \leq \frac{1}{2}\omega_A(P) + \frac{1}{4}\|I + PP^{\sharp_A} + QQ^{\sharp_A}\|_A, \tag{2.8}$$

where $\mathbb{S} = \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}$. Let $\theta \in \mathbb{R}$. It is not difficult to verify that $\Re_{\mathbb{A}}(e^{i\theta}\mathbb{S})$ is an $\mathbb{A}$-self-adjoint operator. So, by (1.13) we have

$$r_{\mathbb{A}}\left(\Re_{\mathbb{A}}(e^{i\theta}\mathbb{S})\right) = \|\Re_{\mathbb{A}}(e^{i\theta}\mathbb{S})\|_{\mathbb{A}}.$$

Now, by using (1.15), we see that

$$\begin{aligned}
r_{\mathbb{A}}\left[\Re_A(e^{i\theta}\mathbb{S})\right] &= \frac{1}{2}r_{\mathbb{A}}(e^{i\theta}\mathbb{S} + e^{-i\theta}\mathbb{S}^{\sharp_{\mathbb{A}}}) \\
&= \frac{1}{2}r_{\mathbb{A}}\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} + e^{-i\theta}\begin{pmatrix} P^{\sharp_{\mathbb{A}}} & 0 \\ Q^{\sharp_{\mathbb{A}}} & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}r_{\mathbb{A}}\left[\begin{pmatrix} e^{i\theta}P + e^{-i\theta}P^{\sharp_A} & e^{i\theta}Q \\ e^{-i\theta}Q^{\sharp_A} & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}r_{\mathbb{A}}\left[\begin{pmatrix} P^{\sharp_A} & e^{i\theta}I \\ Q^{\sharp_A} & 0 \end{pmatrix}\begin{pmatrix} e^{-i\theta}I & 0 \\ P & Q \end{pmatrix}\right].
\end{aligned}$$

Moreover, an application of (1.11) gives

$$\begin{aligned}
r_{\mathbb{A}}\left[\Re_A(e^{i\theta}\mathbb{S})\right] &= \frac{1}{2}r_{\mathbb{A}}\left[\begin{pmatrix} e^{-i\theta}I & 0 \\ P & Q \end{pmatrix}\begin{pmatrix} P^{\sharp_A} & e^{i\theta}I \\ Q^{\sharp_A} & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}r_{\mathbb{A}}\left[\begin{pmatrix} e^{-i\theta}P^{\sharp_A} & I \\ PP^{\sharp_A} + QQ^{\sharp_A} & e^{i\theta}P \end{pmatrix}\right].
\end{aligned}$$

Further, by applying (1.12), we get

$$\begin{aligned}
r_{\mathbb{A}}\left[\Re_A(e^{i\theta}\mathbb{S})\right] &\leq \frac{1}{2}\omega_{\mathbb{A}}\left[\begin{pmatrix} e^{-i\theta}P^{\sharp_A} & I \\ PP^{\sharp_A} + QQ^{\sharp_A} & e^{i\theta}P \end{pmatrix}\right] \\
&\leq \frac{1}{2}\omega_{\mathbb{A}}\left[\begin{pmatrix} e^{-i\theta}P^{\sharp_A} & 0 \\ 0 & e^{i\theta}P \end{pmatrix}\right] + \frac{1}{2}\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & I \\ PP^{\sharp_A} + QQ^{\sharp_A} & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\omega_A(P) + \frac{1}{4}\|I + PP^{\sharp_A} + QQ^{\sharp_A}\|_A \ \text{(by Lemmas 2.1 and 2.4).}
\end{aligned}$$

Hence, we obtain

$$\|\Re_{\mathbb{A}}(e^{i\theta}\mathbb{S})\|_{\mathbb{A}} \leq \frac{1}{2}\omega_A(P) + \frac{1}{4}\|I + PP^{\sharp_A} + QQ^{\sharp_A}\|_A,$$

for every $\theta \in \mathbb{R}$. So, by taking the supremum over all $\theta \in \mathbb{R}$ and then applying (1.8) we obtain (2.8) as required. Let $\mathbb{U} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. In view of (1.15) we have $\mathbb{U}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} 0 & P_{\overline{\mathcal{R}(A)}} \\ P_{\overline{\mathcal{R}(A)}} & 0 \end{pmatrix}$.

Further, it can be seen that $\mathbb{U}$ is $\mathbb{A}$-unitary operator. So, by using (1.7) we get

$$\begin{aligned}\omega_{\mathbb{A}}(\mathbb{T}) &\leq \omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & 0 \\ R & S \end{pmatrix}\right] \\ &= \omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\mathbb{U}^{\sharp_{\mathbb{A}}}\begin{pmatrix} 0 & 0 \\ R & S \end{pmatrix}\mathbb{U}\right] \\ &= \omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} P_{\overline{\mathcal{R}(A)}} & 0 \\ 0 & P_{\overline{\mathcal{R}(A)}} \end{pmatrix}\begin{pmatrix} S & R \\ 0 & 0 \end{pmatrix}\right] \\ &= \omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} S & R \\ 0 & 0 \end{pmatrix}\right].\end{aligned}$$

Finally, by applying (2.8) we obtain

$$\begin{aligned}\omega_{\mathbb{A}}(\mathbb{T}) &\leq \frac{1}{2}\omega_A(P) + \frac{1}{4}\|I + PP^{\sharp_A} + QQ^{\sharp_A}\|_A + \frac{1}{2}\omega_A(S) + \frac{1}{4}\|I + SS^{\sharp_A} + RR^{\sharp_A}\|_A \\ &= \frac{1}{2}\Big(\omega_A(P) + \omega_A(S)\Big) + \frac{1}{4}\left(\|PP^{\sharp_A} + QQ^{\sharp_A}\|_A + \|RR^{\sharp_A} + SS^{\sharp_A}\|_A\right).\end{aligned}$$

This finishes the proof of the theorem. □

The following lemma is useful in proving our next result.

Lemma 2.6 ([17]). *Let* $\mathbb{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ *be such that* $T_{ij} \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ *for all* $i, j \in \{1, 2\}$. *Then,* $\mathbb{T} \in \mathbb{B}_{\mathbb{A}^{1/2}}(\mathcal{H} \oplus \mathcal{H})$ *and*

$$r_{\mathbb{A}}(\mathbb{T}) \leq r\left[\begin{pmatrix} \|T_{11}\|_A & \|T_{12}\|_A \\ \|T_{21}\|_A & \|T_{22}\|_A \end{pmatrix}\right].$$

Theorem 2.7. *Let* $\mathbb{T} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ *be such that* $P, Q, R, S \in \mathbb{B}_A(\mathcal{H})$. *Then,*

$$\omega_{\mathbb{A}}(\mathbb{T}) \leq \frac{1}{2}\left(\|P\|_A + \|S\|_A + \sqrt{\|PP^{\sharp_A} + QQ^{\sharp_A}\|_A} + \sqrt{\|RR^{\sharp_A} + SS^{\sharp_A}\|_A}\right). \tag{2.9}$$

Proof. We first prove that

$$\omega_{\mathbb{A}}(\mathbb{S}) \leq \frac{1}{2}\left(\|P\|_A + \sqrt{\|PP^{\sharp_A} + QQ^{\sharp_A}\|_A}\right), \tag{2.10}$$

where $\mathbb{S} = \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}$. Let $\theta \in \mathbb{R}$. By proceeding as in the proof of Theorem 2.5 and then using Lemma 2.6 we see that

$$\begin{aligned}\|\Re_{\mathbb{A}}(e^{i\theta}\mathbb{S})\|_{\mathbb{A}} &= r_{\mathbb{A}}\left[\Re_{\mathbb{A}}(e^{i\theta}\mathbb{S})\right] \\ &= \frac{1}{2}r_{\mathbb{A}}\left[\begin{pmatrix} e^{-i\theta}P^{\sharp_A} & I \\ PP^{\sharp_A} + QQ^{\sharp_A} & e^{i\theta}P \end{pmatrix}\right] \\ &\leq \frac{1}{2}r\left[\begin{pmatrix} \|P\|_A & 1 \\ \|PP^{\sharp_A} + QQ^{\sharp_A}\|_A & \|P\|_A \end{pmatrix}\right] \\ &= \frac{1}{2}\left(\|P\|_A + \sqrt{\|PP^{\sharp_A} + QQ^{\sharp_A}\|_A}\right).\end{aligned}$$

This immediately proves (2.10) by applying (1.8). Using an argument similar to that used in proof of Theorem 2.5, we get the desired result. □

Before proving our next theorem we have to state the following lemma.

Lemma 2.8 ([7, Theorem 5.1]). *Let $T\in\mathbb{B}(\mathcal{H})$ be an A-selfadjoint operator. Then, for any positive integer n we have*

$$\|T^n\|_A=\|T\|_A^n.$$

Theorem 2.9. *Let $\mathbb{T}=\begin{pmatrix}P & Q\\ R & S\end{pmatrix}$ be such that $P,Q,R,S\in\mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_{\mathbb{A}}(\mathbb{T})\le\sqrt{\omega_A^2(P)+\frac{1}{2}\|Q\|_A\left(\omega_A(P)+\frac{1}{2}\|Q\|_A\right)}+\sqrt{\omega_A^2(S)+\frac{1}{2}\|R\|_A\left(\omega_A(S)+\frac{1}{2}\|R\|_A\right)}.$$

Proof. Let $\mathbb{S}=\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}$. We first prove that

$$\omega_{\mathbb{A}}(\mathbb{S})\le\sqrt{\omega_A^2(P)+\frac{1}{2}\|Q\|_A\left(\omega_A(P)+\frac{1}{2}\|Q\|_A\right)}.\tag{2.11}$$

Let $\theta\in\mathbb{R}$. A straightforward calculation shows that

$$\begin{aligned}\Re_{\mathbb{A}}(e^{i\theta}\mathbb{S})&=\begin{pmatrix}\Re_A(e^{i\theta}P) & \frac{1}{2}e^{i\theta}Q\\ \frac{1}{2}e^{-i\theta}Q^{\sharp_A} & 0\end{pmatrix}\\ &=\begin{pmatrix}\Re_A(e^{i\theta}P) & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & \frac{1}{2}e^{i\theta}Q\\ \frac{1}{2}e^{-i\theta}Q^{\sharp_A} & 0\end{pmatrix}.\end{aligned}$$

This implies that

$$\begin{aligned}\Re_{\mathbb{A}}^2(e^{i\theta}\mathbb{S})=&\begin{pmatrix}\Re_A^2(e^{i\theta}P) & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\frac{1}{4}QQ^{\sharp_A} & 0\\ 0 & \frac{1}{4}Q^{\sharp_A}Q\end{pmatrix}\\ &+\begin{pmatrix}0 & \frac{1}{2}e^{i\theta}\left[\Re_A(e^{i\theta}P)\right]Q\\ \frac{1}{2}e^{-i\theta}Q^{\sharp_A}\left[\Re_A(e^{i\theta}P)\right] & 0\end{pmatrix}.\end{aligned}$$

Thus, by using the triangle inequality together with Lemma 2.2, (1.4) and (1.14) we see that

$$\begin{aligned}\left\|\Re_{\mathbb{A}}^2(e^{i\theta}\mathbb{S})\right\|_{\mathbb{A}}&\le\|\Re_A(e^{i\theta}P)\|_A^2+\frac{1}{4}\|Q\|_A^2+\frac{1}{2}\|\Re_A(e^{i\theta}P)\|_A\|Q\|_A\\ &\le\omega_A^2(P)+\frac{1}{4}\|Q\|_A^2+\frac{1}{2}\omega_A(P)\|Q\|_A,\end{aligned}$$

where the last inequality follows from (1.8). Since $\Re_{\mathbb{A}}(e^{i\theta}\mathbb{S})$ is $\mathbb{A}$-selfadjoint, then an application of Lemma 2.8 gives

$$\|\Re_{\mathbb{A}}(e^{i\theta}\mathbb{S})\|_{\mathbb{A}}^2\le\omega_A^2(P)+\frac{1}{4}\|Q\|_A^2+\frac{1}{2}\omega_A(P)\|Q\|_A,$$

for every $\theta\in\mathbb{R}$. Taking the supremum over all $\theta\in\mathbb{R}$ in the above inequality and then using (1.8) yields that

$$\omega_{\mathbb{A}}^2(\mathbb{S})\le\omega_A^2(P)+\frac{1}{4}\|Q\|_A^2+\frac{1}{2}\omega_A(P)\|Q\|_A.$$

This proves (2.11). Using an argument similar to that used in proof of Theorem 2.5, we get the desired result. □

Next we state the following useful lemmas related to A-selfadjoint operators.

Lemma 2.10. *Let $T,S\in\mathbb{B}(\mathcal{H})$ be two A-selfadjoint operators. If $T-S\ge_A 0$, then*

$$\|T\|_A\ge\|S\|_A.$$

Proof. Since $T - S \geq_A 0$, then $\langle (T-S)x, x\rangle_A \geq 0$ for all $x \in \mathcal{H}$. This gives

$$\langle Tx, x\rangle_A \geq \langle Sx, x\rangle_A, \quad \forall\, x \in \mathcal{H}.$$

So, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality and then using (1.13) we obtain the desired result. □

Lemma 2.11 ([14])**.** *Let $T \in \mathbb{B}_A(\mathcal{H})$ be an A-selfadjoint operator. Then, $T^{2n} \geq_A 0$ for any positive integer n.*

We are now in a position to prove the following theorem.

Theorem 2.12. *Let $\mathbb{T} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_{\mathbb{A}}(\mathbb{T}) \leq \sqrt{2\omega_A^2(P) + \frac{1}{2}\left(\|P^{\sharp_A}Q\|_A + \|Q\|_A^2\right)} + \sqrt{2\omega_A^2(S) + \frac{1}{2}\left(\|S^{\sharp_A}R\|_A + \|R\|_A^2\right)}.$$

Proof. We first prove that

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] \leq \sqrt{2\omega_A^2(P) + \frac{1}{2}\left(\|P^{\sharp_A}Q\|_A + \|Q\|_A^2\right)}. \tag{2.12}$$

Let $\theta \in \mathbb{R}$. By using (1.15), it can be verified that

$$\Re_{\mathbb{A}}\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} \Re_A(e^{i\theta}P) & \frac{1}{2}e^{i\theta}Q \\ \frac{1}{2}e^{-i\theta}Q^{\sharp_A} & 0 \end{pmatrix}$$

and

$$\Im_{\mathbb{A}}\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] = -i\begin{pmatrix} i\Im_A(e^{i\theta}P) & \frac{1}{2}e^{i\theta}Q \\ -\frac{1}{2}e^{-i\theta}Q^{\sharp_A} & 0 \end{pmatrix}.$$

Moreover, by Lemma 2.11, $\Im_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] \geq_{\mathbb{A}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So, we have

$$\Re_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] + \Im_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] - \Re_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] \geq_{\mathbb{A}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, it follows from Lemma 2.10 that

$$\left\|\Re_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right]\right\|_{\mathbb{A}} \leq \left\|\Re_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] + \Im_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right]\right\|_{\mathbb{A}}.$$

This in turn implies, through Lemma 2.8, that

$$\left\|\Re_{\mathbb{A}}\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right]\right\|_{\mathbb{A}}^2 \leq \left\|\Re_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] + \Im_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right]\right\|_{\mathbb{A}}. \tag{2.13}$$

On the other hand, a short calculation reveals that

$$\begin{aligned}
&\Re_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] + \Im_{\mathbb{A}}^2\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} \Re_A^2(e^{i\theta}P) + \Im_A^2(e^{i\theta}P) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{P^{\sharp_A}Q}{2} \\ \frac{Q^{\sharp_A}P}{2} & 0 \end{pmatrix} + \begin{pmatrix} \frac{QQ^{\sharp_A}}{2} & 0 \\ 0 & \frac{Q^{\sharp_A}Q}{2} \end{pmatrix}.
\end{aligned}$$

Hence, by taking into consideration (2.13) and then applying the triangle inequality together with Lemma 2.2 and (1.14) we see that

$$\begin{aligned}
&\left\|\Re_{\mathbb{A}}\left[e^{i\theta}\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right]\right\|_{\mathbb{A}}^2 \\
&\leq \left\|\Re_A^2(e^{i\theta}P) + \Im_A^2(e^{i\theta}P)\right\|_A + \frac{1}{2}\max\{\|P^{\sharp_A}Q\|_A, \|Q^{\sharp_A}P\|_A\} + \frac{1}{2}\|Q\|_A^2 \\
&\leq 2\omega_A^2(P) + \frac{1}{2}\left(\max\{\|P^{\sharp_A}Q\|_A, \|Q^{\sharp_A}P\|_A\} + \|Q\|_A^2\right),
\end{aligned} \tag{2.14}$$

where the last inequality holds by applying (1.4) and (1.8). On the other hand, since $P_{\overline{\mathcal{R}(A)}}A = AP_{\overline{\mathcal{R}(A)}} = A$, then by applying (1.3), we see that

$$\begin{aligned}
\|P^{\sharp_A}Q\|_A &= \|Q^{\sharp_A}P_{\overline{\mathcal{R}(A)}}PP_{\overline{\mathcal{R}(A)}}\|_A \\
&= \sup\left\{|\langle AP_{\overline{\mathcal{R}(A)}}x, (Q^{\sharp_A}P_{\overline{\mathcal{R}(A)}}P)^{\sharp_A}y\rangle|\,;\ x,y\in\mathcal{H},\ \|x\|_A=\|y\|_A=1\right\} \\
&= \sup\left\{|\langle Q^{\sharp_A}P_{\overline{\mathcal{R}(A)}}Px, y\rangle_A|\,;\ x,y\in\mathcal{H},\ \|x\|_A=\|y\|_A=1\right\} \\
&= \sup\left\{|\langle AP_{\overline{\mathcal{R}(A)}}Px, Qy\rangle|\,;\ x,y\in\mathcal{H},\ \|x\|_A=\|y\|_A=1\right\} \\
&= \sup\left\{|\langle Q^{\sharp_A}Px, y\rangle_A|\,;\ x,y\in\mathcal{H},\ \|x\|_A=\|y\|_A=1\right\} \\
&= \|Q^{\sharp_A}P\|_A. \qquad (2.15)
\end{aligned}$$

So, by taking into account (2.14), it follows that

$$\left\|\Re_{\mathbb{A}}\left[e^{i\theta}\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}\right]\right\|_{\mathbb{A}}^2 \le 2\omega_A^2(P) + \frac{1}{2}\left(\|P^{\sharp_A}Q\|_A + \|Q\|_A^2\right),$$

for every $\theta\in\mathbb{R}$. So, by taking the supremum over all $\theta\in\mathbb{R}$ in the above inequality we obtain (2.12) as required. Finally, by using an argument similar to that used in proof of Theorem 2.5, we get the desired inequality. ☐

Our next result reads as follows.

Theorem 2.13. *Let* $\mathbb{T}=\begin{pmatrix}P & Q\\ R & S\end{pmatrix}$ *be such that* $P,Q,R,S\in\mathbb{B}_A(\mathcal{H})$. *Then,*

$$\omega_{\mathbb{A}}(\mathbb{T}) \le \min\{\mu,\nu\},$$

where

$$\mu = \sqrt{\min\{\|P+Q\|_A^2, \|P-Q\|_A^2\} + 2\omega_A(PQ^{\sharp_A})} + \sqrt{\min\{\|R+S\|_A^2, \|R-S\|_A^2\} + 2\omega_A(SR^{\sharp_A})},$$

and

$$\nu = \sqrt{\min\{\|P+R\|_A^2, \|P-R\|_A^2\} + 2\omega_A(P^{\sharp_A}R)} + \sqrt{\min\{\|Q+S\|_A^2, \|Q-S\|_A^2\} + 2\omega_A(S^{\sharp_A}Q)}.$$

Proof. By using (1.5) together with (1.14) and Lemma 2.2 we see that

$$\begin{aligned}
\omega_{\mathbb{A}}\left[\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}\right] &\le \left\|\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}\right\|_{\mathbb{A}} \\
&= \sqrt{\left\|\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}^{\sharp_{\mathbb{A}}}\right\|_{\mathbb{A}}} \\
&= \sqrt{\left\|\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}\begin{pmatrix}P^{\sharp_A} & 0\\ Q^{\sharp_A} & 0\end{pmatrix}\right\|_{\mathbb{A}}} \\
&= \sqrt{\left\|\begin{pmatrix}PP^{\sharp_A}+QQ^{\sharp_A} & 0\\ 0 & 0\end{pmatrix}\right\|_{\mathbb{A}}} \\
&= \sqrt{\|PP^{\sharp_A}+QQ^{\sharp_A}\|_A}. \qquad (2.16)
\end{aligned}$$

Moreover, it is not difficult to verify that

$$PP^{\sharp_A}+QQ^{\sharp_A} = (P\pm Q)(P\pm Q)^{\sharp_A} \mp (PQ^{\sharp_A}+QP^{\sharp_A}).$$

So, since $PP^{\sharp_A}+QQ^{\sharp_A}\geq_A 0$, it follows from (1.13) that

$$\begin{aligned}\|PP^{\sharp_A}+QQ^{\sharp_A}\|_A &= \omega_A(PP^{\sharp_A}+QQ^{\sharp_A})\\ &= \omega_A\Big((P\pm Q)(P\pm Q)^{\sharp_A}\mp(PQ^{\sharp_A}+QP^{\sharp_A})\Big)\\ &\leq \omega_A\Big((P\pm Q)(P\pm Q)^{\sharp_A}\Big)+\omega_A(PQ^{\sharp_A})+\omega_A(QP^{\sharp_A})\\ &= \|P\pm Q\|_A^2+\omega_A(PQ^{\sharp_A})+\omega_A(QP^{\sharp_A}),\end{aligned}$$

where the last equality follows by using (1.13) together with (1.14) since the operator $(P\pm Q)(P\pm Q)^{\sharp_A}$ is A-positive. Further, one observes that

$$\begin{aligned}\omega_A(PQ^{\sharp_A}) &= \omega_A\left((Q^{\sharp_A})^{\sharp_A}P^{\sharp_A}\right)\\ &= \omega_A(P_{\overline{\mathcal{R}(A)}}QP_{\overline{\mathcal{R}(A)}}P^{\sharp_A})=\omega_A(P_{\overline{\mathcal{R}(A)}}QP^{\sharp_A}).\end{aligned}$$

This yields that $\omega_A(PQ^{\sharp_A})=\omega_A(QP^{\sharp_A})$ since $P_{\overline{\mathcal{R}(A)}}A=AP_{\overline{\mathcal{R}(A)}}=A$. Thus, we get

$$\|PP^{\sharp_A}+QQ^{\sharp_A}\|_A\leq\|P\pm Q\|_A^2+2\omega_A(PQ^{\sharp_A}),$$

which, in turn, implies that

$$\|PP^{\sharp_A}+QQ^{\sharp_A}\|_A\leq\min\left(\|P+Q\|_A^2,\|P-Q\|_A^2\right)+2\omega_A(PQ^{\sharp_A}). \tag{2.17}$$

So, a combination of (2.16) together with (2.17) gives

$$\omega_{\mathbb{A}}\left[\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}\right]\leq\sqrt{\min\left(\|P+Q\|_A^2,\|P-Q\|_A^2\right)+2\omega_A(PQ^{\sharp_A})}. \tag{2.18}$$

By considering the $\mathbb{A}$-unitary operator $\mathbb{U}=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$, we see that

$$\begin{aligned}\omega_{\mathbb{A}}\left[\begin{pmatrix}P & Q\\ R & S\end{pmatrix}\right] &\leq \omega_{\mathbb{A}}\left[\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}\right]+\omega_{\mathbb{A}}\left[\begin{pmatrix}0 & 0\\ R & S\end{pmatrix}\right]\\ &= \omega_{\mathbb{A}}\left[\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}\right]+\omega_{\mathbb{A}}\left[\mathbb{U}^{\sharp_{\mathbb{A}}}\begin{pmatrix}S & R\\ 0 & 0\end{pmatrix}\mathbb{U}\right]\\ &= \omega_{\mathbb{A}}\left[\begin{pmatrix}P & Q\\ 0 & 0\end{pmatrix}\right]+\omega_{\mathbb{A}}\left[\begin{pmatrix}S & R\\ 0 & 0\end{pmatrix}\right]\quad\text{(by (1.7))}\\ &\leq \min\left(\|P+Q\|_A^2,\|P-Q\|_A^2\right)+2\omega_A(PQ^{\sharp_A})\\ &\quad+\min\left(\|R+S\|_A^2,\|R-S\|_A^2\right)+2\omega_A(SR^{\sharp_A}).\end{aligned}$$

By observing that $\omega_{\mathbb{A}}\left[\begin{pmatrix}P & Q\\ R & S\end{pmatrix}\right]=\omega_{\mathbb{A}}\left[\begin{pmatrix}P^{\sharp_A} & R^{\sharp_A}\\ Q^{\sharp_A} & S^{\sharp_A}\end{pmatrix}\right]$ and using similar arguments as above we get

$$\begin{aligned}\omega_{\mathbb{A}}\left[\begin{pmatrix}P & Q\\ R & S\end{pmatrix}\right] &= \omega_{\mathbb{A}}\left[\begin{pmatrix}P^{\sharp_A} & R^{\sharp_A}\\ Q^{\sharp_A} & S^{\sharp_A}\end{pmatrix}\right]\\ &\leq \min\left(\left\|P^{\sharp_A}+R^{\sharp_A}\right\|_A^2,\left\|P^{\sharp_A}-R^{\sharp_A}\right\|_A^2\right)+2\omega_A(P^{\sharp_A}(R^{\sharp_A})^{\sharp_A})\\ &\quad+\min\left(\left\|Q^{\sharp_A}+S^{\sharp_A}\right\|_A^2,\left\|Q^{\sharp_A}-S^{\sharp_A}\right\|_A^2\right)+2\omega_A(S^{\sharp_A}(Q^{\sharp_A})^{\sharp_A})\\ &= \min\left(\|P+R\|_A^2,\|P-R\|_A^2\right)+2\omega_A(R^{\sharp_A}P)\\ &\quad+\min\left(\|Q+S\|_A^2,\|Q-S\|_A^2\right)+2\omega_A(Q^{\sharp_A}S).\end{aligned}$$

Hence, the proof is complete since $\omega_A(R^{\sharp_A}P)=\omega_A(P^{\sharp_A}R)$ and $\omega_A(Q^{\sharp_A}S)=\omega_A(S^{\sharp_A}Q)$. ☐

In order to prove a lower bound for $\omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q\\ 0 & 0\end{pmatrix}\right]$, we need the following lemmas.

Lemma 2.14. *Let $T,S\in\mathbb{B}_A(\mathcal{H})$. Then*

$$\max\left\{\|T+S\|_A^2,\|T-S\|_A^2\right\}-\|TT^{\sharp_A}+SS^{\sharp_A}\|_A\leq 2\omega_A\left(TS^{\sharp_A}\right). \tag{2.19}$$

Proof. Let $x\in\mathcal{H}$ be such that $\|x\|_A=1$. If $\Re(z)$ denotes the real part of the complex number z, then we see that

$$\begin{aligned}\left\|T^{\sharp_A}x+S^{\sharp_A}x\right\|_A^2&=\left\|T^{\sharp_A}x\right\|_A^2+2\Re\left(\left\langle S^{\sharp_A}x,T^{\sharp_A}x\right\rangle_A\right)+\left\|S^{\sharp_A}x\right\|_A^2\\&\leq\left\langle\left(TT^{\sharp_A}+SS^{\sharp_A}\right)x,x\right\rangle_A+2\left|\left\langle TS^{\sharp_A}x,x\right\rangle_A\right|\\&\leq\omega_A\left(TT^{\sharp_A}+SS^{\sharp_A}\right)+2\omega_A\left(TS^{\sharp_A}\right)\\&=\left\|TT^{\sharp_A}+SS^{\sharp_A}\right\|_A+2\omega_A\left(TS^{\sharp_A}\right),\end{aligned}$$

where the last equality follows from (1.13) since $TT^{\sharp_A}+SS^{\sharp_A}\geq_A 0$. So, by taking the supremum over all $x\in\mathcal{H}$ with $\|x\|_A=1$ in the above inequality and then using the fact that $\|X\|_A=\|X^{\sharp_A}\|_A$ for all $X\in\mathbb{B}_A(\mathcal{H})$, we get

$$\|T+S\|_A^2\leq\left\|TT^{\sharp_A}+SS^{\sharp_A}\right\|_A+2\omega_A\left(TS^{\sharp_A}\right).$$

Similarly, we prove that

$$\|T-S\|_A^2\leq\left\|TT^{\sharp_A}+SS^{\sharp_A}\right\|_A+2\omega_A\left(TS^{\sharp_A}\right).$$

Hence, we obtain the desired inequality (2.19). ☐

Lemma 2.15. *Let $T,S\in\mathbb{B}(\mathcal{H})$. Then, the following assertions hold*

(1) *If $T\geq_A 0$ and $S\geq_A 0$, then*

$$\|T-S\|_A\leq\max\{\|T\|_A,\|S\|_A\}. \tag{2.20}$$

(2) *If $T,S\in\mathbb{B}_A(\mathcal{H})$, then*

$$2\|T^{\sharp_A}S\|_A\leq\|TT^{\sharp_A}+SS^{\sharp_A}\|_A. \tag{2.21}$$

Proof. (1) Let $Q=T-S$. It is not difficult to check that

$$\|T\|_AI\geq_A T\geq_A Q\quad\text{and}\quad\|S\|_AI\geq_A S\geq_A -Q.$$

This implies, by Lemma 2.10, that $\|Q\|_A\leq\|T\|_A$ and $\|Q\|_A\leq\|S\|_A$. This proves the desired property.

(2) Let $\mathbb{T}=\begin{pmatrix} T & S\\ 0 & 0\end{pmatrix}$. In view of (1.15) we see that

$$\mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}}=\begin{pmatrix} TT^{\sharp_A}+SS^{\sharp_A} & 0\\ 0 & 0\end{pmatrix}\quad\text{and}\quad\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}=\begin{pmatrix} T^{\sharp_A}T & T^{\sharp_A}S\\ S^{\sharp_A}T & S^{\sharp_A}S\end{pmatrix}.$$

Let $\mathbb{U}=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}$. By using (1.15), one gets $\mathbb{U}^{\sharp_{\mathbb{A}}}=\begin{pmatrix} P_{\overline{\mathcal{R}(A)}} & 0\\ 0 & -P_{\overline{\mathcal{R}(A)}}\end{pmatrix}$. So, we verify that $\|\mathbb{U}x\|_{\mathbb{A}}=\|\mathbb{U}^{\sharp_{\mathbb{A}}}x\|_{\mathbb{A}}=\|x\|_{\mathbb{A}}$ for all $x=(x_1,x_2)\in\mathcal{H}\oplus\mathcal{H}$. Hence, $\mathbb{U}$ is $\mathbb{A}$-unitary operator. Moreover, clearly we have $(\mathbb{U}^{\sharp_{\mathbb{A}}})^{\sharp_{\mathbb{A}}}=\mathbb{U}^{\sharp_{\mathbb{A}}}$. In addition, a short calculation shows that

$$(\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T})^{\sharp_{\mathbb{A}}}-\mathbb{U}^{\sharp_{\mathbb{A}}}(\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T})^{\sharp_{\mathbb{A}}}\mathbb{U}^{\sharp_{\mathbb{A}}}=\begin{pmatrix} 0 & 2(S^{\sharp_A}T)^{\sharp_A}\\ 2(T^{\sharp_A}S)^{\sharp_A} & 0\end{pmatrix}.$$

By (2.15), we have $\|T^{\sharp_A}S\|_A = \|S^{\sharp_A}T\|_A$. So, by applying Lemma 2.2 and then using (2.20) we get

$$\begin{aligned}
2\|T^{\sharp_A}S\|_A &= \left\|(\mathbb{T}^{\sharp_\mathbb{A}}\mathbb{T})^{\sharp_\mathbb{A}} - \mathbb{U}^{\sharp_\mathbb{A}}(\mathbb{T}^{\sharp_\mathbb{A}}\mathbb{T})^{\sharp_\mathbb{A}}\mathbb{U}^{\sharp_\mathbb{A}}\right\|_\mathbb{A} \\
&\leq \max\left\{\left\|(\mathbb{T}^{\sharp_\mathbb{A}}\mathbb{T})^{\sharp_\mathbb{A}}\right\|_\mathbb{A}, \left\|\mathbb{U}^{\sharp_\mathbb{A}}(\mathbb{T}^{\sharp_\mathbb{A}}\mathbb{T})^{\sharp_\mathbb{A}}\mathbb{U}^{\sharp_\mathbb{A}}\right\|_\mathbb{A}\right\} \\
&= \max\left\{\left\|\mathbb{T}^{\sharp_\mathbb{A}}\mathbb{T}\right\|_\mathbb{A}, \left\|\mathbb{U}(\mathbb{T}^{\sharp_\mathbb{A}}\mathbb{T})\mathbb{U}\right\|_\mathbb{A}\right\} \quad \text{(by (1.14))} \\
&\leq \left\|\mathbb{T}^{\sharp_\mathbb{A}}\mathbb{T}\right\|_\mathbb{A} \quad (\text{since } \|\mathbb{U}\|_\mathbb{A} = 1) \\
&= \left\|\mathbb{T}\mathbb{T}^{\sharp_\mathbb{A}}\right\|_\mathbb{A} = \|TT^{\sharp_A} + SS^{\sharp_A}\|_A,
\end{aligned}$$

where the last equality follows from (2.15) and Lemma 2.2. Hence, we prove the desired result. □

Lemma 2.16. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\begin{aligned}
&\max\left\{\|T+S\|_A^2, \|T-S\|_A^2\right\} \\
&\geq \frac{\left|\,\|T+S\|_A^2 - \|T-S\|_A^2\right|}{2} + \max\left\{\|T^2+S^2\|_A, \|T^{\sharp_A}T + S^{\sharp_A}S\|_A, \|TT^{\sharp_A} + SS^{\sharp_A}\|_A\right\}.
\end{aligned}$$

Proof. Notice that for any two real numbers x and y we have

$$\max\{x,y\} = \frac{1}{2}\Big(x + y + |x-y|\Big). \tag{2.22}$$

Now, by using (1.14) together with (2.22) we see that

$$\begin{aligned}
&\max\left\{\|T+S\|_A^2, \|T-S\|_A^2\right\} \\
&= \frac{1}{2}\left(\|T+S\|_A^2 + \|T-S\|_A^2 + \left|\,\|T+S\|_A^2 - \|T-S\|_A^2\,\right|\right) \\
&= \frac{1}{2}\left(\left\|(T^{\sharp_A}+S^{\sharp_A})(T+S)\right\|_A + \|(T^{\sharp_A}-S^{\sharp_A})(T-S)\|_A + \left|\,\|T+S\|_A^2 - \|T-S\|_A^2\right|\right) \\
&\geq \frac{1}{2}\left(\left\|(T^{\sharp_A}+S^{\sharp_A})(T+S) + (T^{\sharp_A}-S^{\sharp_A})(T-S)\right\|_A + \left|\,\|T+S\|_A^2 - \|T-S\|_A^2\,\right|\right) \\
&= \left\|T^{\sharp_A}T + S^{\sharp_A}S\right\|_A + \frac{\left|\,\|T+S\|_A^2 - \|T-S\|_A^2\right|}{2}.
\end{aligned} \tag{2.23}$$

By replacing T and S by $T^{\sharp_A}$ and $S^{\sharp_A}$, respectively, in (2.23) and then using the fact that $\|X\|_A = \|X^{\sharp_A}\|_A$ for every $X \in \mathbb{B}_A(\mathcal{H})$ we get

$$\max\left\{\|T+S\|_A^2, \|T-S\|_A^2\right\} \geq \left\|TT^{\sharp_A} + SS^{\sharp_A}\right\|_A + \frac{\left|\,\|T+S\|_A^2 - \|T-S\|_A^2\right|}{2}.$$

On the other hand, by (2.22) one has

$$\begin{aligned}
&\max\left\{\|T+S\|_A^2, \|T-S\|_A^2\right\} \\
&= \frac{1}{2}\left(\|T+S\|_A^2 + \|T-S\|_A^2 + \left|\,\|T+S\|_A^2 - \|T-S\|_A^2\,\right|\right) \\
&\geq \frac{1}{2}\left(\left\|(T+S)^2 + (T-S)^2\right\|_A + \left|\,\|T+S\|_A^2 - \|T-S\|_A^2\,\right|\right) \\
&= \left\|T^2 + S^2\right\|_A + \frac{\left|\,\|T+S\|_A^2 - \|T-S\|_A^2\right|}{2}.
\end{aligned}$$

So, the proof of the lemma is complete. □

Now we are ready to prove the following theorem.

Theorem 2.17. *Let $P, Q \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] \geq \frac{1}{2}\sqrt{\max\Big(\|P+Q\|_A^2, \|P-Q\|_A^2\Big) - 2\omega_A(PQ^{\sharp_A})}. \tag{2.24}$$

Proof. We first prove that

$$\max\Big(\|P+Q\|_A^2, \|P-Q\|_A^2\Big) - 2\omega_A(PQ^{\sharp_A}) \geq 0. \tag{2.25}$$

By applying (2.21) together with the second inequality in (1.5), one observes

$$2\omega_A(PQ^{\sharp_A}) \leq 2\|PQ^{\sharp_A}\|_A \leq \left\|Q^{\sharp_A}(Q^{\sharp_A})^{\sharp_A} + P^{\sharp_A}(P^{\sharp_A})^{\sharp_A}\right\|_A = \left\|P^{\sharp_A}P + Q^{\sharp_A}Q\right\|_A.$$

This implies, through Lemma 2.16, that

$$\begin{aligned}\max\Big(\|P+Q\|_A^2, \|P-Q\|_A^2\Big) &\geq \left\|P^{\sharp_A}P + Q^{\sharp_A}Q\right\|_A + \frac{\big|\,\|P+Q\|_A^2 - \|P-Q\|_A^2\big|}{2} \\ &\geq 2\omega_A(PQ^{\sharp_A}) + \frac{\big|\,\|P+Q\|_A^2 - \|P-Q\|_A^2\big|}{2}.\end{aligned}$$

Hence, (2.25) holds. Now, by using the first inequality in (1.5) we get

$$\begin{aligned}\omega_{\mathbb{A}}^2\left[\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] &\geq \frac{1}{4}\left\|\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right\|_{\mathbb{A}}^2 \\ &= \frac{1}{4}\left\|\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P^{\sharp_A} & 0 \\ Q^{\sharp_A} & 0 \end{pmatrix}\right\|_{\mathbb{A}} \quad \text{(by (1.14))} \\ &= \frac{1}{4}\left\|\begin{pmatrix} PP^{\sharp_A} + QQ^{\sharp_A} & 0 \\ 0 & 0 \end{pmatrix}\right\|_{\mathbb{A}} \\ &= \frac{1}{4}\|PP^{\sharp_A} + QQ^{\sharp_A}\|_A \quad \text{(by Lemma 2.2)} \\ &\geq \frac{1}{4}\left(\max\Big\{\|P+Q\|_A^2, \|P-Q\|_A^2\Big\} - 2\omega_A\Big(PQ^{\sharp_A}\Big)\right),\end{aligned}$$

where the last inequality follows from Lemma 2.14. This finishes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 2.17 and (2.18).

Corollary 2.18. *Let $P, Q \in \mathbb{B}_A(\mathcal{H})$ be such that $APQ^{\sharp_A} = 0$. Then,*

$$\frac{1}{2}\max\Big(\|P+Q\|_A, \|P-Q\|_A\Big) \leq \omega_{\mathbb{A}}\left[\begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}\right] \leq \min\Big(\|P+Q\|_A, \|P-Q\|_A\Big).$$

In particular, if $Q = 0$ we get

$$\frac{1}{2}\|P\|_A \leq \omega_A(P) \leq \|P\|_A.$$

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