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ALTERNATIVE PARTNER CURVES IN THE EUCLIDEAN 3−SPACE

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Abstract. In the present paper, a new type of special curve couple which are called WC^*-partner curves are introduced according to alternative moving frame $\{N, C, W\}$. The distance function between the corresponding points of reference curve and its partner curve is obtained. Besides, the angle function between the vector fields of alternative frame of the curves is expressed by means of alternative curvatures f and g. In addition to these, various characterizations are obtained related to these curves.

1. Introduction

The curves are the fundamental structure of differential geometry. Numerous studies of curves are carried out in 3−dimensional Euclidean space. Two curves which have some special properties at their corresponding points are called curve pairs. Hence, curve pairs are attracted the attention of many researchers [1, 2, 3, 13]. The most famous types of curve pairs are Bertrand partner curves. The Bertrand curves were firstly described by Bertrand Russell in 1850. These curves have the common principal normal vector. The classic characterization for Bertrand curves is that a regular curve α in $\mathbb{E}^3$ is the Bertrand curve if and only if $a\kappa(s)+b\tau(s) = 1$ holds [7]. The other famous curve pair are the Mannheim partner curves. These curves are defined by Mannheim with the equality $\kappa^2+\tau^2 = w^2$ =constant. Another characterization can be made as two curves α and β in $\mathbb{E}^3$ which are called Manneim partner curves if the principal normal vector fields of α coincide with the binormal vector fields of β at the corresponding points of curves [5, 6, 12, 14].

This paper is expected to define a new kind of curve pairs which are called WC^*-partner curves and give various characterization of these curves. For this purpose, an alternative frame on original curve is used and another curve is defined using this frame. First of all, a brief summary of curve theory and alternative frame

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are presented. Afterwards, the definition and main characterizations corporated to distance function and angle function of WC^*-partner curves are introduced.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a regular unit speed curve in the Euclidean $3-$space where s measures its arc length. Also, let $T = \alpha^{'}$ be its unit tangent vector, $N = \dfrac{T^{'}}{\|T^{'}\|}$ be its principal normal vector and $B = T \times N$ be its binormal vector. The triple $\{T, N, B\}$ be the Frenet frame of the curve α. Then the Frenet formula of the curve is given by

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are curvature and torsion of α, respectively [10]. Also, the geodesic curvature of spherical image of principal normal indicatrix of a space curve α is given

$$\sigma = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} (\frac{\tau}{\kappa})^{'}.$$

If we reconstruct the above equation via the harmonic curvature function H which is introduced by Özdamar in [8], we can easily see that

$$\sigma = \frac{H'}{\kappa(1 + H^2)^{3/2}}, \quad H = \frac{\tau}{\kappa}.$$

From the equation (2.1), the unit Darboux vector W of α is as follows

$$W = \frac{1}{\sqrt{\kappa^2 + \tau^2}} (\tau T + \kappa B). \tag{2.2}$$

It is obvious that the Darboux vector is vertical to the principal normal vector field N from equation (2.2). With the help of the vector fields W and N, along $\alpha(s)$, $C = W \times N$ unit vector field is defined. These three orthogonal vectors creates a new frame defined by Uzunoğlu et al. in [11]. This frame is designation by $\{N, C, W\}$ and alternative frame to curve rather than the Frenet frame $\{T, N, B\}$. The alternative frame and derivative formula of the alternative frame are given by

$$\begin{pmatrix} N \\ C \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-\kappa}{\sqrt{\kappa^2 + \tau^2}} & 0 & \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \\ \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} & 0 & \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.3}$$

and

$$\begin{pmatrix} N' \\ C' \\ W' \end{pmatrix} = \begin{pmatrix} 0 & f & 0 \\ -f & 0 & g \\ 0 & -g & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}, \tag{2.4}$$

where

$$f = \kappa\sqrt{1+H^2},\ g = \frac{H^{'}}{1+H^2}.$$

Since the principal normal vector N is common in both frames, following equations are available from the equations (2.1), (2.2) and (2.4),

$$\begin{aligned} C &= -\bar{\kappa}T + \bar{\tau}B \\ W &= \bar{\tau}T + \bar{\kappa}B \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} T &= -\bar{\kappa}C + \bar{\tau}W \\ B &= \bar{\tau}C + \bar{\kappa}W \end{aligned} \tag{2.6}$$

where $\bar{\kappa} = \dfrac{\kappa}{f}$ and $\bar{\tau} = \dfrac{\tau}{f}$.

A regular curve α is called a helix if the tangent lines of the curve makes a constant angle with a fixed direction. This curve is characterized by the property that $\dfrac{\tau}{\kappa}$ is constant [4]. If the principal normal lines of the curve makes a constant angle with a fixed direction, then the curve is called a slant helix and characterized by the equally

$$\frac{g}{f} = \frac{H^{'}}{\kappa(1+H^2)^{3/2}} = \sigma$$

is constant [11]. Then the characterization of a slant helix according to alternative frame is given as follows.

Remark 1. *A regular curve $\alpha(s)$ according to alternative frame $\{N, C, W\}$ with alternative curvatures f and g is a slant helix if and only if $\dfrac{g(s)}{f(s)} = constant$* [11].

3. ALTERNATIVE PARTNER CURVES IN THE EUCLIDEAN 3−SPACE

This section aims to define a new type of partner curves by considering alternative frame and find some characterizations for these curves corporated to distance function between the corresponding points of the curves, curvatures of the curves and angle function.

Definition 1. *Let $\alpha = \alpha(s)$ and $\alpha^* = \alpha^*(s^*)$ be two regular space curves parameterized by its arc length s and s^* with Frenet frames $\{T, N, B\}$, $\{T^*, N^*, B^*\}$, curvatures κ, κ^* and torsions τ, τ^* respectively in the Euclidean $3-$space. Also, let the alternative moving frames and alternative curvatures of curves be $\{N, C, W\}$, f, g and $\{N^*, C^*, W^*\}$, f^*, g^*, respectively. The curves α and α^* are called WC^*-partner curves if the vector fields W and C^{*} coincide i.e., $W = C^*$ holds at the corresponding points of the curves.*

From Definition 1, we can easily write the parametric representation of $\alpha^*(s^*)$ as follows

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)W(s) \tag{3.1}$$

where $\lambda = \lambda(s)$ is the distance function between corresponding points of the curves α and α^*. Because the vector fields W and C^* are the equal, we can represent the relationship between the alternative frames of α and α^*. If $\theta = \theta(s)$ is the angle function between vector fields N and W^*, the following equations are obtained thanks to axis rotation equations.

$$\begin{pmatrix} N^* \\ C^* \\ W^* \end{pmatrix} = \begin{pmatrix} \cos(90-\theta) & \sin(90-\theta) & 0 \\ 0 & 0 & 1 \\ -\sin(90-\theta) & \cos(90-\theta) & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}$$

$$\begin{aligned} N^* &= \sin\theta N + \cos\theta C \\ W^* &= -\cos\theta N + \sin\theta C \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} N &= \sin\theta N^* - \cos\theta W^* \\ C &= \cos\theta N^* + \sin\theta W^*. \end{aligned} \tag{3.3}$$

Theorem 1. *Let $\{\alpha, \alpha^*\}$ be WC^*-partner curves according to alternative frame in Euclidean $3-$space. The distance function $\lambda = \lambda(s)$ between the corresponding points of the α and α^* is as follows,*

$$\lambda(s) = -\frac{\kappa}{fg}.$$

Proof. If we take derivative of the equation (3.1) according to s, we get

$$T^* \frac{ds^*}{ds} = T + \lambda' W + \lambda W'.$$

Using the equations (2.6) and (3.3), we obtain that

$$(-\bar{\kappa}^* C^* + \bar{\tau}^* W^*)\frac{ds^*}{ds} = -(\bar{\kappa} + \lambda g)\cos\theta N^* + (\bar{\tau} + \lambda')C^* - (\bar{\kappa} + \lambda g)\sin\theta W^*.$$

If we consider the above equalities, we can easily see that

$$\lambda(s) = -\frac{\kappa}{fg}.$$

$\square$

Theorem 2. *Let $\{\alpha, \alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. $\{N, C, W, f, g\}$ and $\{N^*, C^*, W^*, f^*, g^*\}$ are the alternative frame elements of the curves α and α^*, respectively. Then the following relation exists among curvatures.*

$$\frac{g^*}{f^*} = -\tan\theta = \text{constant} \quad \text{and} \quad (f^*)^2 + (g^*)^2 = g^2$$

Proof. Since $\{\alpha, \alpha^*\}$ is the WC^*-partner curves, $W = C^*$ and their derivatives are equal.

$$\begin{aligned}(C^*)' &= -f^*N^* + g^*W^*,\\ W' &= -gC.\end{aligned}$$

From the last equation and equation (3.3), we have

$$-f^*N^* + g^*W^* = -g(\cos\theta N^* + \sin\theta W^*)$$

$$\begin{aligned}f^* &= g\cos\theta,\\ g^* &= -g\sin\theta.\end{aligned}$$

So, we obtain that

$$\frac{g^*}{f^*} = -\tan\theta$$

and

$$(f^*)^2 + (g^*)^2 = g^2.$$

□

Theorem 3. *Let $\{\alpha, \alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. $\theta = \theta(s)$ be the angle function between vector fields N and W^*. Then the following relation exists.*

$$\theta = \int_0^s f ds, \quad s = \int_0^{s^*} \frac{f^*}{g\cos\theta} ds^*$$

Proof. From the equation (3.2), we have

$$N^* = \sin\theta N + \cos\theta C$$

If we take the derivative of each side of the above equation according to s, we obtain

$$\frac{dN^*}{ds^*}\frac{ds^*}{ds} = \cos\theta\frac{d\theta}{ds}N + \sin\theta N' - \sin\theta\frac{d\theta}{ds}C + \cos\theta C'$$

$$f^*C^*\frac{ds^*}{ds} = \cos\theta\frac{d\theta}{ds}N + \sin\theta(fC) - \sin\theta\frac{d\theta}{ds}C + \cos\theta(-fN + gW)$$

Because $\{\alpha, \alpha^*\}$ is the WC^*-partner curves, we have

$$f^*W\frac{ds^*}{ds} = (\cos\theta\frac{d\theta}{ds} - f\cos\theta)N + (f\sin\theta - \sin\theta\frac{d\theta}{ds})C + g\cos\theta W$$

$$f^*\frac{ds^*}{ds} = g\cos\theta \text{ and } s = \int_0^{s^*} \frac{f^*}{g\cos\theta} ds^*.$$

Also, from $f\sin\theta - \sin\theta\dfrac{d\theta}{ds} = 0$ and $\cos\theta\dfrac{d\theta}{ds} - f\cos\theta = 0$, we get $f = \dfrac{d\theta}{ds}$ and

$$\theta = \int_0^s f ds.$$

□

Theorem 4. *Let $\{\alpha,\alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. α^* is a helix if and only if*

$$\frac{(\bar\kappa+\lambda g)\sin\theta}{(\bar\tau+\lambda')}$$

is constant.

Proof. If we take the derivative of the equation (3.1) according to parameter s, we have

$$T^*\frac{ds^*}{ds} = T + \lambda^{'}W + \lambda W'$$

and if we use the equation (2.6) and the alternative frame formulas, we get

$$(-\bar\kappa^*C^* + \bar\tau^*W^*)\frac{ds^*}{ds} = -\bar\kappa C + \bar\tau W + \lambda^{'}W - \lambda(gC).$$

From equation (3.2) and $W = C^*$,

$$(-\bar\kappa^*W + \bar\tau^*W^*)\frac{ds^*}{ds} = -\bar\kappa(\cos\theta N^* + \sin\theta W^*) + \bar\tau W + \lambda^{'}W - \lambda g(\cos\theta N^* + \sin\theta W^*)$$

$$\begin{aligned} \bar\kappa^*\frac{ds^*}{ds} &= -(\bar\tau+\lambda') \\ \bar\tau^*\frac{ds^*}{ds} &= -(\bar\kappa+\lambda g)\sin\theta \end{aligned}$$

and

$$\frac{\bar\tau^*}{\bar\kappa^*} = \frac{(\bar\kappa+\lambda g)\sin\theta}{(\bar\tau+\lambda')}. \tag{3.4}$$

Because of $\bar\tau^* = \dfrac{\tau^*}{f^*}$ and $\bar\kappa^* = \dfrac{\kappa^*}{f^*}$, we know that $\dfrac{\bar\tau^*}{\bar\kappa^*} = \dfrac{\tau^*}{\kappa^*}$. So, from equation (3.4), α^* is a helix if and only if

$$\frac{(\bar\kappa+\lambda g)\sin\theta}{(\bar\tau+\lambda')}$$

is constant. □

Theorem 5. *Let $\{\alpha,\alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. α^* is a slant helix if and only if*

$$\frac{g^*}{f^*} = \mathit{constant}$$

Proof. If we use the derivative of the alternative frame, we have

$$\frac{dN^*}{ds^*} = f^* C^*$$

and

$$\frac{dW^*}{ds^*} = -g^* C^*.$$

Using the above two equations, we obtain that

$$\frac{g^*}{f^*} = -\frac{\frac{dW^*}{ds^*}}{\frac{dN^*}{ds^*}}.$$

Also if we take the derivative of the first equality of equation (3.2) according to s, we get

$$\begin{aligned}\frac{dN^*}{ds^*}\frac{ds^*}{ds} &= \cos\theta\frac{d\theta}{ds}N + \sin\theta N' - \sin\theta\frac{d\theta}{ds}C + \cos\theta C', \\ &= \cos\theta\frac{d\theta}{ds}N + \sin\theta(fC) - \sin\theta\frac{d\theta}{ds}C + \cos\theta(-fN + gW), \\ &= \left(\cos\theta\frac{d\theta}{ds} - f\cos\theta\right)N + \left(f\sin\theta - \sin\theta\frac{d\theta}{ds}\right)C + g\cos\theta W \quad (3.5)\end{aligned}$$

From the proof of the Theorem 3, we know that

$$f = \frac{d\theta}{ds}. \tag{3.6}$$

If we use the above equation in (3.5) , we obtain that

$$\frac{dN^*}{ds^*}\frac{ds^*}{ds} = g\cos\theta W. \tag{3.7}$$

Similarly if we take the derivative of the second equality of equation (3.2) according to parameter s, we can easily see that

$$\begin{aligned}\frac{dW^*}{ds^*}\frac{ds^*}{ds} &= \sin\theta\frac{d\theta}{ds}N - \cos\theta N' + \cos\theta\frac{d\theta}{ds}C + \sin\theta C', \\ &= \sin\theta\frac{d\theta}{ds}N - \cos\theta(fC) + \cos\theta\frac{d\theta}{ds}C + \sin\theta(-fN + gW), \\ &= \left(\sin\theta\frac{d\theta}{ds} - f\sin\theta\right)N + \left(-f\cos\theta + \cos\theta\frac{d\theta}{ds}\right)C + g\sin\theta W \quad (3.8)\end{aligned}$$

If we use the equation (3.6) in (3.8) , we have

$$\frac{dW^*}{ds^*}\frac{ds^*}{ds} = g\sin\theta W. \tag{3.9}$$

By proportioning the equations (3.7) and (3.9) , we get

$$\frac{g^*}{f^*} = -\tan\theta = \text{constant}$$

□

Theorem 6. *Let $\{\alpha, \alpha^*\}$ be WC^*-partner curves in Euclidean $3-$space. Then the following relation exists*

$$\frac{g}{f} = \frac{f^*}{f\cos\theta}\frac{ds^*}{ds}$$

Proof. Using alternative frame $\{N, C, W\}$, we have

$$N' = fC \text{ and } W' = -gC.$$

If we calculate the ratio of these two equations, we obtain

$$\frac{g}{f} = -\frac{W'}{N'}. \tag{3.10}$$

From the following equations

$$\frac{dC^*}{ds^*} = -f^*N^* + g^*W^* \quad \text{and} \quad W = C^*,$$

we can see that

$$\frac{dW}{ds^*}\frac{ds^*}{ds} = (-f^*N^* + g^*W^*)\frac{ds^*}{ds}.$$

Also, from equation (3.2) ,

$$\begin{aligned} \frac{dW}{ds^*}\frac{ds^*}{ds} &= [-f^*(\sin\theta N + \cos\theta C) + g^*(-\cos\theta N + \sin\theta C)]\frac{ds^*}{ds} \\ &= [(-f^*\sin\theta - g^*\cos\theta)N + (-f^*\cos\theta + g^*\sin\theta)C]\frac{ds^*}{ds} \end{aligned} \tag{3.11}$$

If we use the equations $f^* = g\cos\theta$ and $g^* = -g\sin\theta$ in Theorem 2, we obtain

$$g^* = -\frac{f^*\sin\theta}{\cos\theta}$$

and if we write this equation in (3.11) , we get

$$\frac{dW}{ds^*}\frac{ds^*}{ds} = \left[\left(-f^*\sin\theta + \frac{f^*\sin\theta}{\cos\theta}\cos\theta\right)N + \left(-f^*\cos\theta - \frac{f^*\sin\theta}{\cos\theta}\sin\theta\right)C\right]\frac{ds^*}{ds}.$$

$$W' = -\frac{f^*}{\cos\theta}\frac{ds^*}{ds}C.$$

Also from the equation (3.10), we have

$$\frac{g}{f} = -\frac{W'}{N'} = -\frac{-\frac{f^*}{\cos\theta}\frac{ds^*}{ds}C}{fC}$$

$$\frac{g}{f} = \frac{f^*}{f\cos\theta}\frac{ds^*}{ds}.$$

So this completes the proof. □

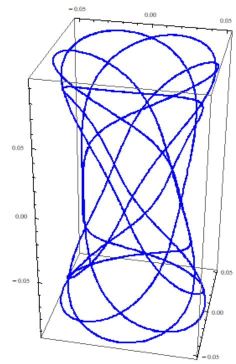

FIGURE 1. The curve α

Example 1. *Let α be spatial curve given by the parametrization ([9])*

$$\alpha(s) = (\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s, \frac{-9}{208}\cos 16s + \frac{1}{117}\cos 36s, \frac{6}{65}\sin 10s).$$

If the necessary arrangements are made, we obtain the curvatures of α as follows

$$\kappa(s) = -24\sin 10s,\ \tau(s) = 24\cos 10s,\ f(s) = 24,\ g(s) = 10.$$

From the Theorem 1, the distance function is obtained as $\lambda(s) = \dfrac{\sin 10s}{10}$. Then the WC^*-partner curve α^* of α is obtained as

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)W(s)$$

$$\begin{aligned} \alpha^*(s) \quad = \quad & (\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{9}{130}\cos 6s \sin 10s - \frac{4}{130}\cos 46s \sin 10s, \\ & \frac{-9}{208}\cos 16s + \frac{1}{117}\cos 36s + \frac{9}{130}\sin 6s \sin 10s - \frac{4}{130}\sin 46s \sin 10s, \\ & \frac{6}{65}\sin 10s + \frac{12}{130}\cos 20s \sin 10s). \end{aligned}$$

Figure 1 shows the graph of the curve α and Figure 2 shows the WC^*-partner curve α^*.

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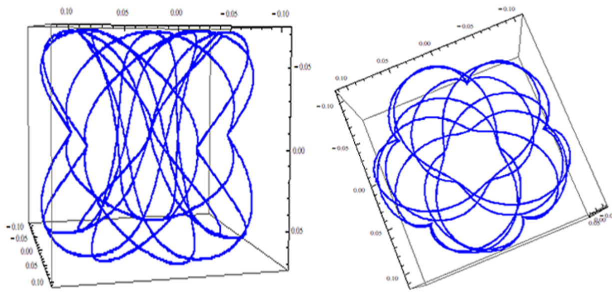

FIGURE 2. $WC^{*}-$partner curve of α

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