ON THE LIPSCHITZ STABILITY OF INVERSE NODAL PROBLEM FOR DIRAC SYSTEM

Emrah YILMAZ and Hikmet KOYUNBAKAN
Fırat University, Department of Mathematics, 23119, Elazığ, TURKEY

ABSTRACT. Inverse nodal problem on Dirac operator is determination problem of the parameters in the boundary conditions, number m and potential function V by using a set of nodal points of a component of two component vector eigenfunctions as the given spectral data. In this study, we solve a stability problem using nodal set of vector eigenfunctions and show that the space of all V functions is homeomorphic to the partition set of all space of asymptotically equivalent nodal sequences induced by an equivalence relation. Moreover, we give a reconstruction formula for the potential function as a limit of a sequence of functions and associated nodal data of one component of vector eigenfunction. Our technique depends on the explicit asymptotic expressions of the nodal parameters and, it is basically similar to [1, 2] which is given for Sturm-Liouville and Hill's operators, respectively.

1. INTRODUCTION

Inverse spectral problems have been a significant research area in mathematical physics. Different methods have been proposed to recover coefficient functions in differential equations by using spectral data [3–10]. Generally, the spectral data have consisted of the eigenvalues and a corresponding sequence of norming constants, or two eigenvalue sequences. In 1988, McLaughlin showed that knowledge of nodal points can determine the potential function of Sturm-Liouville problem up to a constant [11]. This is so called inverse nodal problem. Numerical schemes were then given by Hald and McLaughlin [12] to reconstruct the density function of a vibrating string, the elastic modulus of a vibrating rod, the potential function in Sturm-Liouville problem. Independently, Shen *et al.* [13] studied the relation between nodal points and density function of string equation in 1988. Many results and reconstruction formulas have been derived about inverse nodal problem

---

by several authors [14–18]. Here, we deal with the inverse nodal problem for Dirac system.

Dirac system is a modern presentation of the relativistic quantum mechanics of electrons intended to make new mathematical results accesible to a wider audience. It treats in some depth relativistic invariance of a quantum theory, self-adjointness and spectral theory, qualitative features of relativistic bound and scattering states and the external field problem in quantum electrodynamics, without neglecting the interpretational difficulties and limitations of the theory [19].

Inverse problems for Dirac system had been investigated by Moses [20], Prats and Toll [21], Verde [22], Gasymov and Levitan [23], and Panakhov [24]. It is well known that two spectra uniquely determine the matrix valued potential function in Dirac system [25]. In [26], eigenfunction expansions for one dimensional Dirac operator describing the motion of a particle in quantum mechanics were investigated. In addition, inverse spectral problems for weighted Dirac system were studied in [27].

One studied the properties of the eigenvalues and vector-valued eigenfunctions for the Dirac system with the same spectral parameter in the equations and the boundary conditions [28]. Sampling theory of signal analysis associated with Dirac systems, when the eigenvalue parameter appears linearly in the boundary conditions was investigated in [29]. One investigated a problem for the Dirac differential operators in the case where an eigenparameter not only appears in the differential equation but is also linearly contained in a boundary condition, and proved uniqueness theorems for inverse spectral problem with known collection of eigenvalues and normalizing constants or two spectra [30]. Other than these studies, there are many papers in literature (see [31–35]).

Inverse nodal problems for Dirac system had not been studied until the works of Yang and Huang [36]. They gave reconstruction formulas for one dimensional Dirac operator by using nodal datas. Later years, inverse nodal problem was solved for Dirac system under different boundary conditions [37, 38].

Consider the Dirac system

$$By'(x) + Q(x)y(x) = \lambda y(x), 0 \leq x \leq \pi, \tag{1}$$

with boundary conditions

$$\left(\lambda \cos \alpha + a_0\right) y_1(0) + \left(\lambda \sin \alpha + b_0\right) y_2(0) = 0,$$
$$\left(\lambda \cos \beta + a_1\right) y_1(\pi) + \left(\lambda \sin \beta + b_1\right) y_2(\pi) = 0, \tag{2}$$

where λ is a spectral parameter,

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} V(x) + m & 0 \\ 0 & V(x) - m \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \tag{3}$$

and V is a real valued, continuous function on $[0, \pi]$. Furthermore, $m, a_k, b_k (k = 0, 1), \alpha$ and β are real constants: moreover $-\frac{\pi}{2} \leq \alpha, \beta \leq \frac{\pi}{2}$ [38]. Throughout the

paper [38], Yang and Pivovarchik supposed that

$$a_0 \sin \alpha - b_0 \cos \alpha > 0,$$
$$a_1 \sin \beta - b_1 \cos \beta < 0. \tag{4}$$

The properties of the eigenvalues and eigenfunctions of the problem (1)-(2) were studied in [28]. Under the condition (4), the eigenvalues of the problem (1)-(2) are real and algebraically simple [28]. Considering (3) in (1), we get

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} + \begin{pmatrix} V(x) + m & 0 \\ 0 & V(x) - m \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \lambda \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

and thus, equation (1) is equivalent to a system of two simultaneous first order differential equations

$$y_2'(x, \lambda) + [V(x) + m]\, y_1(x, \lambda) - \lambda y_1(x, \lambda) = 0,$$
$$-y_1'(x, \lambda) + [V(x) - m]\, y_2(x, \lambda) - \lambda y_2(x, \lambda) = 0. \tag{5}$$

In general, potential function of Dirac system (1) has the following form

$$Q(x) = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix},$$

where $p_{ik}(x)$ $(i, k = 1, 2)$ are real valued and continuous functions on $[0, \pi]$. For the case in which $p_{12}(x) = p_{21}(x) = 0$ and $p_{11}(x) = V(x) + m$, $p_{22}(x) = V(x) - m$, where m is the mass of particle, the system (5) is known in relativistic quantum theory as a stationary one dimensional Dirac system or first canonical form of Dirac system [4].

Let $y(x, \lambda_n) = [y_1(x, \lambda_n), y_2(x, \lambda_n)]^T$ be two dimensional vector eigenfunction of the Dirac system (1) related to the eigenvalue $\lambda = \lambda_n$, where T denotes transpose. Assume that $x_n^{j,i}$ are the nodal points of $i-$th component $y_i(x, \lambda_n)$ of the $n-$th eigenfunction $y(x, \lambda_n)$, where $0 < x_n^{1,i} < x_n^{2,i} < ... < x_n^{n-1,i} < \pi$. In other words, $y_i(x_n^{j,i}, \lambda_n) = 0$. Let $I_n^{j,i} = \left(x_n^{j,i}, x_n^{j+1,i}\right)$ be the $j-$th nodal domain, and let

$$l_n^{j,i} = x_n^{j+1,i} - x_n^{j,i},$$

be the associated nodal length. For simplicity, we agree that $x_n^{0,i} = 0$ and $x_n^{|n|+1-i,i} = \pi$. We also define the function $j_{n,i}(x)$ to be the largest index j_i such that $0 \leq x_n^{j,i} \leq x$ for $n > 0$ and $j_{n,i}(x)$ to be the largest index j_i such that $0 \leq x \leq x_n^{j,i}$ for $n < 0$. Thus, $j_i = j_{n,i}(x)$ if and only if $x \in [x_n^{j,i}, x_n^{j+1,i})$ for $n > 0$ and $(x_n^{j+1,i}, x_n^{j,i}]$ for $n < 0$ [36].

Denote $\Lambda^i = \{x_n^{j,i}\}$, $i = 1, 2$. Hence, $\Lambda = \Lambda^1 \cup \Lambda^2$ is called the set of all nodal points of Dirac operator. This set is dense on $[0, \pi]$ [38]. Throughout this study, we'll give all proofs for the first component of the eigenfunction.

The rest of this study is arranged as follows: In remaining part of section 1, we give some properties of Dirac system and quote some important results to use in main theorems. In section 2, we obtain some reconstruction formulas for potential function under different boundary conditions. Finally, we define d_0, $d_{\Sigma_{Dir}}$ to prove

Lipschitz stability of inverse nodal problem. Then, we express Theorem 14 in section 3. By this theorem, we prove the Lipschitz stability of inverse nodal problem for Dirac operator.

Now, we need to remind some conclusions which are given by [38] to use in our main results.

Lemma 1. *[38] The spectrum of the problem (1)-(2) consists of eigenvalues* $\{\lambda_n\}_{n\in\mathbb{Z}}$ *which are all real and algebraically simple behave asymptotically as*

$$\lambda_n = n - 2 + \frac{v}{\pi} + \frac{c}{n} + O\left(\frac{1}{n^2}\right), n \to \infty, \tag{6}$$

and

$$\lambda_{-n} = -n + \frac{v}{\pi} - \frac{c}{n} + O\left(\frac{1}{n^2}\right), n \to \infty,$$

where

$$v = \int_0^\pi V(t)dt + \beta - \alpha,$$

$$c = \frac{m^2}{2} + \frac{m}{2\pi}\left(\sin 2\alpha - \sin 2\beta\right) + \frac{a_0 \sin \alpha - b_0 \cos \alpha}{\pi} - \frac{a_1 \sin \beta - b_1 \cos \beta}{\pi}.$$

Lemma 2. *[38] Let* $y(x, \lambda) = \begin{pmatrix} y_1(x, \lambda) \\ y_2(x, \lambda) \end{pmatrix}$ *be the solution of (1) satisfying the condition*

$$y(0, \lambda) = \begin{pmatrix} -(\lambda \sin \alpha + b_0) \\ \lambda \cos \alpha + a_0 \end{pmatrix}, \tag{7}$$

then, we have

$$y_1(x, \lambda) = -\lambda \sin\left(\lambda x - \int_0^x V(t)dt + \alpha\right) + \frac{m^2}{2}x\cos\left(\lambda x - \int_0^x V(t)dt + \alpha\right)$$

$$- m\cos\alpha \sin\left(\lambda x - \int_0^x V(t)dt\right) - a_0 \sin\left(\lambda x - \int_0^x V(t)dt\right) \tag{8}$$

$$- b_0 \cos\left(\lambda x - \int_0^x V(t)dt\right) + O\left(\frac{e^{\tau x}}{\lambda}\right),$$

and

$$y_2(x, \lambda) = \lambda \cos\left(\lambda x - \int_0^x V(t)dt + \alpha\right) + \frac{m^2}{2}x\sin\left(\lambda x - \int_0^x V(t)dt + \alpha\right)$$

$$+ m \sin \alpha \sin \left(\lambda x - \int_0^x V(t)dt \right) + a_0 \cos \left(\lambda x - \int_0^x V(t)dt \right) \tag{9}$$

$$- b_0 \sin \left(\lambda x - \int_0^x V(t)dt \right) + O \left(\frac{e^{\tau x}}{\lambda} \right),$$

where $\tau = |\mathrm{Im}\,\lambda|$.

Lemma 3. *[38] For sufficiently large $n > 0$, the first component $y_1(x, \lambda_n)$ of the eigenfunction $y(x, \lambda_n)$ for Dirac system has exactly $N(\alpha, \beta)$ nodes in the interval $(0, \pi)$ where*

$$N(\alpha, \beta) = \begin{cases} n - 2, & \text{for } \alpha \geq 0 \text{ and } \beta > 0 \text{ or } \quad \text{for } \alpha < 0 \text{ and } \beta \leq 0 \\ n - 3, & \text{for } \alpha \geq 0 \text{ and } \beta \leq 0 \\ n - 1, & \text{for } \alpha < 0 \text{ and } \beta > 0. \end{cases}$$

Moreover, uniformly with respect to $j \in \{1, 2, ..., N(\alpha, \beta)\}$, the nodal parameters of the problem (1)-(7) has the following asymptotic formulas, respectively for sufficiently large n,

$$x_n^{j,1} = \frac{2\lambda_n^2}{2\lambda_n^2 - m^2} \left[\frac{j\pi}{\lambda_n} + \frac{1}{\lambda_n} \int_0^{x_n^{j,1}} V(t)dt - \frac{\alpha}{\lambda_n} + \frac{m \sin 2\alpha}{2\lambda_n^2} \right.$$

$$\left. + \frac{a_0 \sin \alpha - b_0 \cos \alpha}{\lambda_n^2} + O \left(\frac{1}{\lambda_n^3} \right) \right]$$

and

$$l_n^{j,1} = \frac{2\lambda_n^2}{2\lambda_n^2 - m^2} \left[\frac{\pi}{\lambda_n} + \frac{1}{\lambda_n} \int_{x_n^{j,1}}^{x_n^{j+1,1}} V(t)dt + O \left(\frac{1}{\lambda_n^3} \right) \right],$$

where $n \neq \mp \frac{m}{\sqrt{2}} + 2$. Now, we consider the system (1) with boundary conditions

$$u_1(0) \cos \widetilde{\alpha} + u_2(0) \sin \widetilde{\alpha} = 0,$$

$$u_1(\pi) \cos \widetilde{\beta} + u_2(\pi) \sin \widetilde{\beta} = 0, \tag{10}$$

where $0 \leq \widetilde{\alpha}, \widetilde{\beta} \leq \pi$, and m is positive in (3). It is well known that the spectrum of the system (1) with the boundary conditions (10) includes the eigenvalues $\widetilde{\lambda}_n, n \in \mathbb{Z}$ which are all real and simple, and the sequence $\{\widetilde{\lambda}_n\}$ satisfies the classical asymptotic form [4], [36]

$$\widetilde{\lambda}_n = n + \frac{\widetilde{v}}{\pi} + \frac{\widetilde{c}_1}{n} + O \left(\frac{1}{n^2} \right), \tag{11}$$

where

$$\widetilde{v} = \widetilde{\beta} - \widetilde{\alpha} + \int\limits_0^\pi \widetilde{V}(t)dt, \widetilde{c}_1 = \frac{m(\sin 2\widetilde{\alpha} - \sin 2\widetilde{\beta}) + m^2\pi}{2\pi \cos^2\left(\int\limits_0^\pi \widetilde{V}(t)dt - \widetilde{\alpha} + \widetilde{\beta}\right)}.$$

Let $u(x,\widetilde{\lambda}) = \left(u_1(x,\widetilde{\lambda}), u_2(x,\widetilde{\lambda})\right)^T$ be the solution of the system (1) with initial conditions

$$u_1(0,\widetilde{\lambda}) = \sin\widetilde{\alpha}, u_2(0,\widetilde{\lambda}) = -\cos\widetilde{\alpha}. \tag{12}$$

Then, by successive approximations method, there hold

$$u_1(x,\widetilde{\lambda}) = \sin\left(\widetilde{\lambda}x - \int\limits_0^x \widetilde{V}(t)dt + \widetilde{\alpha}\right) - \frac{U_1}{\widetilde{\lambda}} + O\left(\frac{e^{|\tau\widetilde{\lambda}|x}}{\widetilde{\lambda}^2}\right),$$

$$u_2(x,\widetilde{\lambda}) = -\cos\left(\widetilde{\lambda}x - \int\limits_0^x \widetilde{V}(t)dt + \widetilde{\alpha}\right) - \frac{U_2}{\widetilde{\lambda}} + O\left(\frac{e^{|\tau\widetilde{\lambda}|x}}{\widetilde{\lambda}^2}\right), \tag{13}$$

for large $\left|\widetilde{\lambda}\right|$, where [36]

$$U_1(x,\widetilde{\lambda}) = -m\sin\left(\widetilde{\lambda}x - \int\limits_0^x \widetilde{V}(t)dt\right)\cos\widetilde{\alpha} + \frac{m^2}{2}x\cos\left(\widetilde{\lambda}x - \int\limits_0^x \widetilde{V}(t)dt + \widetilde{\alpha}\right),$$

$$U_2(x,\widetilde{\lambda}) = m\sin\left(\widetilde{\lambda}x - \int\limits_0^x \widetilde{V}(t)dt\right)\sin\widetilde{\alpha} + \frac{m^2}{2}x\sin\left(\widetilde{\lambda}x - \int\limits_0^x \widetilde{V}(t)dt + \widetilde{\alpha}\right). \tag{14}$$

Lemma 4. *[36] For sufficiently large $|n|$, the $i-th$ component $u_i(x,\widetilde{\lambda}_n)$ of the eigenfunction $u(x,\widetilde{\lambda}_n)$ of the problem (1),(2) has exactly $|n| + 1 - i$ nodes in the interval $(0,\pi)$. Moreover, the asymptotic formulas for nodal points of first and second components of the eigenfunction $u(x,\widetilde{\lambda}_n)$ as $|n| \to \infty$ uniformly with respect to $j \in \mathbb{Z}$ are as following*

$$\widetilde{x}_n^{j,1} = \frac{2\widetilde{\lambda}_n^2}{2\widetilde{\lambda}_n^2 - (-1)^j m^2}\left[\frac{j\pi}{\widetilde{\lambda}_n} + \frac{1}{\widetilde{\lambda}_n}\int\limits_0^{\widetilde{x}_n^{j,1}} \widetilde{V}(t)dt - \frac{\widetilde{\alpha}}{\widetilde{\lambda}_n} + \frac{(-1)^j m\sin 2\widetilde{\alpha}}{2\widetilde{\lambda}_n^2} + O\left(\frac{1}{\widetilde{\lambda}_n^3}\right)\right], \tag{15}$$

and

$$\widetilde{x}_n^{j,2} = \frac{2\widetilde{\lambda}_n^2}{2\widetilde{\lambda}_n^2 - (-1)^j m^2}\left[\frac{\left(j - \frac{1}{2}\right)\pi}{\widetilde{\lambda}_n} + \frac{1}{\widetilde{\lambda}_n}\int\limits_0^{\widetilde{x}_n^{j,2}} \widetilde{V}(t)dt - \frac{\widetilde{\alpha}}{\widetilde{\lambda}_n}\right.$$

$$+\frac{(-1)^{j+1}m\sin 2\widetilde{\alpha}}{2\widetilde{\lambda}_n^2}+O\left(\frac{1}{\widetilde{\lambda}_n^3}\right)\Bigg] \qquad (16)$$

where $n\neq \mp\frac{(-1)^{\frac{j}{2}}m}{\sqrt{2}}$. The nodal lengths $\widetilde{l}_n^{j,1}$ for the problem (1), (12) have the following asymptotic expansions

$$\widetilde{l}_n^{j,1} \;=\; \frac{\pi}{\widetilde{\lambda}_n}+\frac{1}{\widetilde{\lambda}_n}\int_{\widetilde{x}_n^{j,1}}^{\widetilde{x}_n^{j+1,1}}\widetilde{V}(t)dt+\frac{\{(-1)^{j+1}-(-1)^j\}m\sin 2\widetilde{\alpha}}{2\widetilde{\lambda}_n^2}$$

$$+\frac{\{(-1)^{j+1}\widetilde{x}_n^{j+1,1}-(-1)^j\widetilde{x}_n^{j,1}\}m^2}{2\widetilde{\lambda}_n^2}+O\left(\frac{1}{\widetilde{\lambda}_n^3}\right).$$

In case of $j=2k$ (or $j=2k+1$), $k\in\mathbb{Z}$; we get

$$\widetilde{l}_n^{j,1}=\frac{2\widetilde{\lambda}_n^2}{2\widetilde{\lambda}_n^2\pm m^2}\left[\frac{\pi}{\widetilde{\lambda}_n}+\frac{1}{\widetilde{\lambda}_n}\int_{\widetilde{x}_n^{j,1}}^{\widetilde{x}_n^{j+1,1}}\widetilde{V}(t)dt\pm\frac{m\sin 2\widetilde{\alpha}}{2\widetilde{\lambda}_n^2}+O\left(\frac{1}{\widetilde{\lambda}_n^3}\right)\right]. \qquad (17)$$

We can easily obtain $\widetilde{l}_n^{j,2}$ similarly as $|n|\rightarrow\infty$ by using definition of nodal lengths and (16). Here, $\{x_n^{j,i}\}$, $\{\widetilde{x}_n^{j,i}\}$, $i=1,2$ and $\{\lambda_n\},\{\widetilde{\lambda}_n\}$ are the nodal sets and eigenvalues of the problems (1), (7) and (1), (12), respectively.

Theorem 5. $V\in L_1(0,\pi)$. Then, for almost every $x\in(0,\pi)$, with $j_i=j_{n,i}(x)$,

$$\lim_{n\rightarrow\infty}\lambda_n\int_{x_n^{j,i}}^{x_n^{j+1,i}}V(t)dt=V(x),\quad \lim_{n\rightarrow\infty}\lambda_n\int_{x_n^{j,i}}^{x_n^{j+1,i}}\cos(2\lambda_n\pi t)V(t)dt=0,$$

where $i=1,2$ and $\lambda_n=n-2$ for the problem (1), (7). We can express the similar theorem for the problem (1), (12).

Proof. It can be proved by similar method given in [1]. The fact that the function V is on the $L_1(0,\pi)$ space is used here. $\qquad\square$

Remark 6. [36] $\{x_n^{j,i}\}\subset\Lambda^i$, $\{\widetilde{x}_n^{j,i}\}\subset\widetilde{\Lambda}^i$ are chosen such that

$$\lim_{n\rightarrow\infty}x_n^{j,i}=x=\lim_{n\rightarrow\infty}\widetilde{x}_n^{j,i},$$

where $i=1,2$ and $x\in[0,\pi]$.

2. Reconstruction of potential function by using nodal points

In this section, we will derive some reconstruction formulas of potential functions V and $\widetilde{V}$, where $l_n^{j,i},\widetilde{l}_n^{j,i}$ and $x_n^{j,i},\widetilde{x}_n^{j,i}$ are nodal lengths and nodal points for the problems (1), (7) and (1), (12), respectively. Here, all of our proofs and definitions will be given for the first component of eigenfunction (That is, for $i=1$).

Theorem 7. *Let $V, \widetilde{V} \in L_1[0, \pi]$ be the potential functions for Dirac system under the conditions (7) and (12), respectively. Define F_n by*
a) for the problem (1), (7),

$$F_n(x) = (n-2) \left\{ \sum_{j=1}^{n-1} \left[n - 2 - \frac{m^2}{2(n-2)} \right] l_n^{j,1} - \pi \right\}.$$

b) for the problem (1), (12),

$$\widetilde{F}_n(x) = n \left\{ \sum_{j=1}^{n-1} \left[n \pm \frac{m^2}{2n} \right] \widetilde{l}_n^{j,i} \pm \frac{m^2}{n} \widetilde{x}_n^{j,i} - \pi \right\},$$

where $l_n^{j,i} = x_n^{j+1,i} - x_n^{j,i}$ and $\widetilde{l}_n^{j,i} = \widetilde{x}_n^{j+1,i} - \widetilde{x}_n^{j,i}$ (j is odd or even). Then, F_n and $\widetilde{F}_n$ converge to V and $\widetilde{V}$ pointwisely almost everywhere, respectively and also in L_1 sense .Moreover, pointwise convergence holds for all the continuity points of V and $\widetilde{V}$.

Proof. a) We will consider the reconstruction formula for the potential function of the problem (1), (7). Observe that, by Lemma 3, we have

$$\frac{\lambda_n}{\pi} l_n^{j,1} - 1 - \frac{m^2 l_n^{j,1}}{2\lambda_n \pi} = \frac{1}{\pi} \int_{x_n^{j,1}}^{x_n^{j+1,1}} V(t)dt + O\left(\frac{1}{\lambda_n^2}\right),$$

and

$$\lambda_n \left[l_n^{j,1} \left(\lambda_n - \frac{m^2}{2\lambda_n} \right) - \pi \right] = \lambda_n \int_{x_n^{j,1}}^{x_n^{j+1,1}} V(t)dt + O\left(\frac{1}{\lambda_n}\right).$$

Then, by using asymptotic expansions for eigenvalues, we obtain

$$\lambda_n \left[l_n^{j,1} \left(\lambda_n - \frac{m^2}{2\lambda_n} \right) - \pi \right] = \left[n - 2 + O\left(\frac{1}{n}\right) \right] \left[l_n^{j,1} \left(n - 2 + O\left(\frac{1}{n}\right) \right. \right.$$

$$\left. \left. - \frac{m^2}{2(n-2) + O\left(\frac{1}{n}\right)} \right) - \pi \right]$$

$$= \left[n - 2 + O\left(\frac{1}{n}\right) \right] \left[l_n^{j,1} \left(n - 2 - \frac{m^2}{2(n-2)} + O\left(\frac{1}{n}\right) \right) - \pi \right]$$

$$= (n-2) \left[l_n^{j,1} \left(n - 2 - \frac{m^2}{2(n-2)} \right) - \pi \right] + o(1).$$

Hence, to prove Theorem 7 (a), it suffices to show Theorem 8.

(b) It can be proved analogously. To complete the proof of Theorem 7 (b), it suffices to express Theorem 9.   □

Theorem 8. *The potential function* $V \in L_1(0, \pi)$ *of the problem (1),(7) satisfies*

$$V(x) = \lim_{n \to \infty} \left[l_n^{j,1} \lambda_n - m^2 \frac{l_n^{j,1}}{2\lambda_n} - \pi \right] \lambda_n,$$

for almost every $x \in (0, \pi)$, *with* $j_1 = j_{n,1}(x)$.

Proof. Lemma 3 yields

$$l_n^{j,1} - \frac{m^2 l_n^{j,1}}{2\lambda_n^2} = \frac{\pi}{\lambda_n} + \frac{\displaystyle\int_{x_n^{j,1}}^{x_n^{j+1,1}} V(t)dt}{\lambda_n} + O\left(\frac{1}{\lambda_n^3}\right),$$

so that

$$\left[\left(l_n^{j,1} - \frac{m^2 l_n^{j,1}}{2\lambda_n^2} \right) \lambda_n - \pi \right] \lambda_n = \lambda_n \int_{x_n^{j,1}}^{x_n^{j+1,1}} V(t)dt + O\left(\frac{1}{\lambda_n}\right). \tag{18}$$

We may assume $x_n^{j,1} \neq x$. By Theorem 5 , if we take limit of both sides of (18) as $n \to \infty$ for almost $x \in (0, \pi)$, we get

$$V(x) = \lim_{n \to \infty} \left[l_n^{j,1} \lambda_n - m^2 \frac{l_n^{j,1}}{2\lambda_n} - \pi \right] \lambda_n.$$

$\square$

Theorem 9. *The potential function* $\widetilde{V} \in L_1(0, \pi)$ *of the problem (1),(12) satisfies*

$$\widetilde{V}(x) = \lim_{n \to \infty} \left[\widetilde{l}_n^{j,1} \widetilde{\lambda}_n \pm m^2 \frac{(\widetilde{x}_n^{j,1} + \widetilde{x}_n^{j+1,1})}{2\widetilde{\lambda}_n} - \pi \right] \widetilde{\lambda}_n \pm m \sin 2\widetilde{\alpha},$$

for almost every $x \in (0, \pi)$, *with* $j_1 = j_{n,1}(x)$.

Proof. It can be proved by using similar process to Theorem 8. $\square$

3. Main Results

In this section, we solve a Lipschitz stability problem for Dirac operator. Lipschitz stability is about a continuity between two metric spaces. So, we have to first construct these spaces. To show continuity, we use a homeomorphism between these spaces. Stability problems were studied by many authors [2, 39, 40]. To solve stability problem, we give a main theorem which execute that the inverse nodal problem for Dirac system is stable with Lipschitz stability. Here and later, we denote the space of all admissible nodal sequences which converge to V by $X = \{X_n^{k,i}\}$, where $L_n^{k,i} = X_n^{k+1,i} - X_n^{k,i}, i = 1, 2$.

Definition 10. *Let* $\mathbb{N}' = \mathbb{N} - \{1\}$. *We denote the space* Ω_{Dir} *of all potential functions of Dirac system and the space* Σ_{Dir} *of all admissible sequences by*

(i):

$\Omega_{Dir} = \{V \in L_1[0, \pi] : V \text{ is the potential function of the Dirac system}\}$,

and $\Sigma_{Dir} = $ The collection of the all double sequences defined as

$X = \left\{ X_n^{k,1} : k = 1, 2, ..., n; n \in \mathbb{N}', 0 < X_n^{1,1} < X_n^{2,1} < ... < X_n^{n-1,1} < \pi \right\}$

for each $n \in \mathbb{N}$.

(ii): Let $X \in \Sigma_{Dir}$ and define $X = \{X_n^{k,1}\}$ where $I_n^{k,1} = \left(X_n^{k,1}, X_n^{k+1,1} \right)$. We say X is quasinodal to some $V \in L_1(0, \pi)$ if X is an admissible sequence of nodes and satisfies (I) and (II) below:

(I) X has the following asymptotics uniformly for k, as $n \to \infty$

$$X_n^{k,1} = \frac{k\pi}{n-2} + O\left(\frac{1}{n}\right), k = 1, 2, ..., n$$

for the problem (1), (7). And the sequence

$$F_n = (n-2)\left\{ \sum_{k=1}^{n-1} \left[n - 2 - \frac{m^2}{2(n-2)} \right] L_n^{k,1} - \pi \right\},$$

converges to V in L_1.

(II) For the problem (1), (12), X has below asymptotics uniformly for k, as $n \to \infty$

$$X_n^{k,1} = \frac{k\pi}{n} + O\left(\frac{1}{n}\right), k = 1, 2, ..., n$$

and the sequence

$$F_n = n\left\{ \sum_{k=1}^{n-1} \left[n \pm \frac{m^2}{2n} \right] L_n^{k,1} \pm \frac{m^2}{n} X_n^{k,1} - \pi \right\},$$

converges to V in L_1. $X \in \Sigma_{Dir}$ is nodal if X satisfies one of the above asymptotic behaviours.

We denote Ω_{Dir} as a collection of all Dirac operators and the space Σ_{Dir} as a collection of all admissible double sequences of nodes such that related functions are convergent in L_1. A pseudometric $d_{\Sigma_{Dir}}$ on Σ_{Dir} will be defined. For convenience, we will use the notation X for the first component. Essentially, $d_{\Sigma_{Dir}}(X, \overline{X})$ is so close to

$$d_0(X, \overline{X}) = \overline{\lim_{n \to \infty}} \pi \left[n - 2 - \frac{m^2}{2(n-2)} \right] \sum_{k=1}^{n-1} \left| L_n^{k,1} - \overline{L}_n^{k,1} \right|,$$

where $n > \frac{m}{\sqrt{2}} + 2$ and $L_n^{k,1} = X_n^{k+1,1} - X_n^{k,1}$, $\overline{L}_n^{k,1} = \overline{X}_n^{k+1,1} - \overline{X}_n^{k,1}$.

If we define $X \sim \overline{X}$ if and only if $d_{\Sigma_{Dir}}(X, \overline{X}) = 0$, then $\sim$ is an equivalence relation on Σ_{Dir} and $d_{\Sigma_{Dir}}$ would be a metric for the partition set $\Sigma_{Dir}^* = \Sigma_{Dir} / \sim$. Let $\Sigma_{Dir_1} \subset \Sigma_{Dir}$ be the subspace of all asymptotically equivalent nodal sequences

and let $\Sigma^*_{Dir_1} = \Sigma_{Dir_1}/ \sim$. Let Φ be a homeomorphism the maps Ω_{Dir} onto $\Sigma^*_{Dir_1}$. We will call Φ as a nodal map.

Lemma 11. *Let* $X, \overline{X} \in \Sigma_{Dir}$.
  a) If X belongs to case I, then

$$L_n^{k,1} = \frac{\pi}{n-2} + O\left(\frac{1}{n}\right), k = 1, 2, ..., n.$$

If X belongs to case II, then

$$L_n^{k,1} = \frac{\pi}{n} + O\left(\frac{1}{n}\right), k = 1, 2, ..., n.$$

  b) $\chi_{n,k} = \left| X_n^{k,1} - \overline{X}_n^{k,1} \right| = O\left(\frac{1}{n}\right)$.
  c) For all $x \in (0, \pi)$, *define* $J_{n,1}(x) = maks\{k : X_n^{k,1} \leq x\}$ *so that* $k = J_{n,1}(x)$ *if and only if* $x \in [X_n^{J_1,1}, X_n^{J_1+1,1}]$. *Then, for sufficiently large n,*

$$\left| J_{n,1}(x) - \overline{J}_{n,1}(x) \right| \leq 1.$$

Proof. a) For case I, we get

$$L_n^{k,1} = X_n^{k+1,1} - X_n^{k,1} = \frac{\pi}{n-2} + O\left(\frac{1}{n}\right),$$

by using the definition of nodal lengths. Similarly, for the case (II), we obtain

$$L_n^{k,1} = X_n^{k+1,1} - X_n^{k,1} = \frac{\pi}{n} + O\left(\frac{1}{n}\right).$$

  b) We only consider case I. The other case is similar. By using asymptotic estimates, we get

$$\left| \chi_{n,k} \right| = \left| X_n^{k,1} - \overline{X}_n^{k,1} \right| \leq \left| X_n^{k,1} - \frac{k\pi}{n-2} \right| + \left| \frac{k\pi}{n-2} - \overline{X}_n^{k,1} \right| = O\left(\frac{1}{n}\right).$$

  c) Fix $x \in (0, \pi)$. Let $J_1 = J_{n,1}(x)$ and $\overline{J}_1 = \overline{J}_{n,1}(x)$. Since

$$X_n^{J_1,1} \leq x \leq X_n^{J_1+1,1} \Rightarrow \frac{J_1\pi}{n-2} + O\left(\frac{1}{n}\right) = X_n^{J_1,1} \leq x \leq X_n^{J_1+1,1} = \frac{(J_1+1)\pi}{n-2} + O\left(\frac{1}{n}\right),$$

and

$$\overline{X}_n^{\overline{J}_1,1} \leq x \leq \overline{X}_n^{\overline{J}_1+1,1} \Rightarrow \frac{\overline{J}_1\pi}{n-2} + O\left(\frac{1}{n}\right) = \overline{X}_n^{\overline{J}_1,1} \leq x \leq \overline{X}_n^{\overline{J}_1+1,1} = \frac{\overline{J}_1\pi}{n-2} + O\left(\frac{1}{n}\right),$$

when n is large enough, $\overline{J}_1 + 1 \geq J_1$ and $J_1 + 1 \geq \overline{J}_1$. Hence, $-1 \leq \overline{J}_1 - J_1 \leq 1$, then $\left| \overline{J}_1 - J_1 \right| \leq 1$. $\qquad \square$

Definition 12. *Suppose that* $X, \overline{X} \in \Sigma_{Dir}$ *with* $L_n^{k,1}$ *and* $\overline{L}_n^{k,1}$ *are their respective grid lengths. Let*

$$S_n\left(X, \overline{X}\right) = \pi \left[n - 2 - \frac{m^2}{2(n-2)}\right] \sum_{k=1}^{n-1} \left|L_n^{k,1} - \overline{L}_n^{k,1}\right|. \qquad (19)$$

Define

$$d_0\left(X, \overline{X}\right) = \overline{\lim_{n \to \infty}} S_n\left(X, \overline{X}\right) \ \ and \ d_{\Sigma_{Dir}}\left(X, \overline{X}\right) = \overline{\lim_{n \to \infty}} \frac{S_n\left(X, \overline{X}\right)}{1 + S_n\left(X, \overline{X}\right)}.$$

We get this metric by evaluating $\left\|V - \overline{V}\right\|_1$ in Theorem 14 This definition was first made by [1]. Since the function $f(x) = \dfrac{x}{1+x}$ is monotonic, we have

$$d_{\Sigma_{Dir}}\left(X, \overline{X}\right) = \frac{d_0\left(X, \overline{X}\right)}{1 + d_0\left(X, \overline{X}\right)} \in [0, \pi],$$

admitting that if $d_0\left(X, \overline{X}\right) = \infty$, then $d_{\Sigma_{Dir}}\left(X, \overline{X}\right) = 1$. Conversely

$$d_0\left(X, \overline{X}\right) = \frac{d_{\Sigma_{Dir}}\left(X, \overline{X}\right)}{1 - d_{\Sigma_{Dir}}\left(X, \overline{X}\right)}.$$

We can easily prove this equality by using Law's method [1, 2].

Lemma 13. *Let* $X, \overline{X} \in \Sigma_{Dir}$.
  a) $d_{\Sigma_{Dir}}$ *is a pseudometric on* Σ_{Dir}.
  b) If X *and* $\overline{X}$ *belong to different cases,* $d_{\Sigma_{Dir}}\left(X, \overline{X}\right) = 1$.

Proof. It can be proved similar way with in [1]. While realizing the conditions for being metric in the first proof, the space of all admissible nodal sequences is used in the second proof.   □

Stability problems for Sturm-Liouville and Hill's operators were studied in [1, 2] respectively. Now, we prove the stability of the inverse nodal problem for Dirac operator with Lipschitz stability. The below theorem guarantees the Lipschitz stability of inverse nodal problem for Dirac operator.

Theorem 14. *The metric spaces* $\left(\Omega_{Dir}, \|.\|_1\right)$ *and* $\left(\Sigma_{Dir_1}^* / \sim, d_{\Sigma_{Dir}}\right)$ *are homeomorphic to each other. Here,* $\sim$ *is the equivalence relation induced by* $d_{\Sigma_{Dir}}$. *Furthermore*

$$\left\|V - \overline{V}\right\|_1 = \frac{d_{\Sigma_{Dir}}\left(X, \overline{X}\right)}{1 - d_{\Sigma_{Dir}}\left(X, \overline{X}\right)},$$

where $d_{\Sigma_{Dir}}\left(X, \overline{X}\right) < 1$.

Proof. By Lemma 13 we only need to consider when $X, \overline{X} \in \Sigma_{Dir}$ belong to same case. Without loss of generality, let $X, \overline{X}$ belong to case I. In this case, we should denote

$$\left\| V - \overline{V} \right\|_1 = d_0 \left(X, \overline{X} \right).$$

According to the Theorem 7, F_n and $\overline{F}_n$ converge to V and $\overline{V}$, respectively. If we use the definition of norm in L_1 for the functions V and $\overline{V}$, we have

$$\left\| V - \overline{V} \right\|_1 = (n-2) \left[n - 2 - \frac{m^2}{2(n-2)} \right] \int_0^\pi \left| L_n^{J_{n,1}(x),1} - \overline{L}_n^{\overline{J}_{n,1}(x),1} \right| dx + o(1).$$

Hence by Fatou's Lemma,

$$\left\| V - \overline{V} \right\|_1 \leq (n-2) \left[n - 2 - \frac{m^2}{2(n-2)} \right] \int_0^\pi \left| L_n^{J_{n,1}(x),1} - \overline{L}_n^{J_{n,1}(x),1} \right| dx$$

$$+ (n-2) \left[n - 2 - \frac{m^2}{2(n-2)} \right] \int_0^\pi \left| \overline{L}_n^{J_{n,1}(x),1} - \overline{L}_n^{\overline{J}_{n,1}(x),1} \right| dx. \qquad (20)$$

Here, first and second terms can be written as

$$\int_0^\pi \left| L_n^{J_{n,1}(x),1} - \overline{L}_n^{J_{n,1}(x),1} \right| dx = \frac{\pi}{n-2} \sum_{k=1}^{n-1} \left| L_n^{k,1} - \overline{L}_n^{k,1} \right| + o\left(\frac{1}{n^2} \right),$$

and

$$\int_0^\pi \left| \overline{L}_n^{J_{n,1}(x),1} - \overline{L}_n^{\overline{J}_{n,1}(x),1} \right| dx = o\left(\frac{1}{n^3} \right).$$

If we consider last equalities in (20), we get

$$\left\| V - \overline{V} \right\|_1 \leq (n-2) \left[n - 2 - \frac{m^2}{2(n-2)} \right] o\left(\frac{1}{n^3} \right)$$

$$+ (n-2) \left[n - 2 - \frac{m^2}{2(n-2)} \right] \left[\frac{\pi}{n-2} \sum_{k=1}^{n-1} \left| L_n^{k,1} - \overline{L}_n^{k,1} \right| + o\left(\frac{1}{n^2} \right) \right],$$

and

$$\left\| V - \overline{V} \right\|_1 \leq \pi \left[n - 2 - \frac{m^2}{2(n-2)} \right] \sum_{k=1}^{n-1} \left| L_n^{k,1} - \overline{L}_n^{k,1} \right| + o(1). \qquad (21)$$

Similarly, we can get

$$\left\| V - \overline{V} \right\|_1 \geq \pi \left[n - 2 - \frac{m^2}{2(n-2)} \right] \sum_{k=1}^{n-1} \left| L_n^{k,1} - \overline{L}_n^{k,1} \right| + o(1). \qquad (22)$$

If we consider (21) and (22) together, we obtain

$$\left\| V - \overline{V} \right\|_1 = \pi \left[n - 2 - \frac{m^2}{2(n-2)} \right] \sum_{k=1}^{n-1} \left| L_n^{k,1} - \overline{L}_n^{k,1} \right|.$$

The proof is complete after taking the limit as $n \to \infty$. $\square$

4. CONCLUSION

In this study, Lipschitz stability of inverse nodal problem is proved for Dirac operator by using zeros of the first component function of two dimensional vector eigenfunction. Proofs are made for the first component of the eigenfunction throughout the study. Such a way was followed, since the proofs for the second component have similar behavior. Especially, two metric spaces were defined and it was shown that they were homeomorphic to each other. These results are new and can be generalized.

Authors Contribution Statement All authors contributed equally to the article.

Declaration of Competing Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

REFERENCES

[1] Law, C. K., Tsay, J. On the well-posedness of the inverse nodal problem, *Inverse Problems*, 17(5) (2001), 1493-1512.

[2] Cheng, Y. H., Law, C. K., The inverse nodal problem for Hill's equation, *Inverse Problems*, 22(3) (2006), 891-901.

[3] Ambarzumyan, V. A., Über eine Frage der Eigenwerttheorie, *Zeitschrift für Physik*, 53 (1929), 690-695.

[4] Levitan, B. M., Sargsyan, I. S., Introduction to spectral theory: self adjoint ordinary differential operators, *American Mathematical Society*, Providence, Rhode Island, 1975.

[5] McLaughlin, J. R., Analytical methods for recovering coefficients in differential equations from spectral data, *SIAM*, 28(1) (1986), 53-72.

[6] Pöschel, J., Trubowitz, E., Inverse spectral theory, *Academic Press*, Orlando, 1987.

[7] Pivovarchik, V., Direct and inverse three-point Sturm-Liouville problem with parameter-dependent boundary conditions, *Asymptotic Analysis*, 26(3) (2001), 219–238.

[8] Shieh, C. T., Buterin, S. A., Ignatiev, M., On Hochstadt-Lieberman theorem for Sturm-Liouville operators, *Far East Journal of Applied Mathematics*, 52(2) (2011), 131-146.

[9] Rundell, W., Sacks, P. E., The reconstruction of Sturm-Liouville operators, *Inverse Problems,* 8(3) (1992), 457–482.

[10] Hryniv, R., Pronska, N., Inverse spectral problems for energy-dependent Sturm-Liouville equations, *Inverse Problems*, 28(8) (2012), 085008.

[11] McLaughlin, J. R., Inverse spectral theory using nodal points as data-a uniqueness result, *Journal of Differential Equations*, 73(2) (1988), 354-362.

[12] Hald, O. H., McLaughlin, J. R., Solution of inverse nodal problems, *Inverse Problems*, 5(3) (1989), 307-347.

[13] Shen, C. L., On the nodal sets of the eigenfunctions of the string equation, *SIAM Journal on Mathematical Analysis*, 19 (1988), 1419-1424.

[14] Browne, P. J., Sleeman, B. D., Inverse nodal problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions, *Inverse Problems*, 12(4) (1996), 377-381.

[15] Kuryshova, Y. V., Shieh, C. T., An inverse nodal problem for integro-differential operators, *Journal of Inverse and III-posed Problems*, 18(4) (2010), 357–369.

[16] Yilmaz, E., Koyunbakan, H., Reconstruction of potential function and its derivatives for Sturm-Liouville problem with eigenvalues in boundary condition, *Inverse Problems in Science and Engineering*, 18(7) (2010), 935–944.

[17] Guo, Y., Wei, G., Inverse nodal problem for Dirac equations with boundary conditions polynomially dependent on the spectral parameter, *Results in Mathematics*, 67(1-2) (2015), 95-110.

[18] Law, C. K., Yang, C. F., Reconstructing the potential function and its derivatives using nodal data, *Inverse Problems*, 14(2) (1998), 299-312.

[19] Panakhov, E. S., Yilmaz, E., Koyunbakan, H., Inverse nodal problem for Dirac operator, *World Applied Sciences Journal*, 11(8) (2010), 906-911.

[20] Moses, H. E., Calculation of the scattering potential from reflection coefficients, *Physical Review*, 102(2) (1956), 559-567.

[21] Prats, F., Toll, J. S., Construction of the Dirac equation central potential from phase shifts and bound states, *Physical Review*, 113(1) (1959), 363-370.

[22] Verde, M., The inversion problem in wave mechanics and dispersion relations, *Nuclear Physics*, 9 (1958), 255-266.

[23] Gasymov, M. G., Levitan, B. M., The inverse problem for the Dirac System, *Doklady Akademi NAauk SSSR*, 167 (1966), 967-970.

[24] Panakhov, E. S., The defining of Dirac system in two incompletely set collection of eigenvalues, *Doklady Akademi AzSSR*, 5 (1985), 8-12.

[25] Gasymov, M. G. Dzhabiev, T. T., Determination of a system of Dirac differential equations using two spectra, *Transactions of the summer school on spectral Theory of Operators*, (1975), 46-71.

[26] Joa, I., Minkin, A., Eigenfunction estimate for a Dirac operator, *Acta Mathematica Hungarica*, 76(4) (1997), 337-349.

[27] Watson, B. A., Inverse spectral problems for weighted Dirac systems, *Inverse Problems*, 15(3) (1999), 793-805.

[28] Kerimov, N. B., A boundary value problem for the Dirac system with a spectral parameter in the boundary conditions, *Differential Equations*, 38(2) (2002), 164-174.

[29] Annaby, M. H., Tharwat, M. M., On sampling and Dirac systems with eigenparameter in the boundary conditions, *Journal of Applied Mathematics and Computing*, 36(1-2) (2011), 291-317.

[30] Amirov, R. Kh., Keskin, B., Özkan, A. S., Direct and inverse problems for the Dirac operator with a spectral parameter linearly contained in a boundary condition, *Ukrainian Mathematical Journal*, 61(9) (2009), 1365-1379.

[31] Albeverio, S., Hryniv, R. O., Mykytyuk, Ya. V., Inverse spectral problems for Dirac operators with summable potentials, *Russian Journal of Mathematical Physics*, 12(4) (2005), 406–423.

[32] Hryniv, R. O., Mykytyuk, Ya. V., On zeros of some entire functions, *Transactions of the American Mathematical Society*, 361(4) (2009), 2207–2223.

[33] Mykytyuk, Ya. V., Puyda, D. V., Inverse spectral problems for Dirac operators on a finite interval, *Journal of Mathematical Analysis and Applications*, 386(1) (2012), 177–194.

[34] Thaller, B., The Dirac Equation, Springer, Berlin, 1992.

[35] Puyda, D. V., Inverse spectral problems for Dirac operators with summable matrix-valued potentials, *Integral Equations and Operator Theory*, 74(3) (2012), 417-450.

[36] Yang, C. F., Huang, Z. Y., Reconstruction of the Dirac operator from nodal data, *Integral Equations and Operator Theory*, 66(4) (2010), 539-551.

[37] Gulsen, T., Yilmaz, E., Koyunbakan, H., Inverse nodal problem for $p-$laplacian Dirac system, *Mathematical Methods in the Applied Sciences*, 40(2017), 2329–2335.

[38] Yang, C. F., Pivovarchik, V. N., Inverse nodal problem for Dirac system with spectral parameter in boundary conditions, *Complex Analysis and Operator Theory*, 7(4) (2013), 1211-1230.

[39] Marchenko, V. A., Maslov, K. V., Stability of the problem of recovering the Sturm-Liouville operator from the spectral function, *Mathematics of the USSR Sbornik*, 81(4) (1970), 525-551.

[40] McLaughlin, J. R., Stability theorems for two inverse spectral problems, *Inverse Problems*, 4(2) (1988), 529-540.