https://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 70, Number 1, Pages 265–268 (2021)
DOI: 10.31801/cfsuasmas.733614
ISSN 1303–5991 E-ISSN 2618–6470

COMMUNICATIONS SERIES A1

Received by the editors: May 7, 2020; Accepted: December 19, 2020

PSEUDO PURE-INJECTIVE OBJECTS

Mustafa Kemal BERKTAŞ

Department of Mathematics, Uşak University, Uşak, TURKEY

Abstract. We show that if M and N are pure essentially equivalent objects in a finitely accessible additive category $\mathcal{A}$ such that M is pseudo pure-N-injective and N is pseudo pure-M-injective, then $M \cong N$.

1. Introduction

Recently, the famous Schröder-Bernstein problem in set theory has been solved for automorphism-invariant modules (equivalently pseudo-injective modules [8]) by Guil et al. [10]. They prove that if M, N are automorphism invariant modules such that there are monomorphisms $f : M \to N$ and $g : N \to M$, then $M \cong N$ ([10, Theorem 3.1]). This result is an extension of a result by Alahmadi et al. ([2, Corollary 2.3]). They prove that if M and N are automorphism-invariant modules of finite Goldie dimension such that there is a monomorphism from M to N and a monomorphism from N to M, then $M \cong N$. The present paper contains a generalization of [2, Corollary 2.3] to finitely accessible additive categories (Corollary 5). Then we show that if M and N are pure essentially equivalent objects in a finitely accessible additive category such that M is pseudo pure-N-injective and N is pseudo pure-M-injective, then $M \cong N$ (Theorem 6).

Following [7], an additive category $\mathcal{A}$ is called *finitely accessible (or locally finitely presented)* if it has direct limits, the class of finitely presented objects $\mathcal{A}_0$ is skeletally small and every object is a direct limit of finitely presented objects. A sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in the finitely accessible additive category $\mathcal{A}$ (with $gf = 0$) is called *pure exact* provided it induces an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(F, X) \to \mathrm{Hom}_{\mathcal{A}}(F, Y) \to \mathrm{Hom}_{\mathcal{A}}(F, Z) \to 0$$

2020 *Mathematics Subject Classification.* Primary 18C35; Secondary 18E05.

Keywords and phrases. Schröder-Bernstein problem, pseudo-injective object, pure Goldie dimension, finitely accessible additive category.

mkb@usak.edu.tr
0000-0003-4395-9521.

for all finitely presented objects F of $\mathcal{A}_0$. In this case, f is called a *pure-monomorphism* and g a *pure-epimorphism.* An object M of $\mathcal{A}$ is called *pure-injective* if every pure exact sequence in $\mathcal{A}$ with the first term M splits. Throughout $\mathcal{A}$ will denote a finitely accessible additive category.

2. Results

Let A, A' and A'' be objects in $\mathcal{A}$. A pure monomorphism $p : A \to A'$ is said to be *pure essential* if whenever $f : A' \to A''$ is a morphism such that fp is a pure monomorphism, then f also must be a pure monomorphism ([4]).

Lemma 1. *Let $f : A \to B$ be a pure essential monomorphism in $\mathcal{A}$. If f splits, then $A \cong B$.*

Proof. Assume f splits. Then there exists an epimorphism $g : B \to A$ such that $gf = 1_A$. Since f is pure essential, g must be a monomorphism. □

Lemma 2. *Let $f : A \to B$ and $g : B \to C$ be pure monomorphisms in $\mathcal{A}$. If gf is a pure essential monomorphism, then g is a pure essential monomorphism.*

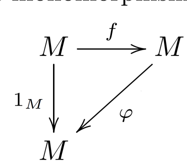
Proof. Let $h : C \to D$ be pure monomorphism in $\mathcal{A}$ such that hg is a pure monomorphism. Then hgf is a pure monomorphism. Since gf is pure essential, h is a pure monomorphism. □

A non-zero object in $\mathcal{A}$ is *pure uniform* if all non-zero subobjects are pure essential. An object A of $\mathcal{A}$ is said to have *finite pure Goldie dimension* if A has a pure essential subobject that is the finite direct sum of indecomposable pure subobjects whose every non-zero subobject is pure essential([5]).

Let M, N be objects in $\mathcal{A}$. Recall from [6] that M is called *pseudo pure-N-injective* if every pure-monomorphisms $f : Y \to N$ and $g : Y \to M$ in $\mathcal{A}$, where Y is any object in $\mathcal{A}$, there exists a homomorphism $\varphi : N \to M$ such that $\varphi f = g$. If M is pseudo pure-M-injective, then M is called *pseudo pure-injective.* Clearly every pure-injective object is pseudo pure-injective.

Lemma 3. *Let M be a pseudo pure-injective object of finite pure Goldie dimension in $\mathcal{A}$. Then every pure monomorphism $f : M \to M$ is an isomorphism.*

Proof. Let $f : M \to M$ be a pure monomorphism. Consider the following diagram.



Since M is pseudo pure-injective, there exists $\varphi : M \to M$ such that $\varphi f = 1_M$. Hence f splits. Since M has finite pure Goldie dimension, f is pure essential by Lemma 2 and f is an isomorphism by Lemma 1. □

Recall that a ring R is called *semilocal* if $R/J(R)$ is a semisimple artinian ring where $J(R)$ denotes the Jacobson radical of R.

Now we are ready to give the following different generalized form of [4, Theorem 5] and [9, Corollary 4.5].

Corollary 4. *Let M be a pseudo pure-injective object of finite pure Goldie dimension in $\mathcal{A}$. Then endomorphism ring of M is semilocal.*

Proof. This follows from [4, Theorem 5] by using Lemma 3. □

Now we give the slight generalization of [2, Corollary 2.3]. In [10], they remark that; their result ([10, Theorem 3.1]) can not be applied, for instance, to flat modules. It is well known that for an associative ring R with identity the category of flat right R-modules is an accessible category (see [1, Chapter 2]).

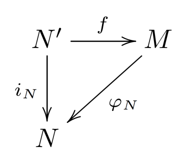
Corollary 5. *Let M, N be two pseudo pure-injective objects of finite pure Goldie dimension and let $f : M \to N$ and $g : N \to M$ be pure-monomorphisms in $\mathcal{A}$. Then $M \cong N$.*

Proof. Let M, N be two pseudo pure-injective objects of finite pure Goldie dimension and let $f : M \to N$ and $g : N \to M$ be pure-monomorphisms in $\mathcal{A}$. Then fg is an endomorphism of N and fg is a pure monomorphism. By Lemma 3, it is an automorphism of N. Thus f is a pure epimorphism and so f is an isomorphism. □

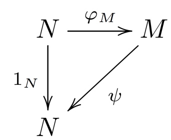
Recall from [5] (also see [11]) that two objects M and N in $\mathcal{A}$ are *pure essentially equivalent* if there exist pure essential subobjects M' of M and N' of N such that $M' \cong N'$. Let M and N be two objects such that both are finite direct sums of pure uniform objects. Notice that, using [3, Theorem 1], if there are pure monomorphisms $f : M \to N$ and $g : N \to M$, then M and N are pure essentially equivalent.

Theorem 6. *Let M and N be two pure essentially equivalent objects in $\mathcal{A}$. If M is pseudo pure-N-injective and N is pseudo pure-M-injective, then $M \cong N$.*

Proof. Assume M and N are pure essentially equivalent and N is pseudo pure-M-injective. Let M' and N' be pure essential subobjects of M and N such that $M' \cong N'$, respectively. Then we have a commutative diagram



such that $\varphi_N f = i_N$ where f is the composite morphism of $i_M : M' \to M$ and the isomorphism $s : N' \to M'$. Notice that i_N (i_M) is pure essential. Since f is pure essential, φ_N is a pure monomorphism. Similarly, there exists a pure monomorphism $\varphi_M : N \to M$. Since N is pseudo pure-M-injective , we have a commutative diagram



such that $\psi\varphi_M = 1_N$. This means that φ_M splits. Similarly, φ_N splits. We know that i_N is a pure essential monomorphism. Therefore φ_N (also φ_M) is pure essential by Lemma 2 and we obtain that $M \cong N$ by Lemma 1. □

Declaration of Competing Interests The author has no competing interest to declare.

References

[1] Adámek, J. and Rosicky, J., Locally presentable and accessible categories, Cambridge University Press, Cambridge, 1994.

[2] Alahmadi, A., Facchini, A. and Tung, N. K., Automorphism-invariant modules, *Rend. Semin. Mat Univ. Padova,* 133 (2015), 241–259.

[3] Berktaş, M. K., A uniqueness theorem in a finitely accessible additive category, *Algebr. Represent. Theor.,* 17 (2014), 1009–1012.

[4] Berktaş, M. K., On objects with a semilocal endomorphism rings in finitely accessible additive categories, *Algebr. Represent. Theor.,* 18 (2015), 1389–1393.

[5] Berktaş, M. K., On pure Goldie dimensions, *Comm. Algebra,* 45 (2017), 3334–3339.

[6] Berktaş, M. K. and Keskin Tütüncü, D., The Schröder-Bernstein problem for objects in Grothendieck categories, preprint.

[7] Crawley-Boevey, W., Locally finitely presented additive categories, *Comm. Algebra,* 22 (1994), 1641–1674.

[8] Er, N., Singh, S. and Srivastava, A. K., Rings and modules which are stable under automorphisms of their injective hulls, *J. Algebra,* 379 (2013), 223–229.

[9] Facchini, A. and Herbera, D., Local morphisms and modules with a semilocal endomorphism ring, *Algebr. Represent. Theor.,* 9 (2006), 403–422.

[10] Guil Asensio, P. A., Kaleboğaz, B. and Srivastava A. K., The Schröder-Bernstein problem for modules, *J. Algebra* 498 (2018), 153-164.

[11] Krause, H. Uniqueness of uniform decompositions in abelian categories, *J. Pure Appl. Algebra,* 183 (2003), 125–128.