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Research Article

On the ranks of certain ideals of monotone contractions

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Abstract

Let T_n be the (full) transformation semigroup, and let OCT_n and $ORCT_n$ be its subsemigroups of isotone contractions and of monotone contractions on a finite chain $X_n = \{1, \ldots, n\}$ under its natural order, respectively. In this study, we obtain the ranks of the ideals $OCT_{n,r} = \{\alpha \in OCT_n : |\mathrm{im}\,(\alpha)| \leq r\}$ and $ORCT_{n,r} = \{\alpha \in ORCT_n : |\mathrm{im}\,(\alpha)| \leq r\}$ for $1 \leq r \leq n-1$.

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1. Introduction

For $n \in \mathbb{Z}^+$ let $X_n = \{1, \ldots, n\}$ be a finite chain, under its natural order, and let T_n be the *(full) transformation semigroup*, the semigroup of all (full) transformations on X_n with usual composition. It is well known that every finite semigroup is embeddable in a transformation semigroup T_n for any appropriate n, which is correspond to Cayley's theorem for finite symmetric group S_n, the group of all permutations on X_n. Hence, the studies on transformation semigroups and their subsemigroups have certain important roles for semigroup theory like as the studies on symmetric groups for group theory.

An element $\alpha \in T_n$ is said to be *isotone* or *order-preserving* (*antitone* or *order-reversing*) if $x \leq y \Rightarrow x\alpha \leq y\alpha$ ($x \leq y \Rightarrow x\alpha \geq y\alpha$) for all $x, y \in X_n$, and said to be *monotone* if α is isotone or antitone. Notice that if $|\mathrm{im}\,(\alpha)| = 1$ then α is both isotone and antitone, and so monotone. It is easy to see that the product of two isotone transformations and also the product of two antitone transformations is isotone; and the product of any isotone transformation with any antitone transformation (in each order) is antitone. Then, as known, the subsets

$$\begin{aligned} O_n &= \{\alpha \in T_n : \alpha \text{ is an isotone transformation on } X_n\} \text{ and} \\ OR_n &= \{\alpha \in T_n : \alpha \text{ is a monotone transformation on } X_n\} \end{aligned}$$

are subsemigroups of T_n. An other notable element in T_n is contraction. An element $\alpha \in T_n$ is called a *contraction* if $|x\alpha - y\alpha| \leq |x - y|$ for all $x, y \in X_n$, and also the subset

$$CT_n = \{\alpha \in T_n : \alpha \text{ is a contraction on } X_n\}$$

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is a subsemigroup of T_n. Furthermore, let OCT_n and $ORCT_n$ be the subsemigroups of X_n, consists of all isotone contractions and of all monotone contractions, respectively, that is

$$OCT_n = O_n \cap CT_n \text{ and } ORCT_n = OR_n \cap CT_n$$

for $1 \leq r \leq n$, let

$$\begin{aligned} OCT_{n,r} &= \{\alpha \in OCT_n : |\mathrm{im}\,(\alpha)| \leq r\} \text{ and} \\ ORCT_{n,r} &= \{\alpha \in ORCT_n : |\mathrm{im}\,(\alpha)| \leq r\} \end{aligned}$$

which are clearly subsemigroups, even ideals, of OCT_n and of $ORCT_n$, respectively. Also, we have $OCT_{n,r} \leq ORCT_{n,r}$ for $1 \leq r \leq n$.

For any non-empty subset U of any semigroup S, the subsemigroup generated by U is defined as the smallest subsemigroup of S containing U and denoted by $\langle U \rangle$. Moreover, if $S = \langle U \rangle$ then U is said to be *a generating set* of S, and S is said to be *the semigroup generated by* U. Also, the *rank* of a finitely generated semigroup S, a semigroup generated by some finite subsets, is defined by

$$\mathrm{rank}\,(S) = \min\{\,|U| : \langle U \rangle = S\},$$

and any generating set of S with cardinality $\mathrm{rank}\,(S)$ is called a *minimal* generating set of S.

As stated in [7], although the notion "contraction" first appeared in [10], algebraic and combinatorial properties of the semigroups CT_n and OCT_n were investigated first by Adeshola in [1]. Then Adeshola and Umar investigated the cardinalities of some equivalences on OCT_n and $ORCT_n$ in [2]; and Garba, Ibrahim and Imam presented characterizations of Green's relations on CT_n and starred Green's relations on both CT_n and OCT_n in [7]. Ibrahim, Imam, Adeshola and Bakare investigate the local and global U-depth for any generating set U of OCT_n as well as the status of OCT_n in [14]. An other interesting lack for these semigroups is their ranks. To find the rank of an arbitrary semigroup is an important problem in semigroup theory, similar to find the dimension of an arbitrary group in group theory. Howie, with various co-authors, wrote a lot of studies on ranks of semigroups, see [8, 9, 11, 13] for examples. Then, this problem has gained importance for other researchers working in this field and then many more papers on ranks of semigroups have been written, see [3, 5, 6, 17] for examples. Recently, Toker showed that $\mathrm{rank}\,(OCT_{n,n-1}) = n - 1$, and $\mathrm{rank}\,(OCT_n) = n$ for $n \geq 3$, and that

$$\mathrm{rank}\,(ORCT_n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is an odd number} \\ \frac{n+2}{2} & \text{if } n \text{ is an even number} \end{cases}$$

for $n \geq 1$ in [15]. Now, we examine the ideals $OCT_{n,r}$ and $ORCT_{n,r}$ for $1 \leq r \leq n - 1$. Although the part of the motivation of this paper is to state a useful method for finding minimal generating sets of the implied ideals, the main motivation of this paper is to calculate their ranks. Finally, we show that

$$\mathrm{rank}\,(OCT_{n,r}) = \mathrm{rank}\,(ORCT_{n,r}) = \begin{cases} n & \text{for } r = 1 \\ \binom{n-1}{r-1} & \text{for } 2 \leq r \leq n - 1 \end{cases}$$

in this study.

2. Preliminaries

For any $\alpha \in T_n$ the *height* and the *kernel* of α are defined by

$$\begin{aligned} \mathrm{h}\,(\alpha) &= |\mathrm{im}\,(\alpha)| \text{ and} \\ \ker(\alpha) &= \{(x, y) : x, y \in X_n \text{ and } x\alpha = y\alpha\}, \end{aligned}$$

respectively. It is well known that $\ker(\alpha)$ is an equivalence relation on X_n, and that the set of the equivalence classes obtained by $\ker(\alpha)$, say

$$X_n/\ker(\alpha) = \{y\alpha^{-1} : y \in \mathrm{im}\,(\alpha)\},$$

is a partition of X_n, called the *kernel partition* of α and denoted by $\mathrm{kp}\,(\alpha)$. For $\alpha, \beta \in T_n$ it is clear that

$$\mathrm{kp}\,(\alpha) = \mathrm{kp}\,(\beta) \Leftrightarrow \ker(\alpha) = \ker(\beta).$$

Let $P = \{I_1, \ldots, I_p\}$ be a partition of X_n for any $1 \leq p \leq n$. Then we write $I_i < I_j$ if $x < y$ for all $x \in I_i$ and for all $y \in I_j$ for $1 \leq i, j \leq p$. Without loss of generality, if $P = \{I_1 < \cdots < I_p\}$ then P is called an *ordered partition*. Moreover, a subset $\{a_1, \ldots, a_p\}$ of X_n is called a *representative set* (or a *transversal* or a *cross-section*) of P if $|\{a_1, \ldots, a_p\} \cap I_i| = 1$ for each $1 \leq i \leq p$. Also, a subset $\emptyset \neq C \subseteq X_n$ is called a *convex subset* if

$$x, y \in C \text{ and } x \leq z \leq y \Rightarrow z \in C.$$

Now let $\alpha \in ORCT_n = OR_n \cap CT_n$ with $\mathrm{h}\,(\alpha) = p$ $(1 \leq p \leq n)$. Since $\alpha \in OR_n$, it is well known that the kernel classes of α are convex ordered subsets of X_n (see [6, p187] for example), that is there exist $x_1, \ldots, x_{p-1} \in X_n$ such that the kernel classes of α are $I_i = \{x_{i-1} + 1, \ldots, x_i\}$ for $1 \leq i \leq p$ where $x_0 = 0$ and $x_p = n$, and so the kernel partition of $\alpha \in OR_n$ is $\mathrm{kp}\,(\alpha) = P = \{I_1 < \cdots < I_p\}$. Moreover, since $\alpha \in CT_n$ it is also well known that $\mathrm{im}\,(\alpha)$ is a convex subset of X_n (see [2, Lemma 1.2] for example), that is there exist $a \in X_n$ such that $\mathrm{im}\,(\alpha) = A = \{a, a+1, \ldots, a+p-1\}$. Then α has the following tabular form:

$$\alpha = \begin{pmatrix} I_1 & I_2 & \cdots & I_p \\ a & a+1 & \cdots & a+p-1 \end{pmatrix}, \text{ or shortly } \alpha = \begin{pmatrix} P \\ A \end{pmatrix}$$

if α is isotone, and

$$\alpha = \begin{pmatrix} I_1 & I_2 & \cdots & I_p \\ a+p-1 & a+p-2 & \cdots & a \end{pmatrix} \text{ or shortly } \alpha = \begin{pmatrix} P \\ A^R \end{pmatrix}$$

if α is antitone. For any $1 \leq p \leq n$, there exist $\binom{n-1}{p-1}$ many convex ordered partition of X_n into p subsets, and $n - p + 1$ many convex subset of X_n with p elements. Therefore, it is easy to see that $|OCT_{n,1}| = |ORCT_{n,1}| = n$ and that

$$|OCT_{n,r}| = \sum_{p=1}^{r} \binom{n-1}{p-1}(n-p+1) \text{ and}$$

$$|ORCT_{n,r}| = n + 2\sum_{p=2}^{r} \binom{n-1}{p-1}(n-p+1)$$

for $2 \leq r \leq n-1$.

As usual, an element x in any semigroup S is called an *idempotent* if $x^2 = x$, and the set of all idempotents in any subset $\emptyset \neq U \subseteq S$ denoted by $E(U)$. Then it is well known that

$$\alpha \in E(T_n) \Leftrightarrow \mathrm{im}\,(\alpha) = \mathrm{fix}\,(\alpha) = \{x \in X_n : x\alpha = x\}.$$

In particular,

$$\alpha \in E(OCT_n) \Leftrightarrow \mathrm{im}\,(\alpha) \text{ is a representative set of } \mathrm{kp}\,(\alpha),$$

and clearly for any $\alpha \in OCT_n$, $\mathrm{im}\,(\alpha) = \{a, a+1, \ldots, a+p-1\}$ $(1 \leq p \leq n)$ is a representative set of $\mathrm{kp}\,(\alpha)$ if and only if

$$\mathrm{kp}\,(\alpha) = \{\{1, \ldots, a\}, \{a+1\}, \ldots, \{a+p-2\}, \{a+p-1, \ldots, n\}\}.$$

Notice that, for each convex subset $\emptyset \neq A \subseteq X_n$ there exists a unique convex ordered partition P of X_n such that A is a representative set of P, and so there exists a unique idempotent in OCT_n such that $\operatorname{im}(\alpha) = A$. Thus, since there exist $n-p+1$ many convex subsets of X_n with cardinality p and since each idempotent have to be isotone, it follows that

$$|E(OCT_{n,r})| = |E(ORCT_{n,r})| = \sum_{p=1}^{r}(n-p+1),$$

and so

$$|E(OCT_n)| = |E(ORCT_n)| = \sum_{p=1}^{n}(n-p+1) = \frac{n(n+1)}{2}$$

which is first appeared in [2, Corollary 2.14].

After stating some combinatorial results in $OCT_{n,r}$ and $ORCT_{n,r}$, now we give some properties about their algebraic structures. In general, one of the most common interest on algebraic structure of any semigroup S is to examine the characterization of the Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{D}, \mathcal{H}$ on S, especially if a semigroup S is regular (for the definitions and certain properties of Green's equivalences and for the other undefined terms in semigroup theory see [4, 12] for examples). For non-regular semigroups, it is more common to examine the characterization of the starred Green's equivalences $\mathcal{L}^*, \mathcal{R}^*, \mathcal{J}^*, \mathcal{H}^*, \mathcal{D}^*$, the generalization of Green's equivalences. On any semigroup S, the starred Green's equivalence $\mathcal{L}^*$ ($\mathcal{R}^*$) is defined for $a, b \in S$ by the rule that $a\mathcal{L}^*b$ ($a\mathcal{R}^*b$) if and only if $a\mathcal{L}b$ ($a\mathcal{R}b$) on some over-semigroup of S (a semigroup containing S as a subsemigroup). These equivalences also have the following characterizations:

$$a\mathcal{L}^*b \Leftrightarrow ax = ay \text{ if and only if } bx = by \text{ for all } x, y \in S^1$$

$$a\mathcal{R}^*b \Leftrightarrow xa = ya \text{ if and only if } xb = yb \text{ for all } x, y \in S^1$$

for $a, b \in S$ where S^1 is the monoid obtained from S by adjoining an identity if necessary. Moreover, the equivalences $\mathcal{H}^*$ and $\mathcal{D}^*$ are defined as the intersection and the join of the equivalences $\mathcal{L}^*$ and $\mathcal{R}^*$, respectively, say $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$ and $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$, and it is well known that $\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^*$.

Let $S \in \{OCT_{n,r}, ORCT_{n,r}\}$ for $1 \leq r \leq n-1$ unless otherwise stated. For $\alpha, \beta \in S$, it is a routine matter to prove, as in [7, Theorem 4.1], that

(*i*) $\alpha\mathcal{L}^*\beta \Leftrightarrow \operatorname{im}(\alpha) = \operatorname{im}(\beta)$,
(*ii*) $\alpha\mathcal{R}^*\beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$,
(*iii*) $\alpha\mathcal{H}^*\beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$ and $\operatorname{im}(\alpha) = \operatorname{im}(\beta)$, and
(*iv*) $\alpha\mathcal{D}^*\beta \Leftrightarrow |\operatorname{im}(\alpha)| = |\operatorname{im}(\beta)|$,

and also $\mathcal{D}^* = \mathcal{J}^*$. Then, for $1 \leq p \leq r \leq n-1$, we denote the starred Green's $\mathcal{D}^*$-class of all elements in S of height p by D_p^*, that is

$$D_p^* = \{\alpha \in S : |\operatorname{im}(\alpha)| = p\}.$$

It is clear that there exist r many $\mathcal{D}^*$-classes, namely $D_1^*, \ldots, D_r^*$, and S is the disjoint union of $D_1^*, \ldots, D_r^*$. Moreover, there exist $n-p+1$ many $\mathcal{L}^*$-classes and $\binom{n-1}{p-1}$ many $\mathcal{R}^*$-classes in D_p^* for each $1 \leq p \leq r$. As also shown in [15] that, it is a routine matter to show that $\mathcal{D}_p^* \subseteq \langle \mathcal{D}_{p+1}^* \rangle$, more explicitly, for any $\alpha \in \mathcal{D}_p^*$ there exist $\beta, \gamma \in \mathcal{D}_{p+1}^*$ such that $\alpha = \beta\gamma$, for any $1 \leq p \leq n-2$. Therefore, for any subset $\emptyset \neq U \subseteq S$,

$$S = \langle U \rangle \Leftrightarrow \mathcal{D}_r^* \subseteq \langle U \rangle.$$

Also, it is clear that $\mathcal{D}_r^*$ can be generated only by its own elements. Hence, it is enough to examine only the non-empty subsets of $\mathcal{D}_r^*$ to find a (minimal) generating set of S.

Next we give a lemma which is useful for this manuscript.

Lemma 2.1. *For $1 \leq p \leq n-1$ and $2 \leq k$ let $\alpha_1, \ldots, \alpha_k \in D_p^*$ in S where $S \in \{OCT_{n,r}, ORCT_{n,r}\}$ for $1 \leq r \leq n-1$. Then*

$$\begin{aligned} \alpha_1 \cdots \alpha_k \in D_p^* \quad &\Leftrightarrow \quad \alpha_i\alpha_{i+1} \in D_p^* \text{ for each } 1 \leq i \leq k-1 \\ &\Leftrightarrow \quad \mathrm{im}\,(\alpha_i) \text{ is a representative set of } \mathrm{kp}\,(\alpha_{i+1}) \text{ for each } 1 \leq i \leq k-1. \end{aligned}$$

Proof. The proof is similar to the proof of [3, Lemma 2]. □

Now we give some definitions about digraphs. A digraph (directed graph) is an ordered pair $\Pi = (V, E)$ where V is a set whose elements are called *vertices* and $E \subseteq V \times V$ is a set of ordered pairs whose elements are called *arrows* or *directed edges*. For some elements $u_1, \ldots, u_k \in V$ ($k \geq 2$) that do not need to be different if $(u_1, u_2), (u_2, u_3), \ldots, (u_{k-1}, u_k) \in E$, then $u_1 \to u_2 \to \cdots \to u_k$ is called *a walk from u_1 to u_k*. In particular, for distinct vertices $u_1, \ldots, u_k \in V$ where $k \geq 1$, the closed walk $u_1 \to \cdots \to u_k \to u_1$ is called a *cycle*. For any walk $u_1 \to \cdots \to u_k$ ($2 \leq k$), the ordered product $u_1u_2 \cdots u_k$ is called a *consecutive product*. (For unexplained terms about digraphs, see [16] for example.)

Finally, we define a new digraph $\mathfrak{D}_U$ for any $\emptyset \neq U \subseteq D_r^*$ which will be used in the main theorem of this manuscript. Let $\emptyset \neq U \subseteq D_r^*$ in S where $S \in \{OCT_{n,r}, ORCT_{n,r}\}$ for $1 \leq r \leq n-1$. Then the digraph $\mathfrak{D}_U = (U, E)$ is defined by

$$\begin{aligned} E &= \{(\alpha, \beta) : \alpha\beta \in D_r^*\} \\ &= \{(\alpha, \beta) : \mathrm{im}\,(\alpha) \text{ is a representative set of } \mathrm{kp}\,(\beta)\}. \end{aligned}$$

Notice that, for any $\alpha \in ORCT_{n,r}$, $\mathrm{im}\,(\alpha) = \{a, a+1, \ldots, a+r-1\}$ is a representative set of $\mathrm{kp}\,(\beta)$ if and only if $\mathrm{kp}\,(\beta) = \{\{1, \ldots, a\}, \{a+1\}, \ldots, \{a+r-2\}, \{a+r-1, \ldots, n\}\}$.

3. Rank of $OCT_{n,r}$

Theorem 3.1. *Let $1 \leq r \leq n-1$ and let $\emptyset \neq U \subseteq D_r^*$. Then U is a generating set of $OCT_{n,r}$ if and only if, for each convex ordered partition P of X_n into r subsets and for each convex subset A of X_n with cardinality r, there exist $\alpha, \beta \in U$ such that*

(i) $\mathrm{kp}\,(\alpha) = P$,

(ii) $\mathrm{im}\,(\beta) = A$ *and*

(iii) $\alpha = \beta$ *or there exists a walk from α to β in the digraph $\mathfrak{D}_U$.*

Proof. ($\Rightarrow$) For any convex ordered partition P of X_n into r subsets and any convex subset A of X_n with cardinality r ($1 \leq r \leq n-1$), consider the unique isotone contraction $\gamma \in D_r^*$ with kernel partition P and image set A. If $\gamma \in U$, then the result is clear. Now let $\gamma \in D_r^* \setminus U$. Since $\emptyset \neq U \subseteq D_r^*$ is a generating set of $OCT_{n,r}$, then there exist $\alpha_1, \ldots, \alpha_m \in U$ such that $\alpha_1 \cdots \alpha_m = \gamma$ for $m \geq 2$. Then we have $\ker(\alpha_1) \subseteq \ker(\gamma)$, $\mathrm{im}\,(\gamma) \subseteq \mathrm{im}\,(\alpha_m)$ and $\alpha_1, \ldots, \alpha_m, \gamma \in D_r^*$, and so $\mathrm{kp}\,(\alpha_1) = \mathrm{kp}\,(\gamma) = P$ and $\mathrm{im}\,(\alpha_m) = \mathrm{im}\,(\gamma) = A$. Thus the first two conditions are satisfied. From Lemma 2.1, also we have $\alpha_i\alpha_{i+1} \in D_r^*$ for each $1 \leq i \leq m-1$, and so $\alpha_1 \to \cdots \to \alpha_m$ is a walk from α_1 to α_m in $\mathfrak{D}_U$.

($\Leftarrow$) Let $\gamma \in D_r^*$ and let $\mathrm{kp}\,(\gamma) = P$ and $\mathrm{im}\,(\gamma) = A$. From the assumptions, there exist $\alpha, \beta \in U$ such that $\mathrm{kp}\,(\alpha) = P = \mathrm{kp}\,(\gamma)$, $\mathrm{im}\,(\beta) = A = \mathrm{im}\,(\gamma)$, and $\alpha = \beta$ or there exists a walk from α to β in $\mathfrak{D}_U$. If $\alpha = \beta$ clearly $\alpha = \beta = \gamma \in U$, otherwise, let $\alpha = \alpha_1 \to \cdots \to \alpha_m = \beta$ ($m \geq 2$) be a walk from α to β in $\mathfrak{D}_U$, and let ξ be the consecutive product of all elements on this walk, say $\xi = \alpha_1 \cdots \alpha_m$. Similarly, we have $\ker(\alpha) \subseteq \ker(\xi)$ and $\mathrm{im}\,(\xi) \subseteq \mathrm{im}\,(\beta)$. Moreover, it follows from the definition of the digraph $\mathfrak{D}_U$ and Lemma 2.1 that $\xi \in D_r^*$, and so $\mathrm{kp}\,(\xi) = \mathrm{kp}\,(\alpha) = \mathrm{kp}\,(\gamma)$ and $\mathrm{im}\,(\xi) = \mathrm{im}\,(\beta) = \mathrm{im}\,(\gamma)$. Hence, $\gamma = \xi \in \langle U \rangle$, and so $D_r^* \subseteq \langle U \rangle$, as required. □

In conclusion, if $OCT_{n,r} = \langle U \rangle$ for any $\emptyset \neq U \subseteq D_r^*$, then for each $\alpha \in \mathcal{D}_r^*$ there exists at least one element in U which is $\mathcal{L}^*$-equivalent to α and there exists at least one element in U which is $\mathcal{R}^*$-equivalent to α. Thus U must cover the $\mathcal{L}^*$-classes and also the $\mathcal{R}^*$-classes in D_r^*. Hence $\mathrm{rank}\,(OCT_{n,1}) \geq n$ and $\mathrm{rank}\,(OCT_{n,r}) \geq \binom{n-1}{r-1}$ for $2 \leq r \leq n-1$.

Corollary 3.2. *$OCT_{n,1} = \{\begin{pmatrix} X_n \\ \{1\} \end{pmatrix}, \ldots, \begin{pmatrix} X_n \\ \{n\} \end{pmatrix}\}$ and there is no proper generating set of $OCT_{n,1}$.*

Lemma 3.3. *For $2 \leq r \leq n-1$, let $m = n-r+1$ and let $I_1, \ldots, I_m$ be all of the convex subsets of X_n with cardinality r. Then there exist m different convex ordered partitions $P_1, \ldots, P_m$ of X_n into r subsets such that I_i is a representative set of P_i for each $1 \leq i \leq m$.*

Proof. For each $1 \leq i \leq m$, let $I_i = \{i, i+1, \ldots, i+r-1\}$ and let $P_i = \{\{1, \ldots, i\}, \{i+1\}, \ldots, \{i+r-2\}, \{i+r-1, \ldots, n\}\}$. Then the result is clear. □

Lemma 3.4. *For $2 \leq r \leq n-1$, let $m = n-r+1$, and let $I_1, \ldots, I_m$ be all of the convex subsets of X_n with cardinality r. Then there exist $\alpha_1, \ldots, \alpha_m \in D_r^*$ in $OCT_{n,r}$ such that*

(i) $\operatorname{kp}(\alpha_i) \neq \operatorname{kp}(\alpha_j)$ *if* $1 \leq i \neq j \leq m$ *and*

(ii) $\operatorname{im}(\alpha_i) = I_i$ *is a representative set of* $\operatorname{kp}(\alpha_{i+1})$, *for each* $1 \leq i \leq m$ *where* $\alpha_{m+1} = \alpha_1$.

Proof. Let $m = n-r+1$, and let $I_1, \ldots, I_m$ be all of the convex subsets of X_n with cardinality r for $2 \leq r \leq n-1$. Then it follows from Lemma 3.3 that there exist m different convex ordered partitions $P_1, \ldots, P_m$ of X_n into r subsets such that I_i is a representative set of P_i for each $1 \leq i \leq m$. Without loss of generality, let $\alpha_i \in D_r^*$ be the unique isotone contraction represented by $\operatorname{kp}(\alpha_i) = P_{i-1}$ and $\operatorname{im}(\alpha_i) = I_i$ for each $1 < i \leq m$ where $P_0 = P_m$. Then the result is clear. □

For $2 \leq r \leq n-1$, let $I_1, \ldots, I_m$ be all of the convex subsets of X_n with cardinality r, and let $R_1^*, \ldots, R_t^*$ be all of the starred Green's $\mathcal{R}^*$-classes in D_r^* where $m = n-r+1$ and $t = \binom{n-1}{r-1}$. It follows from Lemma 3.4 that there exist $\alpha_1, \ldots, \alpha_m \in D_r^*$ in $OCT_{n,r}$ such that

(i) $\operatorname{kp}(\alpha_i) \neq \operatorname{kp}(\alpha_j)$ if $1 \leq i \neq j \leq m$ and

(ii) $\operatorname{im}(\alpha_i) = I_i$ is a representative set of $\operatorname{kp}(\alpha_{i+1})$, for each $1 \leq i \leq m$ where $\alpha_{m+1} = \alpha_1$.

Without loss of generality, we may assume that $\alpha_i \in R_i^*$ for $1 \leq i \leq m$, and take an arbitrary contraction α_{m+j} from R_{m+j}^* for each $1 \leq j \leq t-m$. Then consider the set $U = \{\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_t\}$ which satisfies the first two conditions of Theorem 3.1. Also, it is easy to see that,

$$\alpha_1 \to \cdots \to \alpha_m \to \alpha_1$$

is a cycle in $\mathfrak{D}_U$ and, for each $1 \leq j \leq t-m$, there exists $1 \leq i \leq m$ such that $\alpha_{m+j} \to \alpha_i$ is a walk in $\mathfrak{D}_U$. Therefore, for any $\gamma \in D_r^* \setminus U$, there exist $\alpha_k, \alpha_l \in U$ such that $\operatorname{kp}(\alpha_k) = \operatorname{kp}(\gamma)$, $\operatorname{im}(\alpha_l) = \operatorname{im}(\gamma)$, and that there exists a walk from α_k to α_l in $\mathfrak{D}_U$. Indeed to generate $\gamma \in D_r^* \setminus U$, we can use the consecutive product of all isotone contractions on a suitable walk from the vertex α_k to the vertex α_l. Then it follows from Theorem 3.1 that U is a generating set of $OCT_{n,r}$, and so we have the following theorem.

Theorem 3.5.

$$\operatorname{rank}(OCT_{n,r}) = \begin{cases} n & \text{for } r = 1 \\ \binom{n-1}{r-1} & \text{for } 2 \leq r \leq n-1 \end{cases}.$$

Proof. The result follows from Corollary 3.2 and the fact $\operatorname{rank}(OCT_{n,r}) \geq \binom{n-1}{r-1}$ for $2 \leq r \leq n-1$. □

4. Rank of $ORCT_{n,r}$

Notice that $OCT_{n,1} = ORCT_{n,1}$. Therefore, unless otherwise stated, in this section we consider the case $2 \leq r \leq n-1$. Also, recall that for any subset $\emptyset \neq U \subseteq ORCT_{n,r}$, U is a generating set of $ORCT_{n,r}$ if and only if $\mathfrak{D}_r^* \subseteq \langle U \rangle$.

Lemma 4.1. *$ORCT_{n,r} = \langle R(D_r^*) \rangle$ where $R(D_r^*) = \{\alpha \in D_r^* : \alpha$ is antitone$\}$.*

Proof. Let $\alpha \in D_r^*$ be a isotone contraction with the following tabular form:

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a & a+1 & \cdots & a+r-1 \end{pmatrix}.$$

Then clearly we have $\alpha = \beta\gamma$ where

$$\beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a+r-1 & a+r-2 & \cdots & a \end{pmatrix} \in R(D_r^*) \text{ and}$$

$$\gamma = \begin{pmatrix} \{1,\ldots,a\} & \{a+1\} & \cdots & \{a+r-2\} & \{a+r-1,\ldots,n\} \\ a+r-1 & a+r-2 & \cdots & a+1 & a \end{pmatrix} \in R(D_r^*).$$

Thus, $D_r^* \subseteq \langle R(D_r^*) \rangle$, and so $ORCT_{n,r} = \langle R(D_r^*) \rangle$. □

As a result, $\emptyset \neq U \subseteq D_r^*$ is a generating set of $ORCT_{n,r}$ if and only if $R(D_r^*) \subseteq \langle U \rangle$.

Theorem 4.2. *For $2 \leq r \leq n-1$, $\emptyset \neq U \subseteq D_r^*$ is a generating set of $ORCT_{n,r}$ if and only if, for each convex ordered partition P of X_n into r subsets and for each convex subset A of X_n with cardinality r, there exist $\alpha, \beta \in U$ such that*

(*i*) $\operatorname{kp}(\alpha) = P$,
(*ii*) $\operatorname{im}(\beta) = A$, *and*
(*iii*) *$\alpha = \beta \in R(D_r^*)$ or there exists a walk from α to β in the digraph $\mathfrak{D}_U$ such that the number of vertex on the walk which is antitone is an odd number.*

Proof. ($\Rightarrow$) For any convex ordered partition P of X_n into r subsets and any convex subset A of X_n with cardinality r ($2 \leq r \leq n-1$), consider the unique antitone contraction $\gamma \in R(D_r^*)$ with kernel partition P and image set A. If $\gamma \in U$, then the result is clear. Now let $\gamma \in R(D_r^*) \setminus U$. Since $\emptyset \neq U \subseteq D_r^*$ is a generating set of $ORCT_{n,r}$, then $R(D_r^*) \subseteq \langle U \rangle$, and so there exist $\alpha_1, \ldots, \alpha_m \in U$ such that $\alpha_1 \cdots \alpha_m = \gamma$ for $m \geq 2$. Then we have $\ker(\alpha_1) \subseteq \ker(\gamma)$, $\operatorname{im}(\gamma) \subseteq \operatorname{im}(\alpha_m)$ and $\alpha_1, \ldots, \alpha_m, \gamma \in D_r^*$, and so $\operatorname{kp}(\alpha_1) = \operatorname{kp}(\gamma) = P$ and $\operatorname{im}(\alpha_m) = \operatorname{im}(\gamma) = A$. Thus the first two conditions are satisfied. Moreover, we have $\alpha_i\alpha_{i+1} \in D_r^*$ for each $1 \leq i \leq m-1$ from Lemma 2.1, and so $\alpha_1 \to \cdots \to \alpha_m$ is a walk in the digraph $\mathfrak{D}_U$. Moreover, since $\gamma \in R(D_r^*)$, it is easy to see that the number of vertex on the walk which is antitone has to be an odd number.

($\Leftarrow$) Let $\gamma \in R(D_r^*)$ and let $\operatorname{kp}(\gamma) = P$ and $\operatorname{im}(\gamma) = A$. Then from the assumptions, there exist $\alpha, \beta \in U$ such that $\operatorname{kp}(\alpha) = P = \operatorname{kp}(\gamma)$, $\operatorname{im}(\beta) = A = \operatorname{im}(\gamma)$, and $\alpha = \beta \in R(D_r^*)$ or there exists a walk from α to β in the digraph $\mathfrak{D}_U$ such that the number of vertex on the walk which is antitone is an odd number. If $\alpha = \beta \in R(D_r^*)$ clearly $\alpha = \beta = \gamma \in U$, otherwise, let ξ be the consecutive product of all elements on the walk. Then, as in the proof of Theorem 3.1, $\operatorname{kp}(\xi) = \operatorname{kp}(\alpha) = \operatorname{kp}(\gamma)$ and $\operatorname{im}(\xi) = \operatorname{im}(\beta) = \operatorname{im}(\gamma)$. Moreover, since the number of vertex on the walk which is antitone is an odd number, we have $\xi \in R(D_r^*)$. Thus, $\gamma = \xi \in \langle U \rangle$, and so $R(D_r^*) \subseteq \langle U \rangle$, as required. □

Lemma 4.3. *For $2 \leq r \leq n-1$ let $m = n-r+1$ and let $I_1, \ldots, I_m$ be all of the convex subsets of X_n with cardinality r. Then there exist $\alpha_1, \ldots, \alpha_m \in D_r^*$ in $ORCT_{n,r}$ such that*

(*i*) $\operatorname{kp}(\alpha_i) \neq \operatorname{kp}(\alpha_j)$ *if $1 \leq i \neq j \leq m$ and*
(*ii*) *$\operatorname{im}(\alpha_i) = I_i$ is a representative set of $\operatorname{kp}(\alpha_{i+1})$, for each $1 \leq i \leq m$ where $\alpha_{m+1} = \alpha_1$.*

Proof. It can be proved as in Lemma 3.4. □

Let $I_1, \dots, I_m$ be all of the convex subsets of X_n with cardinality r, and let $R_1^*, \dots, R_t^*$ be all of the starred Green's $\mathcal{R}^*$-classes in D_r^* where $m = n - r + 1$ and $t = \binom{n-1}{r-1}$. It follows from Lemma 4.3 that there exist $\alpha_1, \dots, \alpha_m \in D_r^*$ in $ORCT_{n,r}$ such that

(*i*) $\mathrm{kp}\,(\alpha_i) \neq \mathrm{kp}\,(\alpha_j)$ if $1 \leq i \neq j \leq m$ and

(*ii*) $\mathrm{im}\,(\alpha_i) = I_i$ is a representative set of $\mathrm{kp}\,(\alpha_{i+1})$, for each $1 \leq i \leq m$ where $\alpha_{m+1} = \alpha_1$.

Without loss of generality, we can take α_1 as an antitone contraction and α_i as an isotone contraction for each $2 \leq i \leq m$. Also, we may assume that $\alpha_i \in R_i^*$ for each $1 \leq i \leq m$, and take an arbitrary isotone contraction α_{m+j} from R_{m+j}^* for each $1 \leq j \leq t - m$. Then consider the set $U = \{\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_t\}$. It is easy to see that

$$\alpha_1 \to \cdots \to \alpha_m \to \alpha_1$$

is a cycle in the digraph $\mathfrak{D}_U$. Moreover, for each $1 \leq j \leq t - m$, there exists $1 \leq d_j \leq m$ such that $\mathrm{im}\,(\alpha_{m+j}) = \mathrm{im}\,(\alpha_{d_j})$. If $1 \leq d_j \leq m - 1$ then clearly $\alpha_{m+j} \to \alpha_{d_j+1}$ is a walk in $\mathfrak{D}_U$, and if $d_j = m$ then clearly $\alpha_{m+j} \to \alpha_1$ is a walk in $\mathfrak{D}_U$. Hence, for any convex ordered partition P of X_n into r subsets and for any convex subset A of X_n with cardinality r, there exists a unique $\alpha_k \in U$ such that $\mathrm{kp}\,(\alpha_k) = P$ and there exist unique $\alpha_l \in \{\alpha_1, \dots, \alpha_m\}$ such that $\mathrm{im}\,(\alpha_l) = A$. Therefore, first two conditions of Theorem 4.2 are satisfied. Now we examine the last condition of Theorem 4.2. There exist three cases.

Case 1: When $\alpha_k = \alpha_l = \alpha_1$, the result is clear since $\alpha_1 \in R(D_r^*)$.

Case 2: When $\alpha_k = \alpha_l \neq \alpha_1$, consider the walk

$$\alpha_l \to \alpha_{l+1} \to \cdots \to \alpha_m \to \alpha_1 \to \cdots \to \alpha_{l-1} \to \alpha_l,$$

which contains only one antitone map. Hence, the third condition of Theorem 4.2 is satisfied.

Case 3: When $\alpha_k \neq \alpha_l$, there exist p $(p \geq 0)$ distinct elements $\beta_1, \dots, \beta_p \in U \setminus \{\alpha_k, \alpha_l\}$ such that the shortest walk from α_k to α_l is

$$\alpha_k \to \beta_1 \to \cdots \to \beta_p \to \alpha_l.$$

If $\alpha_1 \in \{\alpha_k, \beta_1, \dots, \beta_p, \alpha_l\}$, then the third condition of Theorem 4.2 is satisfied. If $\alpha_1 \notin \{\alpha_k, \beta_1, \dots, \beta_p, \alpha_l\}$ then consider the walk

$$\alpha_k \to \beta_1 \to \cdots \to \beta_p \to \alpha_l \to \alpha_{l+1} \to \cdots \to \alpha_m \to \alpha_1 \to \alpha_2 \to \cdots \to \alpha_{l-1} \to \alpha_l,$$

which satisfies the third condition of Theorem 4.2.

Therefore, from Theorem 4.2, U is a generating set of $ORCT_{n,r}$, and so we have the following theorem.

Theorem 4.4. $\mathrm{rank}\,(ORCT_{n,r}) = \begin{cases} n & \textit{for } r = 1 \\ \binom{n-1}{r-1} & \textit{for } 2 \leq r \leq n-1 \end{cases}.$

Proof. The result follows from Corollary 3.2 and the facts $ORCT_{n,1} = OCT_{n,1}$ and $\mathrm{rank}\,(ORCT_{n,r}) \geq \binom{n-1}{r-1}$ for $2 \leq r \leq n - 1$. □

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