Turk. J. Math. Comput. Sci.
12(2)(2020) 166–175

DOI : 10.47000/tjmcs.734243

Timelike Ruled Surfaces in de-Sitter 3-Space

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Received: 08-05-2020 • Accepted: 28-11-2020

Abstract. In this paper, timelike ruled surfaces are studied in de-Sitter space S_1^3. A ruled surface in the de-Sitter space S_1^3 is obtained by moving a geodesic along a curve. Developable ruled surface, striction point, striction curve, dispersion parameter and orthogonal trajectory concepts are investigated for the obtained ruled surface.

2010 AMS Classification: 53A35

Keywords: Straight geodesic, Base curve, Central point, Central curve.

1. Introduction

Let $U \subset \mathbb{R}^2$ be open subset, and let $x : U \to S_1^3$ be an embedding. If the vector subspace $\tilde{U}$ which generated by $\{x_{u_1}, x_{u_2}\}$ contain at least a timelike vector field then x is called timelike surface in S_1^3 in [1]. In [7, 8], Turgut and Hacısalihoglu studied timelike ruled surfaces in the Minkowski 3-space. They showed that these surface are obtained by a timelike straight lines moving along a spacelike curves in [7, 8].

Let $x : M \longrightarrow \mathbb{R}_1^4$ be an immersion of a surface M into $\mathbb{R}_1^4$. We say that x is timelike (*resp*. spacelike, lightlike) if the induced metric on M via x is Lorentzian (*resp. Riemannian, degenerated*) in [3, 4]. If $\langle x, x\rangle = 1$, then x is an immersion of S_1^3 in [1]. A ruled surface is a surface generated by a straight line l moving along a curve α [8]. The various positions of the generating line l are called the rulings of the surface.

In this paper timelike ruled surface is investigated in de-Sitter 3-space S_1^3. A ruled surface is a surface obtained by a geodesic d_s^α moving along a curve α in [2, 5]. Thus ruled surface has a parametrization in S_1^3 as follows

$$\varphi(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,Z(s)$$

where α is called the base curve and Z is called the director vector of d_s^α.

2. Preliminaries

Let R_1^4 be 4-dimensional vector space equipped with the scalar product $\langle , \rangle$ which is defined by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

R_1^4 is 4-dimensional vector space equipped with the scalar product $\langle , \rangle$, than R_1^4 is called Lorentzian 4-space or 4-dimensional Minkowski space in [1, 6]. The Lorentzian norm (length) of x is defined to be

$$\|x\| = |\langle x, x\rangle|^{\frac{1}{2}} .$$

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If $\left(x_0^i, x_1^i, x_2^i, x_3^i\right)$ is the coordinate of x_i with respect to canonical basis $\{e_0, e_1, e_2, e_3\}$ of R_1^4, then the lorentzian cross product $x_1 \times x_2 \times x_3$ is defined by the symbolic determinant

$$x_1 \wedge x_2 \wedge x_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix}.$$

On can easly see that

$$\langle x_1 \wedge x_2 \wedge x_3, x_4 \rangle = \det (x_1, x_2, x_3, x_4).$$

We have follow of pseudo-sphere in R_1^4 [1]:

$$\text{de Sitter 3- space: } S_1^3 = \left\{ x \in R_1^4 \,|\langle x, x \rangle = 1 \right\}.$$

3. Timelike Ruled Surfaces in de-Sitter Space S_1^3

Now let's investigate the timelike ruled surfaces that its base curve is a spacelike curve and its direction geodesic is a timelike geodesic in the de-Sitter space S_1^3.

Let α be a differentiable curve with unit speed in de-Sitter space S_1^3, then it is defined by

$$\alpha : I \to S_1^3 \subset R_1^4 \,,\, \alpha (s) = (\alpha_1 (s), \alpha_2 (s), \alpha_3 (s), \alpha_4 (s)) \;, \forall s \in I,$$

where $\{0\} \subset I \subset \mathbb{R}$. In here

$$\langle \alpha (s), \alpha (s) \rangle = 1,$$

and since α base curve is a spacelike curve, then is

$$\langle \alpha' (s), \alpha' (s) \rangle = 1.$$

Let's assume that

$$\langle \alpha (s), Z (s) \rangle = 0, \forall s \in I,$$

where

$$Z : I \to H^3, Z (s) = (z_1 (s), z_2 (s), z_3 (s), z_4 (s))$$

and

$$\langle Z (s), Z (s) \rangle = -1.$$

Then, a geodesic d_s^α in de-Sitter space S_1^3 has a parametrization

$$d_s^\alpha : \mathbb{R} \to S_1^3, d_s^\alpha (t) = (\cosh t) \,\alpha (s) + (\sinh t) \,Z (s),$$

where $\alpha (s)$ is a initial point and $Z (s)$ is the direction vector of d_s^α. Here frenet components of base curve $\alpha (s)$ are $\{T_\alpha, N_\alpha, B_\alpha, \kappa_d, \tau_d\}$. Let T_d be tangent of geodesic d_s^α at the point $\alpha (s)$ and assume that T_d and T_α are linearly independent for all $s \in I$. Then, we obtain $(I \times \mathbb{R}, \varphi)$ parametrized by $\varphi : I \times \mathbb{R} \to S_1^3$

$$\varphi (s, t) = (\cosh t) \,\alpha (s) + (\sinh t) \,Z (s).$$

This $(I \times \mathbb{R}, \varphi)$ surface is called a ruled surface produced by the geodesic d_s^α.Let us denote this ruled surface with M. Then we can give the following definition.

Definition 3.1. The surface obtained by moving a given d_s^α geodesic along a given α curve is called the ruled surface in the de-Sitter space S_1^3. Where d_s^α is the direction geodesic of the ruled surface and the α curve is called base curve of ruled surface.

Let us find orthonormal base of tangent space $\chi (M)$ of ruled surface M along curve α. If T is unit tangent vector of curve α and Z is unit director vector of goedesic d_s^α, then we can choose timelike vector field such that

$$Y = \tilde{T}_d - \left\langle \tilde{T}_d, T \right\rangle T$$

that is orthogonal to T in this plane, where

$$\tilde{T}_d = \frac{T_d}{\|T_d\|}$$

is the unit tangent of geodesic d_s^α and

$$T_d = (\cosh t) \,T_{\alpha(s)} + (\sinh t) \,T_{Z(s)}.$$

Also, if we take $X = \dfrac{Y}{\|Y\|}$, then

$$\|X\| = -1, \langle X, T\rangle = 0 \text{ and } \langle T, T\rangle = 1.$$

Thus, $\{X, T\}$ are the orthonormal vectors of $\chi(M)$. Also,

$$\xi = \varphi \times T \times X$$

is normal vector of ruled surface M in de-Sitter space S^3_1. That is

$$\xi \in \chi^{\perp}(M),$$
$$\chi\left(S^3_1\right) = Sp\{X, T\} \oplus Sp\{\xi\},$$
$$\chi\left(R^4_1\right) = Sp\{X, T\} \oplus Sp\{\xi, \varphi\}.$$

In this case, system $\{\varphi, T, X, \xi\}$ is orthonormal base of M.

Now let investigate alteration of this system along curve α. The levi-civita connection of R^4_1, S^3_1 and M is denoted $\bar{\bar{D}}, \bar{D}$ and D, respectively.

$$\begin{cases} \bar{\bar{D}}_X Y = \bar{D}_X Y - \langle X, Y\rangle \alpha, & \tilde{A}(X) = \bar{\bar{D}}_X \alpha = I(X), \\ \bar{D}_X Y = D_X Y - \langle A(X), Y\rangle \xi, & A(X) = \bar{D}_X \xi \end{cases}$$

is written from Gauss equation. In de-Sitter space S^3_1, let's derive the $\{T, X, \xi\}$ orthonormal frame along curve α. In this case we get the system in S^3_1

$$\begin{cases} \bar{D}_T T &= aX + b\xi, \\ \bar{D}_T X &= aT + c\xi, \\ \bar{D}_T \xi &= -bT + cX. \end{cases}$$

The matrix representation of this system is

$$\begin{bmatrix} \bar{D}_T T \\ \bar{D}_T X \\ \bar{D}_T \xi \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ -b & c & 0 \end{bmatrix} \begin{bmatrix} T \\ X \\ \xi \end{bmatrix},$$

where

$$a = \left\langle \bar{D}_T X, T\right\rangle, b = \left\langle \bar{D}_T T, \xi\right\rangle \text{ and } c = \left\langle \bar{D}_T X, \xi\right\rangle.$$

Now, in R^4_1, let's derive the $\{\varphi, T, X, \xi\}$ orthonormal frame along curve α. In this case we get the system in R^4_1

$$\begin{cases} \bar{\bar{D}}_T \varphi &= (\cosh t + a \sinh t) T + (c \sinh t) \xi, \\ \bar{\bar{D}}_T T &= -\varphi + aX + c\xi, \\ \bar{\bar{D}}_T X &= aT + c\xi, \\ \bar{\bar{D}}_T \xi &= -bT + cX. \end{cases} \tag{3.1}$$

The matrix form of system (3.1)

$$\begin{bmatrix} \bar{\bar{D}}_T \varphi \\ \bar{\bar{D}}_T T \\ \bar{\bar{D}}_T X \\ \bar{\bar{D}}_T \xi \end{bmatrix} = \begin{bmatrix} 0 & \cosh t + a\,\mathrm{Si}\,nht & 0 & c \sinh t \\ -1 & 0 & a & b \\ 0 & a & 0 & c \\ 0 & -b & c & 0 \end{bmatrix} \begin{bmatrix} \varphi \\ T \\ X \\ \xi \end{bmatrix}.$$

For ruled surface M that is given by parametrization

$$\varphi : I \times \mathbb{R} \to S^3_1,\ \varphi(s, t) = (\cosh t)\,\alpha(s) + (\sinh t)\,X(s)$$

$$\begin{cases} E &= \langle \varphi_s, \varphi_s\rangle = (\cosh t + a \sinh t)^2 + c^2 \sinh^2 t, \\ F &= \langle \varphi_s, \varphi_t\rangle = 0, \\ G &= \langle \varphi_t, \varphi_t\rangle = -1. \end{cases}$$

Since $\langle \xi, \xi \rangle = F^2 - EG = E$ and $E > 0$, then $\langle \xi, \xi \rangle > 0$. That is, since ξ is spacelike vector, then ruled surface M is a timelike surface. Let us denote domain of t by J.

$$\varphi_{t_o} : I \times \{t_0\} \to M \ , \ \varphi_{t_0}(s, t_0) = (\cosh t_0)\, \alpha(s) + (\sinh t_0)\, X(s)$$

determines a curve of ruled surface M where t is constant in its domain. The tangent vector field of this curve is

$$A = (\cosh t_0 + a \sinh t_0)\, T(s) + c\, (\sinh t_0)\, \xi(s)\,.$$

Since $\langle A, A \rangle = E$ and $E > 0$, then A is spacelike vector. Thus φ_{t_0} curve is spacelike curve and also

$$\langle X, A \rangle = 0.$$

4. Developable Timelike Ruled Surfaces in de-Sitter Space S_1^3

Definition 4.1. If the tangent planes of a ruled timelike surface in S_1^3 are the same along its main geodesics, then this timelike ruled surface is called developable timelike ruled surface.

Theorem 4.2. *Let M be timelike ruled surface in de-Sitter space S_1^3. Then the tangent planes are the same along a main geodesic if and only if $c = 0$.*

Proof. Let M be a timelike ruled surface, and suppose that tangent planes of this ruled surface are the same along one of its main geodesics. We consider the tangent vector field

$$A = (\cosh t_0 + a \sinh t_0)\, T(s) + c\, (\sinh t_0)\, \xi(s)$$

of curve $\varphi_{t_0} : I \times \{t_0\} \to M$ which is passed through by $t_0 \in I$. Since φ_{t_0} is parametre curve of M, the vector A is in the tangent plane of the surface M. Hence $c = 0$.

Conversely, assume that $c = 0$. In this case, since

$$A = (\cosh t_0 + a \sinh t_0)\, T(s)$$

and

$$T_{\varphi(t_0, s)} M = sp\{T, X\} = sp\{T, A\}$$

then the tangent planes are the same along one of its main geodesics. □

Corollary 4.3. *The timelike ruled surface M in de-Sitter space S_1^3 is developable surface if and only if $c = 0$.*

Corollary 4.4. *For a timelike ruled surface M in de-Sitter space S_1^3,*

$$b = -\det\left(T, X, \varphi, \overline{\overline{D}}_T T\right) \ \text{and} \ c = -\det\left(T, X, \varphi, \overline{\overline{D}}_T X\right).$$

5. A Striction Point and Position Vector of A Striction Point of Timelike Ruled Surface in de-Sitter Space S_1^3

Definition 5.1. Let undevelopable timelike ruled surface be given in de-Sitter space S_1^3. If there exists common perpendicular of two neighbour main geodesic of timelike ruled surface, the foot of this perpendicular on principal geodesic is called striction or central point.

Definition 5.2. When the main geodesic of undevelopable timelike ruled surface in de-Sitter space S_1^3 creates the timelike ruled surface through base curve, the geometrical place of the striction points of ruled surface is called striction curve of M.

If w be the distance between the striction point of the undevelopable timelike ruled surface and base curve, then position vector $\bar{\alpha}(s)$ can be defined by

$$\bar{\alpha}(s, w) = (\cosh w)\, \alpha(s) + (\sinh w)\, X(s)\,,$$

where $\alpha(s)$ is position vector of base curve and $X(s)$ is direction vector of main geodesic.

The parametre w can be written as the combination of position vector of base curve and direction vector of timelike ruled surface. Let first two of three neighbour geodesic of timelike ruled surface be

$$d_s^{\alpha} = (\cosh t)\, \alpha(s) + (\sinh t)\, X(s)$$

and

$$d^{\alpha}_{s+\Delta s} = (\cosh t)\,\alpha\,(s+\Delta s) + (\sinh t)\,X\,(s+\Delta s)\,,$$

where $X(s)$ and $X(s) + \bar{D}_{T(s)}X(s)$ are the direction vectors of these main geodesic, respectively. Also let P, P' and Q, Q' be the feet on the main geodesic of the common perpendicular of the neighbour geodesic. Thus P and Q are two different striction points. The direction of common perpendecular first two main geodesics are linearly dependent to the vector

$$\alpha\,(s) \times X\,(s) \times \left[X\,(s) + \bar{D}_{T(s)}X\,(s)\right].$$

Therefore

$$\alpha\,(s) \times X\,(s) \times \left[X\,(s) + \bar{D}_{T(s)}X\,(s)\right] = \alpha\,(s) \times X\,(s) \times \bar{D}_{T(s)}X\,(s)\,.$$

The vector $\overrightarrow{PQ}$ coincides with the vector $\overrightarrow{PP'}$ in the limiting position, and $\overrightarrow{PQ}$ will be the tangent vector of the striction curve. Since

$$\left\langle X\,(s)\,, \overrightarrow{PQ}\right\rangle = 0 \text{ and } \left\langle X\,(s) + \bar{D}_{T(s)}X\,(s)\,, \overrightarrow{PQ}\right\rangle = 0$$

then we obtain

$$\left\langle \bar{D}_{T(s)}X\,(s)\,, \overrightarrow{PQ}\right\rangle = 0.$$

Thus

$$\left\langle \bar{D}_{T(s)}X\,(s)\,, \bar{D}_{T(s)}\bar{\alpha}\,(s)\right\rangle = 0. \tag{5.1}$$

On the other hand, since

$$\bar{D}_{T(s)}\bar{\alpha}\,(s) = \overline{\overline{D}}_{T(s)}\bar{\alpha}\,(s) + \langle T\,(s)\,, \bar{\alpha}\,(s)\rangle\,\bar{\alpha}\,(s)$$

we obtain

$$\bar{D}_{T(s)}\bar{\alpha}\,(s) = \overline{\overline{D}}_{T(s)}\bar{\alpha}\,(s)\,.$$

Consequently, from (5.1)

$$\left\langle \overline{\overline{D}}_{T(s)}X\,(s)\,, \overline{\overline{D}}_{T(s)}\bar{\alpha}\,(s)\right\rangle = 0$$

and then

$$\frac{\sinh w}{\cosh w} = -\frac{a}{a^2 + c^2}$$

or

$$w = \operatorname{arctan} h\left(-\frac{a}{a^2 + c^2}\right).$$

So, the position vector of striction curve is

$$\bar{\alpha}\,(s, w) = (\cosh w)\,\alpha\,(s) - \frac{a}{a^2 + c^2}\,(\cosh w)\,X\,(s)\,. \tag{5.2}$$

Theorem 5.3. *The distance between the striction point of the undevelopable timelike ruled surface and base curve is constant, that is*

$$w = \operatorname{arctan} h\left(-\frac{a}{a^2 + c^2}\right)$$

is constant.

Proof. Since

$$\langle X\,(s)\,, PQ\rangle = 0,$$

we obtain

$$\left\langle X\,(s)\,, \bar{D}_{T(s)}\bar{\alpha}\,(s)\right\rangle = 0$$

and

$$\bar{D}_{T(s)}\bar{\alpha}\,(s) = \overline{\overline{D}}_{T(s)}\bar{\alpha}\,(s)\,.$$

Thus

$$\left\langle X\,(s)\,, \overline{\overline{D}}_{T(s)}\bar{\alpha}\,(s)\right\rangle = 0$$

and

$$(-\cosh w)\,\frac{dw}{ds} = 0.$$

As a result

$$\frac{dw}{ds} = 0$$

and so, w is constant. □

Theorem 5.4. *Striction curve of a timelike ruled surface which is undevelopable in de-Sitter space S^3_1 is independent of choosing base curve.*

Proof. Let us denote two timelike ruled surfaces in de-Sitter space S^3_1 by

$$\varphi(t,s) = (\cosh t)\,\alpha(s) + (\sinh t)\,X(s),$$
$$\varphi(t,s) = (\cosh t)\,\beta(s) + (\sinh t)\,X(s),$$

where α and β are two different base curve of the timelike ruled surface in S^3_1. Then the striction curves of timelike ruled surface are

$$\bar{\alpha}(s) = (\cosh t)\,\alpha(s) - \frac{a}{a^2+c^2}(\cosh t)\,X(s),$$
$$\bar{\beta}(s) = (\cosh t)\,\beta(s) - \frac{a}{a^2+c^2}(\cosh t)\,X(s).$$

If we subtract $\bar{\beta}(s)$ from $\bar{\alpha}(s)$ and use (5.2), we obtain

$$\bar{\alpha}(s) - \bar{\beta}(s) = 0$$

which was what we wanted. □

Theorem 5.5. *Let M be undevelopable timelike ruled surface in de-Sitter space S^3_1. The point $\varphi(s,v_0)$ is striction point on main geodesic which passes through $\alpha(s)$ point if and only if $\bar{D}_{T(s)}X(s)$ is a normal vector of tangent plane on $\varphi(s,v_0)$ point.*

Proof. Let M be undevelopable timelike ruled surface in de-Sitter space S^3_1. Suppose that $\bar{D}_{T(s)}X(s)$ is a normal vector of tangent plane on $\varphi(s,v_0)$ point. Since tangent vector field of $\varphi_{v_0} : I\times\{v_0\} \to M$ curve is

$$A = (\cosh v_0 + a\sinh v_0)\,T(s) + c(\sinh v_0)\,\xi(s)$$

then

$$\left\langle \bar{D}_{T(s)}X(s), A\right\rangle = 0.$$

Thus, we obtain

$$\frac{\sinh v_o}{\cosh v_0} = -\frac{a}{a^2+c^2}.$$

Therefore $\varphi(s,v_0)$ is a striction point of M.

Conversely, suppose that $\varphi(s,v_0)$ is a striction point with main geodesic passing through the points $\alpha(s)$. Thus,

$$\left\langle \bar{D}_{T(s)}X(s), X(s)\right\rangle = 0,$$
$$\left\langle \bar{D}_{T(s)}X(s), A\right\rangle = a(\cosh v_0 + a\sinh v_0) + c^2\sinh v_0.$$

Since $\varphi(s,v_0)$ is striction point, then we get

$$a(\cosh v_0 + a\sinh v_0) + c^2\sinh v_0 = 0.$$

Hence, we obtain

$$\left\langle \bar{D}_{T(s)}X(s), A\right\rangle = 0.$$

So, $\bar{D}_{T(s)}X(s)$ is a normal vector of tangent plane at $\varphi(s,v_0)$. □

Remark 5.6. Let $\bar{D}_{T(s)}X(s)$ be a normal vector of the tangent plane on the striction point. Since

$$\left\langle \bar{D}_{T(s)}X(s), \bar{D}_{T(s)}X(s)\right\rangle = a^2 + c^2 > 0.$$

$\bar{D}_{T(s)}X(s)$ is a spacelike normal vector field.

Theorem 5.7. *Let M be undevelopable timelike ruled surface in de-Sitter space S^3_1. The striction curve $\bar{\alpha}(s)$ is a spacelike curve.*

Proof. We need to show that tangent vector field of striction curve $\bar{\alpha}$ is a spacelike vector field. It is clear that

$$\left\langle \overline{\overline{D}}_{T(s)}\bar{\alpha}(s), \overline{\overline{D}}_{T(s)}\bar{\alpha}(s) \right\rangle = \frac{c^2}{a^2+c^2}\cosh^2 w > 0,$$

where

$$\overline{\overline{D}}_{T(s)}\bar{\alpha}(s) = (\cosh w)\,\overline{\overline{D}}_{T(s)}\alpha(s) - \frac{a}{a^2+c^2}(\cosh w)\,\overline{\overline{D}}_{T(s)}X(s).$$

Since $\left\langle \overline{\overline{D}}_{T(s)}\bar{\alpha}(s), \overline{\overline{D}}_{T(s)}\bar{\alpha}(s) \right\rangle > 0$, then $\bar{\alpha}(s)$ is spacelike curve. □

6. Dispersion Paremeter of Timelike Ruled Surface in de-Sitter S_1^3 Space

Let the base curve of a timelike ruled surface M be the striction curve in de-Sitter space S_1^3 Then, the distance from the striction point to the base curve is

$$w = \operatorname{arctan} h\left(-\frac{a}{a^2+c^2}\right) = 0.$$

Hence, we have

$$a = 0,$$

and since

$$\bar{D}_{T(s)}X(s) = aT(s) + c\xi(s),$$

the vector field $\bar{D}_{T(s)}X(s)$ and normal of surface $\xi(s)$ are linearly independent. Therefore, there exists $\lambda \in \mathbb{R}$ for the equality

$$\xi(s) = \lambda \bar{D}_{T(s)}X(s).$$

On the other hand, since

$$\xi(s) = \varphi \times X \times T$$

and

$$\varphi = (\cosh t)\,\alpha(s) + (\sinh t)\,X(s),$$

we have

$$\xi(s) = (\cosh t)\left[\alpha(s) \times X(s) \times T(s)\right].$$

Therefore, we have

$$\lambda \bar{D}_{T(s)}X(s) = (\cosh t)\left[\alpha(s) \times X(s) \times T(s)\right].$$

If we take scalar product with $\bar{D}_{T(s)}X(s)$ of both sides of the above equality, then we have

$$\lambda = (\cosh t)\,\frac{\det\left(\alpha(s), T(s), X(s), \bar{D}_{T(s)}X(s)\right)}{\left\langle \bar{D}_{T(s)}X(s), \bar{D}_{T(s)}X(s)\right\rangle},$$

where λ is called dispersion parameter of timelike ruled surface in de-Sitter space S_1^3.

Theorem 6.1. *The timelike ruled surface M in de-Sitter space S_1^3 is developable if and only if dispersion parameter of M is zero.*

Proof. From Corollary-1 and Corollary-2, we get

$$c = -\det\left(T(s), X(s), \alpha(s), \overline{\overline{D}}_{T(s)}X(s)\right) = 0$$

and so it is clear from the definition of the dispersion parameter that

$$\lambda = (\cosh t)\,\frac{\det\left(\alpha(s), T(s), X(s), \bar{D}_{T(s)}X(s)\right)}{\left\langle \bar{D}_{T(s)}X(s), \bar{D}_{T(s)}X(s)\right\rangle} = 0.$$ □

Definition 6.2. If there exists a curve that cuts vertically each main geodesic of the ruled surface in de-Sitter space S_1^3, then this curve is called orthogonal trajector of timelike ruled surface in de-Sitter space S_1^3.

Theorem 6.3. *Let M be a timelike ruled surface in de-Sitter space S_1^3. There is only one orthogonal trajector which passes through every point of M.*

Proof. Let M be a timelike ruled surface in given by the parametrization $\varphi : I \times J \to S_1^3 \subset R_1^4$,

$$\varphi(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,Z(s).$$

Then, the orthogonal trajectory of M is $\beta : \tilde{I} \subset I \to M$,

$$\beta(s) = [\cosh f(s)]\,\alpha(s) + [\sinh f(s)]\,Z(s).$$

Since

$$\left\langle \bar{D}_{T(s)}\beta(s), Z(s)\right\rangle = 0,$$

we get

$$f(s) = -\int \left\langle \overline{\overline{D}}_{T(s)}\alpha(s), Z(s)\right\rangle ds + h,$$

where $\langle Z(s), Z(s)\rangle = -1$. If we take

$$F(s) = -\int \left\langle \overline{\overline{D}}_{T(s)}\alpha(s), Z(s)\right\rangle ds,$$

we get

$$f(s) = F(s) + h.$$

Since h is choosen arbitrary, there are a lot of curves that satisfy the condition

$$\left\langle \bar{D}_{T(s)}\beta(s), Z(s)\right\rangle = 0.$$

Let us now find $s \in \mathbb{R}$ such that

$$P_0 = [\cosh(F(s)+h)]\,\alpha(s) + [\sinh(F(s)+h)]\,Z(s).$$

We get

$$[\cosh f(s)]\,\alpha(s) + [\sinh f(s)]\,Z(s) = [\cosh v_0]\,\alpha(s_0) + [\sinh v_0]\,Z(s_0).$$

So,

$$\alpha(s_0) = \alpha(s),\; v_0 = f(s).$$

If we choose interval I such that α is one to one, then we get

$$s = s_0.$$

Thus,

$$h = f(s_0) - F(s_0).$$

Consequently, there exists only one orthogonal trajectory passing through the point P_0. Therefore, $\tilde{I}$ must be equal to I. □

Theorem 6.4. *Let M be undevelopable timelike ruled surface in de-Sitter space S_1^3. The shortest distance along orthogonal trajectors beetween of any two main geodesic of M is distance measured along curve $\varphi_t : I \to M$ corresponding to* $t = \frac{1}{2}\operatorname{arctan} h\left(-\frac{2a}{1+a^2+c^2}\right)$.

Proof. Let us take two geodesic passing through points $\alpha(s_1)$ and $\alpha(s_2)$ where $s_1, s_2 \in I$ and $s_1 < s_2$. Also, let us denote distance obtained along orthogonal trajector t =constant between these lines by $d(t)$. Then,

$$d(t) = \int_{s_1}^{s_2} \|A\|\,ds = \sqrt{(\cosh t + a\sinh t)^2 + c^2\sinh^2 t}\,(s_2 - s_1),$$

where

$$A = (\cosh t + a\sinh t)\,T(s) + c\,(\sinh t)\,\xi(s).$$

If $d'(t) = 0$, then $d(t)$ takes maximum value. Hence we get

$$t = \frac{1}{2}\operatorname{arctan} h\left(-\frac{2a}{1+a^2+c^2}\right).$$ □

Theorem 6.5. *Let M be timelike ruled surface in de-Sitter space S_1^3. The geodesic of M are both asymtotic and geodesic curves.*

Proof. Let X be tangent vector field of a geodesic of a timelike ruled surface M. Since every geodesic in ruled surface M, it is a geodesic S^3_1. Thus we get

$$\bar{D}_X X = 0.$$

From Gauss equation, we also get

$$\bar{D}_X X = D_X X + \langle S(X), X\rangle \xi.$$

Thus

$$D_X X = -\langle S(X), X\rangle \xi.$$

Therefore

$$D_X X \in \chi(M) \text{ and } \langle S(X), X\rangle \xi \in \chi^{\perp}(M).$$

Since metric on M is nondegenered, we get

$$\chi\left(S^3_1\right) = \chi(M) \oplus \chi^{\perp}(M) \text{ and } \chi(M) \cap \chi^{\perp}(M) = \{0\}.$$

Thus

$$D_X X = 0 \text{ and } \langle S(X), X\rangle = 0.$$

The proof is completed. □

Theorem 6.6. *Let M be timelike ruled surface in de-Sitter space S^3_1. Then*

$$K(p) \geq 0 \text{ for all } p \in M$$

where K is Gauss curvature function of M.

Proof. Let X be tangent vector field of the main geodesic at point $p \in M$ and take the orthonormal basis $\{X, Y\}$ of $\chi(M)$. Since M is timelike ruled surface, X ,Y are timelike and spacelike vector fields, respectivelly. The weingarthen operator S of M can be written

$$\begin{aligned} S(X) &= -\langle S(X), X\rangle X + \langle S(X), Y\rangle Y, \\ S(Y) &= -\langle S(Y), X\rangle X + \langle S(Y), Y\rangle Y. \end{aligned}$$

In this case, the matrix

$$S = \begin{bmatrix} -\langle S(X), X\rangle & \langle S(X), Y\rangle \\ -\langle S(Y), X\rangle & \langle S(Y), Y\rangle \end{bmatrix}$$

is corresponding to weingarthen operator S. On the other hand , the weingarthen operator S is selfadjoint,

$$\langle S(Y), X\rangle = \langle Y, S(X)\rangle .$$

Also, by Theorem 6.5

$$\langle S(X), X\rangle = 0, \langle S(Y), Y\rangle = 0.$$

Hence, we get

$$K = \det S = \langle S(X), Y\rangle^2$$

from definition Gauss curvature. The proof is completed. □

Theorem 6.7. *Let M be a timelike ruled surface in de-Sitter space S^3_1. Then*

$$\begin{cases} \varphi \times T \times X &= \xi, \\ T \times X \times \xi &= -\varphi, \\ \xi \times \varphi \times T &= X, \\ X \times \xi \times \varphi &= T, \end{cases}$$

where T is unit tangent vector of base curve, φ is position vector of M, X is unit tangent vector field of main geodesic of M and ξ is unit normal vector field of M.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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