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Eigenvalue Estimates Using Harmonic 1−Form of Constant Length for The $Spin^c$ Dirac Operator

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ABSTRACT: In this paper, we obtain a lower bound for the eigenvalue of the $Spin^c$ Dirac operator on an $(d \geq 3)$ −dimensional compact Riemannian $Spin^c$ −manifold admitting a non−zero harmonic 1 −form of constant length. Then we show that, in the limiting case, this 1 −form is parallel.

Keywords: Spin and $Spin^c$ geometry, Dirac operator, Estimation of eigenvalues.

Sabit Uzunluklu Harmonik 1−Form Kullanılarak $Spin^c$ Dirac Operatörünün Özdeğerlerine Tahminler

ÖZET: Bu makalede, sıfır olmayan sabit uzunluklu harmonik 1-formu kabul eden $(d \geq 3)$ −boyutlu kompakt bir Riemann $Spin^c$ −manifoldu üzerinde tanımlı $Spin^c$ Dirac operatörünün öz değeri için alt sınır elde ettik. Daha sonra, limit durumunda harmonik 1 −formun paralel olduğunu gösterdik.

Anahtar Kelimeler: Spin ve $Spin^c$ geometry, Dirac operatörü, Öz değer tahminleri.

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INTRODUCTION

The Dirac operator is an important tool that provides information about the topology and geometry of the compact Riemannian $Spin^c$ −manifold and compact Riemannian $Spin$ manifold. Due to this feature of the Dirac operator, many authors have been systematically worked on it. One of these studies is to give a lower bound to the the square of the eigenvalue of the Dirac operator. In 1963 A. Lichnerowicz (Lichnerowicz, 1963) presented the following formula called Schrödinger−Lichnerwicz formula

$$D^2 = \Delta + \frac{R}{4} \tag{1}$$

where Δ is the Laplacian acting on any spinor field and R is the scalar curvature of (M, g). By using (1) A. Lichnerowicz obtained the following estimates for the eigenvalue of the Dirac operator D,

$$\lambda^2 \geq \frac{1}{4} \inf_M R. \tag{2}$$

In (Friedrich, 1980) T. Friedrich proved that on a Spin manifold (M, g) of dimension $d \geq 2$, any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{d}{4(d-1)} \inf_M R. \tag{3}$$

The proof is based on the modified spinorial Levi−Civita connection

$$\nabla_V^f \varphi = \nabla_V \varphi + fV \cdot \varphi, \tag{4}$$

where $f \in C^\infty(M, \mathbb{R})$. The limiting case of (3) implies that the existence of Killing spinor, i.e a spinor φ satisfying the equation:

$$\nabla_V \varphi + fV \cdot \varphi = 0, \ \forall \ V \in \chi(M). \tag{5}$$

In dimensions 2, C. Bär (Bär, 1992) obtained a bound to the eigenvalue λ of the Dirac operator according to the Euler−Poincare characteristic $\mathcal{X}(M)$ of M as follows:

$$\lambda^2 \geq \frac{2\pi\mathcal{X}(M)}{Area(M,g)}. \tag{6}$$

Later on, by using the conformal covariance of the Dirac operator, O. Hijazi improved the inequality (3), on a Spin manifold (M, g) of dimension $d \geq 3$,

$$\lambda^2 \geq \frac{d}{4(d-1)} \mu_1, \tag{7}$$

here μ_1 denotes the first eigenvalue of the Yamabe operator L given by

$$L := 4\frac{d-1}{d-2}\Delta_g + R \tag{8}$$

and Δ_g denotes the positive Laplacian acting on functions. In 1995, O. Hijazi (Hijazi, 1995) modified the spinorial Levi−Civita connection in the direction of symmetric endomorphism l_φ as

$$\nabla_V^{l_\varphi}\varphi = \nabla_V\varphi + l_\varphi(V)\cdot\varphi \tag{9}$$

Then, he obtained that

$$\lambda^2 \geq \inf_M\left(\frac{R}{4} + |l_\varphi|^2\right). \tag{10}$$

Also, O. Hijazi has shown that using the modified spinorial Levi−Civita connection given in (9) and the conformal covariance of the Dirac operator on a Spin manifold (M, g), any eigenvalue of the Dirac operator is satisfied

$$\lambda^2 \geq \begin{cases} \frac{1}{4}\mu_1 + \inf_M|l_\varphi|^2, & if\ \ d \geq 3 \\ \frac{\pi\mathcal{X}(M)}{Area(M,g)} + \inf_M|l_\varphi|^2, & if\ \ d = 2, \end{cases} \tag{11}$$

where μ_1 is the first eigenvalue of the Wamabe operator L. In the limiting case of (10), O. Hijazi obtained the following relations:

$$(trl_\varphi)^2 = \frac{R}{4} + |l_\varphi|^2,$$

$$grad(trl_\varphi) = -div(trl_\varphi), \tag{12}$$

where (trl_φ) is the trace part of l_φ. Subsequently, G. Habib (Habib, 2007) modified the spinorial Levi-Civita connection (9) in the direction of the skew−symmetric endomorphism q_φ of the 2 −tensor E, as

$$\nabla_V^{l_\varphi}\varphi = \nabla_V\varphi + l_\varphi(V)\cdot\varphi + q_\varphi(V)\cdot\varphi. \tag{13}$$

By using (13), he improved (10) as follows:

$$\lambda^2 \geq \inf_M\left(\frac{R}{4} + |l_\varphi|^2 + |q_\varphi|^2\right). \tag{14}$$

As mentioned above, many studies have been done to improve lower bound (3), but the fundemantal question is: When is the equalitW in (3) hold? Accordingly, O. Hijazi (Hijazi, 1986) and A. Lichnerowicz (Lichnerowicz, 1988; Lichnerowicz 1987) noticed that the equalitW in (3) cannot hold on the Spin manifolds admitting a non−zero parallel r −form for some $r \in \{1,2,\ldots,d-1\}$. Under this assumption, A. Moroianu and L. Ornea (Moroianu et al., 2004) enhanced the lower bound obtained in (3) on a d −dimensional Spin manifolds admitting a non−trivial harmonic 1 −form of constant length as follows:

$$\lambda^2 \geq \frac{d-1}{d-2}\inf_M R. \tag{15}$$

In the limiting case, they show that, the universal cover of the manifold is isometric to the $\mathbb{R}\times N$ where N is a manifold admitting Killing spinors. In this paper, we consider the same assumption for the compact d −dimensional $Spin^c$ −manifold admitting a non−trivial harmonic 1 −form of constant length. Before mentioning to this assumption, we briefly touch on what kind of studies are done to obtain lower bound estimates for the eigenvalue of the $Spin^c$ Dirac operator defined .

All the inequalities mentioned above is obtained on the Spin manifold. This paper deals onlW with the eigenvalue of the Dirac operator defined on the $Spin^c -$ manifold.

In 1999, A. Moroianu and M. Herzlich (Moroianu et al., 1999) proved that on a compact Riemannian $Spin^c$ manifold of dimension $n \geq 3$, any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{d}{4(d-1)}\mu_1, \tag{16}$$

where μ_1 denotes the first eigenvalue of the perturbed Yamabe operator L^Ω given by

$$L^\Omega := L - c_d|\Omega|_g, \tag{17}$$

and Ω is the curvature form of the line bundle $\mathcal{L}$. In (Herzlich et al., 1999), theW showed that there are no generalized Killing spinors on a Spin $^c -$ manifold of dimension $d \geq 4$, except the usual Killing spinors.

Using the modified spinorial Levi−Civita connection in the direction of $l_\varphi + q_\varphi$, R. Nakad (Nakad, 2010) proved that, on a compact $Spin^c$ −manifold of dimension $d \geq 2$ any eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geq \inf_M \left(\frac{R}{4} - \frac{c_d}{4}|\Omega|_g + |l_\varphi|^2 + |q_\varphi|^2\right), \tag{18}$$

where $c_d = 2[\frac{d}{2}]^{\frac{1}{2}}$. Also, by considering the deformation of the spinorial Levi−Civita conection in the direction of the symmetric endomorphism l_Φ given in (9), he obtained

$$\lambda^2 \geq \inf_M \left(\frac{R}{4} - \frac{c_d}{4}|\Omega|_g + |l_\varphi|^2\right). \tag{19}$$

Furthermore, using the modified spinorial Levi−Civita connecion in the direction of l_Φ and conformal covariance of the Dirac operator, he has shown that on a on a $Spin^c$ −manifold, any eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geq \begin{cases} \frac{1}{4}\mu_1 + \inf_M |l_\varphi|^2, & if \;\; d \geq 3 \\ \frac{\pi\chi(M)}{Area(M,g)} + \inf_M |l_\varphi|^2, & if \;\; d = 2, \end{cases} \tag{20}$$

where μ_1 denotes first eigenvalue of the perturbed Yamabe operator L^Ω. Then, in the limiting case, he obtained the following relations

$$(trl_\varphi)^2 = \frac{R}{4} - [\frac{d}{2}]^{1/2}|\Omega|_g + |l_\varphi|^2,$$

$$grad(trl_\varphi) = -div(trl_\varphi), \tag{21}$$

where (trl_φ) is the trace part of l_φ.

In this paper, we show that any eigenvalue λ of D on an $(d \geq 3)$ −dimensional $Spin^c$ −manifold admitting a non−zero harmonic 1 −form of constant lenght satisfies

$$\lambda^2 \geq \frac{d-1}{4(d-2)} \inf_M \left(R - c_d|\Omega|_g\right). \quad (22)$$

Furthermore, in the limiting case, this 1 −form is parallel.

In the following section, some basic notions concerning Riemannian $Spin^c$-manifold and Dirac operator is introduced.

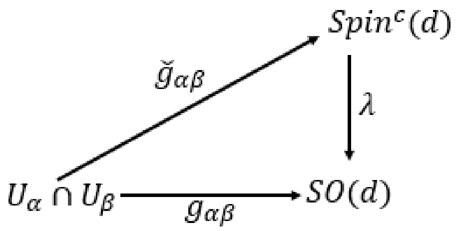
MATERIALS AND METHODS

$Spin^c$ Geometry and the Dirac operator

Definitions of $Spin^c$ −structures on (M, g) are obtained as follows: The structure group of d −dimensional compact Riemannian manifold (M, g) is $SO(d)$ and there is an open covering $\{U_\alpha\}_{\alpha\in A}$ with the transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow SO(d)$ for (M, g). Accordingly, if there exists another collection of transition functions

$$\tilde{g}_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow Spin^c(d)$$

such that the following diagram commutes



that is, $\lambda \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition $\tilde{g}_{\alpha\beta}(x) \circ \tilde{g}_{\beta\gamma}(x) = \tilde{g}_{\alpha\gamma}(x)$ on $U_\alpha \cap U_\beta \cap U_\gamma$ is satisfied, then M is called $Spin^c$ manifold. Along with the $Spin^c$ manifold (M, g), one can construct two principal bundles such as $P_{SO(d)}$, P_{Spin^c} (Friedrich, 2000).

On a Riemannian $Spin^c$ manifold, an associated spinor bundle $\mathbb{S} = P_{Spin^c} \times_d \Delta_d$ can be constructed by using spinor representations

$$\kappa_n: Spin^c(d) \mapsto Aut(\Delta_d)$$

where Δ_d is the irreducible representation of Clifford algebra (Friedrich, 2000). The sections of $\mathbb{S}$ are called spinor fields. The spinor bundle $\mathbb{S}$ carries a natural Hermitian product, denoted by (,) and satisfies

$$(V \cdot \varphi, \psi) = -(\varphi, V \cdot \psi)$$

for every $V \in \chi(M)$ and $\varphi, \psi \in \Gamma(\mathbb{S})$.

The following bundle map κ is obtained by globalising κ_n as follows:

$$\kappa : TM \to End(\mathbb{S}).$$

and Clifford multiplication of a vector field V with the spinor field φ is defined by

$$V \cdot \varphi := \kappa(V)(\varphi).$$

By using the map κ, the bundle map ρ, which associates each 2 −form to an endomorphism of $\mathbb{S}$, can be defined on the orthonormal frame $\{e_1, e_2, \dots, e_d\}$ as follows:

$$\begin{array}{ccc} \rho : \Lambda^2(T^*M) & \to & End(\mathbb{S}) \\ \eta = \sum_{i<j} \eta_{ij} e^i \wedge e^j & \to & \rho(\eta) = \sum_{i<j} \eta_{ij} \kappa(e_i)\kappa(e_j). \end{array}$$

Also ρ can be extended to a complex valued 2 −forms (Salamon,1999), such that

$$\rho : \Lambda^2(T^*M) \otimes \mathbb{C} \to End(\mathbb{S}).$$

On the spinor bundle $\mathbb{S}$, $V \in \chi(\mathrm{M})$ and $\varphi, \psi \in \Gamma(\mathbb{S})$, the following properties is satisfied: (Salamon,1999),

$$V(\varphi, \psi) = (\nabla_V \varphi, \psi) + (\varphi, \nabla_V \psi)$$

$$\nabla_V(\alpha \cdot \varphi) = (\nabla_V \alpha) \cdot \varphi + \alpha \cdot \nabla_V \varphi$$

$$V \cdot \alpha = V \wedge \alpha - V \lrcorner \alpha, \tag{23}$$

where " ∧ " and " ⌟" denotes the exterior product and interior product with V, respectively. The Dirac operator induced by the Levi−Civita connection ∇^g, is defined as follows:

$$D =\cdot \circ \nabla : \Gamma(\mathbb{S}) \overset{\nabla}{\to} \Gamma(T^*M \otimes \mathbb{S}) \overset{g}{\cong} \Gamma(TM \otimes \mathbb{S}) \overset{\cdot}{\to} \Gamma(\mathbb{S})$$

where the isomorphism between T^*M and TM determined by the metric g. ∇ is a spinorial connection on the spinor bundle $\mathbb{S}$.

Let $e = \{e_1, e_2, \dots, e_d\}$ be an orthonormal frame on $U \subset M$. Accordingly, Dirac operator locally can be written as

$$D\varphi = \sum_{i=1}^{d} e_i \cdot \nabla_{e_i} \varphi \tag{24}$$

Also, Schrödinger−Lichnerowicz formula is given by

$$D^2 \varphi = \nabla^* \nabla \varphi + \frac{R}{4} \Psi + \frac{i}{2} \rho(\Omega) \varphi. \tag{25}$$

On the spinor bundle $\mathbb{S}$, $\mathcal{R}$ denotes the spinorial curvature associated with the connection Ω as:

$$\mathcal{R}_{V,W} \varphi = \frac{1}{4} \sum_{i,j=1}^{d} g(R_{V,W} e_i, e_j) e_i \cdot e_j \cdot \Psi + \frac{i}{2} \Omega(V, W) \cdot \varphi \tag{26}$$

where $V, W \in \chi(M)$ and $\varphi \in \Gamma(\mathbb{S})$.

In the $Spin^c$ case, the Ricci is discrebe as

$$\sum_j e_j \cdot \mathcal{R}_{V,W}\varphi = \frac{1}{2}Ric(V)\cdot\varphi - \frac{i}{2}(V \lrcorner \Omega)\cdot\varphi. \tag{27}$$

Lemma 2.1 *On the d −dimensional Spin ^{c} − manifold, for any spinor field $\varphi \in \Gamma(\mathbb{S})$ and a real 2 −form Ω, we have*

$$(i\rho(\Omega)\varphi,\varphi) \geq -\frac{c_d}{2}|\Omega|_g|\varphi|^2, \tag{28}$$

where $|\Omega|_g$ denotes the norm of Ω with respect to the Riemannian metric g (Herzlich et al., 1999).

RESULTS AND DISCUSSION

Eigenvalue Estimates

In this section, for a given non−zero harmonic 1 −form of constant length, we give a lower bound estimate to the eigenvalue λ of the Spin ^{c} Dirac operator. Then, by considering limiting case we obtain that harmonic 1 −form is parallel.

Theorem 3.1 *Assume that (M^d, g) is a $(d \geq 3)$ − dimensional Spin ^{c} −manifold admitting a non zero harmonic 1 −form of constant length. Then, the following estimate is satisfied*

$$\lambda^2 \geq \frac{d-1}{4(d-2)}\inf_M\left(R - c_d|\Omega|_g\right), \tag{29}$$

where $c_d = 2[\frac{d}{2}]^{1/2}$. Also, in equality case for some eigenvalue λ, ζ is parallel.

Proof. Assume that ζ is a dual vector field of a harmonic 1 −form ω of unit length on a Spin ^{c} − manifold (M^d, g). Considering Penrose−like operator $T: \chi(M)\Gamma(\mathbb{S}) \longrightarrow \Gamma(\mathbb{S})$,

$$T_V\varphi = \nabla_V\varphi + \frac{1}{d-1}V\cdot D\varphi - \frac{1}{d-1} < V,\zeta > \zeta\cdot D\varphi - < V,\zeta > \nabla_\zeta\varphi, \tag{30}$$

where $V \in \chi(M)$ and $\varphi \in \Gamma(\mathbb{S})$.

Taking $V = e_i$ in (30) and performing its Hermitian inner product with itself, yields

$$\begin{aligned}|T_{e_i}\varphi|^2 = {} & |\nabla_{e_i}\varphi|^2 - \frac{2}{d-1}Re(e_i\cdot\nabla_{e_i}\varphi, D\varphi) - \frac{2}{d-1}Re\left(\nabla_{e_i}\varphi, < e_i,\zeta > \zeta\cdot D\varphi\right) \\ & - 2Re\left(\nabla_{e_i}\varphi, < e_i,\zeta > \nabla_\zeta\varphi\right) - \frac{2}{(d-1)^2}Re(e_i\cdot D\varphi, < e_i,\zeta > \zeta\cdot D\varphi) \\ & + \frac{|D\varphi|^2}{(d-1)^2} - \frac{2}{(d-1)}Re(e_i\cdot\ D\varphi, < e_i,\zeta > \nabla_\zeta\varphi) \\ & + \frac{1}{(d-1)^2}(< e_i,\zeta > \zeta\cdot D\varphi, < e_i,\zeta > \zeta\cdot D\varphi) \\ & + \frac{2}{(d-1)}Re(< e_i,\zeta > \zeta\ \cdot D\varphi, < e_i,\zeta > \nabla_\zeta\varphi)\end{aligned}$$

$$+(< e_i, \zeta > \nabla_\zeta \varphi, < e_i, \zeta > \nabla_\zeta \varphi). \tag{31}$$

Summing over i and using the fact that $< \zeta, \zeta >= 1$, gives

$$|T\varphi|^2 = |\nabla\varphi|^2 - \frac{2}{d-1}|D\varphi|^2 + \frac{2}{d-1}Re(\zeta \cdot \nabla_\zeta \varphi, D\varphi) - 2|\nabla_\zeta \varphi|^2 - \frac{2}{(d-1)^2}|D\varphi|^2$$

$$+\frac{d}{(d-1)^2}|D\varphi|^2 + \frac{2}{(d-1)}Re(D\varphi, \zeta \cdot \nabla_\zeta \varphi) + \frac{1}{(d-1)^2}|D\varphi|^2$$

$$-\frac{2}{(d-1)}Re(D\varphi, \zeta \cdot \nabla_\zeta \varphi) + |\nabla_\zeta \varphi|^2$$

$$= |\nabla\varphi|^2 - |\nabla_\zeta \varphi|^2 + \frac{2}{(d-1)}Re(D\varphi, \zeta \cdot \nabla_\zeta \varphi) - \frac{1}{(d-1)^2}|D\varphi|^2. \tag{32}$$

Recall that, the harmonicitW of ζ satisfies

$$D(\zeta \cdot \varphi) = -\zeta \cdot D\varphi - 2\nabla_\zeta \varphi. \tag{33}$$

Square norm of (33) is

$$|D(\zeta \cdot \varphi)|^2 = |\zeta \cdot D\varphi|^2 + 4|\nabla_\zeta \varphi|^2 - 4Re(D\varphi, \zeta \cdot \nabla_\zeta \varphi). \tag{34}$$

Integrating (32) over M and using (34), we obtain

$$\int_M |T\varphi|^2 v_g = \int_M (|\nabla\varphi|^2 - |\nabla_\zeta \varphi|^2 + \frac{1}{2(d-1)}|\zeta \cdot D\varphi|^2 + \frac{2}{d-1}|\nabla_\zeta \varphi|^2$$

$$-\frac{1}{2(d-1)}|D(\zeta \cdot \varphi)|^2 - \frac{1}{(d-1)^2}|D\varphi|^2)v_g, \tag{35}$$

where v_g is the volume element induced by g.

Inserting (25) in the above equalitW, we get

$$\int_M |T\varphi|^2 v_g = \int_M ((\frac{d-2}{d-1})|D\varphi|^2 - \frac{R}{4}|\Phi|^2 - (\frac{i}{2}\rho(\Omega)\varphi, \varphi) - (\frac{d-3}{d-1})|\nabla_\zeta \varphi|^2$$

$$-\frac{1}{2(d-1)}(|D(\zeta \cdot \varphi)|^2 - |\zeta \cdot D\varphi|^2))v_g \tag{36}$$

Using the following Rayleigh inequality (Moroianu et al., 2004) and $< \zeta, \zeta >= 1$,

$$\lambda^2 \le \frac{\int_M |D\Psi|^2 v_g}{\int_M |\Psi|^2 v_g}, \tag{37}$$

where $\Psi = \zeta \cdot \varphi$, we have

$$\int_M \left((\frac{d-2}{d-1})\lambda^2 - \frac{R}{4} - (\frac{i}{2}\rho(\Omega)\varphi, \varphi)\right)|\varphi|^2 v_g = \int_M \ (|T\varphi|^2 + \left(\frac{d-3}{d-1}\right)|\nabla_\zeta \varphi|^2$$

$$+\frac{1}{2(d-1)}(|D(\zeta \cdot \varphi)|^2 - |\zeta \cdot D\varphi|^2))v_g \ge 0, \tag{38}$$

which implies the inequality (29).

Consider the limiting case of (29). Let λ be an eigenvalue of D to which is attached an eigenspinor φ. Then, $T\varphi = 0$, implies

$$\nabla_{e_i}\varphi + \frac{\lambda}{n-1} e_i \cdot \varphi - \frac{\lambda}{n-1} < e_i, \zeta > \zeta \cdot \varphi - < e_i, \zeta > \nabla_\zeta \varphi = 0. \quad (39)$$

Performing its Clifford multiplication by e_i and using $< \zeta, \zeta >= 1$, Wields

$$0 = \sum_{i=1}^{n} \left(e_i \cdot \nabla_{e_i}\varphi + \frac{\lambda}{d-1} e_i \cdot e_i \cdot \varphi - \frac{\lambda}{d-1} < e_i, \zeta > e_i \cdot \zeta \cdot \varphi - < e_i, \zeta > e_i \cdot \nabla_\zeta \varphi \right)$$

$$= \lambda\Phi - \frac{d}{d-1}\lambda\Phi + \frac{1}{n-1}\lambda\Phi - \zeta \cdot \nabla_\zeta \Phi$$

$$= -\zeta \cdot \nabla_\zeta \varphi. \quad (40)$$

EqualitW (40) implies that $\nabla_\zeta \varphi = 0$.

As in (Moroianu et al., 2004), φ satisfies the Killing type equation

$$\nabla_V \varphi = -\frac{\lambda}{d-1} V \cdot \varphi + \frac{\lambda}{d-1} < V, \zeta > \zeta \cdot \varphi. \quad (41)$$

Before we give an explicit form of the curvature tensor $\mathcal{R}$ defined in (26), we compute:

$$\nabla_V \nabla_W \varphi = \nabla_V \left(-\frac{\lambda}{d-1} (W - < W, \zeta > \zeta) \cdot \varphi \right)$$

$$= -\frac{\lambda}{d-1} \nabla_V W \cdot \varphi + \frac{\lambda}{d-1} \nabla_V < W, \zeta > \zeta \cdot \varphi + \frac{\lambda}{d-1} < W, \zeta > \nabla_V \zeta \cdot \varphi$$

$$+ \frac{\lambda^2}{(d-1)^2} (W - < W, \zeta > \zeta)(V - < V, \zeta > \zeta) \cdot \varphi \quad (42)$$

Also, $\nabla_W \nabla_V \varphi$ can be calculated in the same way. Now, we can compute the explicit form of $\mathcal{R}$ as follows:

$$\mathcal{R}_{V,W}\varphi = \nabla_{[V,W]}\varphi - [\nabla_V, \nabla_W]\varphi$$

$$= \frac{\lambda}{d-1} (< \nabla_V W, \zeta > - \nabla_V < W, \zeta >) \zeta \cdot \varphi - \frac{\lambda}{d-1} (< \nabla_W V, \zeta > - \nabla_W < V, \zeta >) \zeta \cdot \varphi$$

$$- \frac{\lambda}{d-1} < W, \zeta > \nabla_V \zeta \cdot \varphi + \frac{\lambda}{d-1} < V, \zeta > \nabla_W \zeta \cdot \varphi + \frac{\lambda^2}{(d-1)^2} ((V - < V, \zeta > \zeta)$$

$$(W - < W, \zeta > \zeta) - (W - < W, \zeta > \zeta)(V - < V, \zeta) \zeta >) \cdot \varphi$$

$$= \frac{\lambda}{d-1} (< V, \nabla_W \zeta > \zeta - < W, \nabla_V \zeta > \zeta + < V, \zeta > \nabla_W \zeta - < W, \zeta > \nabla_V \zeta) \cdot \varphi$$

$$+ \frac{\lambda^2}{(d-1)^2} ((V - < V, \zeta > \zeta)(W - < W, \zeta > \zeta) - (W - < W, \zeta) \zeta > (V - < V, \zeta > \zeta)) \cdot \varphi \quad (43)$$

Taking $W = e_j$ and $V = \zeta$. Then performing Cliffor multiplication with e_j, we get

$$\sum_{j=1}^{d} e_j \cdot \mathcal{R}_{\zeta,e_j}\varphi = \frac{1}{2}Ric(\zeta)\cdot\varphi - \frac{i}{2}(\zeta \lrcorner\Omega)\cdot\varphi$$

$$= \frac{\lambda}{d-1}\sum_{j=1}^{d}\Big(<\zeta,\nabla_{e_j}\zeta> e_j\cdot\zeta - <e_j,\nabla_\zeta\zeta> e_j\cdot\zeta + <\zeta,\zeta> e_j\cdot \nabla_{e_j}\zeta - <e_j,\zeta> e_j\cdot\nabla_\zeta\zeta\Big)\cdot\varphi$$

$$= \frac{\lambda}{d-1}\sum_{j=1}^{d}(<\zeta,\nabla_{e_j}\zeta> e_j\cdot\zeta - <e_j,\nabla_\zeta\zeta> e_j\cdot\zeta + e_j\cdot\nabla_{e_j}\zeta - \zeta\cdot\nabla_\zeta\zeta)\cdot\varphi \quad (44)$$

The harmonicity of the vector field ζ means that $<\nabla_V\zeta, W> - <\nabla_W\zeta, V> = 0$ for all $V, Y \in \chi(M)$. In case of $V = \zeta$, one can easilW show that $<\nabla_\zeta\zeta, W> = 0$ which implies that $\nabla_\zeta\zeta = 0$. Accordingly, (44) is vanished. This means that

$$\sum_{j=1}^{n} e_j \cdot \mathcal{R}_{\zeta,e_j}\varphi = \frac{1}{2}Ric(\zeta)\cdot\varphi - \frac{i}{2}(\zeta \lrcorner\Omega)\cdot\varphi = 0. \quad (45)$$

Considering scalar product of (41) with φ. After seperating real and imaginary parts of this scalar product, we obtain $Ric(\zeta) = 0$ and $(\zeta \lrcorner\Omega) = 0$. Accordingly, ζ is parallel, i.e., $\nabla\zeta = 0$ (Lawson et al., 1989).

CONCLUSION

In this paper, ζ is using to give an optimal estimates for the eigenvalue of the *Spin* ^{c} Dirac operator.

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