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FIRST ORDER MAXIMAL DISSIPATIVE SINGULAR DIFFERENTIAL OPERATORS

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Abstract. In this paper, using the Calkin-Gorbachuk method, the general form of all maximal dissipative extensions of the minimal operator generated by first order linear multipoint symmetric singular differential-operator expression in the direct sum of Hilbert space of vector-functions has been found. Later on, the structure of spectrum of these extensions is researched. Finally, the results are supported by an application.

1. Introduction

Operator theory is important to understand the nature of the spectral properties of an operator associated with a boundary value problem acting on a Hilbert space. To obtain such an information as is well known that the corresponding inner product is useful. A linear closed densely defined operator $T : D(T) \subset X \to X$ in a Hilbert space X is called to be dissipative if and only if

$$Im(T\psi, \psi)_X \geq 0, \ \psi \in D(T),$$

where $Im(\ \cdot\ ,\ \cdot\ )$ and $D(T)$ denote the imaginary part of the inner product and the domain of the operator T, respectively (see [3]). If a dissipative operator has no any proper dissipative extension, then it is called maximal dissipative [3]. A direct result on dissipative operators is that their spectrum lies in the closed upper half-plane. Therefore, open lower half-plane does not belong to the spectrum of T. Maximal dissipative operators play a very important role in mathematics and physics. In physics, there are many interesting applications of the dissipative operators in areas like hydrodynamic, laser and nuclear scattering theories.

Remember that the general theory of self-adjoint extensions of linear densely-defined closed symmetric operators in any Hilbert space was mentioned in the well-known work of Neumann [9]. The complete informations of Vishik's and Birman's

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investigations on the all non-negative selfadjoint extensions of a positive closed symmetric operator have been given by Fischbacher in [2].

The functional model theory of Nagy and Foias [6] is a basic method for investigation the spectral properties of dissipative operators. The maximal dissipative extensions and their spectral analysis of the minimal operator having equal deficiency indices generated by formally symmetric differential-operator expression in one finite or infinite interval case in the Hilbert space of vector-functions have been researched by Gorbachuk [3]. This method has been generalized in terms of boundary values by Rofe-Beketov, Kholkin in [8].

In the present study, in Section 3, using the Calkin-Gorbachuk method, the representation of all maximal dissipative extensions of the minimal operator generated by the first order linear symmetric differential-operator expression in the direct sum of Hilbert spaces of vector-functions in two infinite interval case is obtained. Later on, in Section 4, we also investigate the structure of spectrum of these dissipative extensions.

2. Statement of the problem

Let X be a separable Hilbert space and $a_1, a_2 \in \mathbb{R}$ such that $a_1 < a_2$. In the Hilbert spaces

$$\mathcal{X} = L^2(X, (-\infty, a_1)) \oplus L^2(X, (a_2, \infty))$$

of vector-functions on $(-\infty, a_1) \cup (a_2, \infty)$, consider the following linear multipoint differential operator expression for first order of the form

$$l(\nu) = (l_1(\nu_1), l_2(\nu_2)),\ \nu = (\nu_1, \nu_2),$$

where

$$l_1(\nu_1) = i\nu_1' + \Omega_1 \nu_1,$$
$$l_2(\nu_2) = i\nu_2' + \Omega_2 \nu_2,$$

where $\Omega_m : D(\Omega_m) \subset X \to X,\ m = 1, 2$ are linear selfadjoint operators.

The minimal Υ_0^1 and Υ_0^2 operators corresponding to differential operator expression $l_1(\,\cdot\,)$ and $l_2(\,\cdot\,)$ in $L^2(X, (-\infty, a_1))$ and $L^2(X, (a_2, \infty))$ can be constructed by using the same technique in [4], respectively. The operators $\Upsilon^1 = (\Upsilon_0^1)^*,\ \Upsilon^2 = (\Upsilon_0^2)^*$ are maximal operators corresponding to $l_1(\,\cdot\,)$ and $l_2(\,\cdot\,)$ in $L^2(X, (-\infty, a_1))$ and $L^2(X, (a_2, \infty))$, respectively. In this case, the operators

$$\Upsilon_0 = \Upsilon_0^1 \oplus \Upsilon_0^2 \text{ and } \Upsilon = \Upsilon^1 \oplus \Upsilon^2$$

in the Hilbert space $\mathcal{X}$ are called minimal and maximal operators corresponding to differential operator expression $l(\,\cdot\,)$, respectively.

We have that the domains of the operators Υ and Υ_0 are of the form

$$D(\Upsilon) = \{\nu \in \mathcal{X} : l(\nu) \in \mathcal{X}\},$$
$$D(\Upsilon_0) = \{\nu \in D(\Upsilon) : \nu_1(a_1) = \nu_2(a_2) = 0\}.$$

Our aim in this paper is to obtain all maximal dissipative extensions of the minimal operator Υ_0 in $\mathcal{X}$ in terms of boundary values and investigate the spectrum of them. Then, we give an application of obtained results to the concrete model.

3. Representation of maximal dissipative extensions

In this section, we will study the abstract representation of all maximal dissipative extensions of Υ_0 in terms of boundary values using the Calkin-Gorbachuk method.

Firstly, let us define the deficiency indices of any symmetric operator in a Hilbert space.

Definition 1. [7] *Let T be a symmetric operator, λ be an arbitrary non-real number and $\mathfrak{X}$ be a Hilbert space. We denote by $\mathcal{R}_{\overline{\lambda}}$ and $\mathcal{R}_\lambda$ the ranges of the operator $\left(T-\overline{\lambda}I\right)$ and $(T-\lambda I)$, respectively, where I is identity operator on $\mathfrak{X}$. Clearly, $\mathcal{R}_{\overline{\lambda}}$ and $\mathcal{R}_\lambda$ are subspaces of $\mathfrak{X}$, which need not necessarily be closed. We call $\left(\mathfrak{X}-\mathcal{R}_{\overline{\lambda}}\right)$ and $(\mathfrak{X}-\mathcal{R}_\lambda)$, which are their orthogonal complements, the deficiency spaces of the operator T and we denote them by $\mathcal{N}_{\overline{\lambda}}$ and $\mathcal{N}_\lambda$, respectively: thus*

$$\mathcal{N}_{\overline{\lambda}}=\mathfrak{X}-\mathcal{R}_{\overline{\lambda}},\quad \mathcal{N}_\lambda=\mathfrak{X}-\mathcal{R}_\lambda.$$

The numbers

$$n_{\overline{\lambda}}=dim\mathcal{N}_{\overline{\lambda}},\quad n_\lambda=dim\mathcal{N}_\lambda$$

are called deficiency indices of the operator T.

Let us prove the following auxiliary result we will need:

Lemma 2. *The deficiency indices of Υ_0 are of the form*

$$(n_+(\Upsilon_0),n_-(\Upsilon_0))=(dimX,dimX).$$

Proof. Here, without loss generality it will be assumed that $\Omega_1=\Omega_2=0$. The general solution of the differential equations can be given as follows:

$$i\nu_{1\pm}'(\xi)=\mp i\nu_{1\pm}(\xi),\ \xi<a_1,$$
$$i\nu_{2\pm}'(\xi)=\mp i\nu_{2\pm}(\xi),\ \xi>a_2$$

where

$$\begin{aligned}\nu_{1\pm}(\xi)&=exp\left(\mp(\xi-a_1)\right)\kappa_1,\ \kappa_1\in X,\ \xi<a_1,\\ \nu_{2\pm}(\xi)&=exp\left(\mp(\xi-a_2)\right)\kappa_2,\ \kappa_2\in X,\ \xi>a_2,\end{aligned}$$

respectively. Hence, we have

$$\begin{aligned}n_+(\Upsilon_0^1)&=dimKer(\Upsilon^1+iI)=0,\\ n_-(\Upsilon_0^1)&=dimKer(\Upsilon^1-iI)=dimX,\\ n_+(\Upsilon_0^2)&=dimKer(\Upsilon^2+iI)=dimX,\\ n_-(\Upsilon_0^2)&=dimKer(\Upsilon^2-iI)=0,\end{aligned}$$

where I is identity operator in the corresponding space. Therefore, we get

$$\begin{aligned} n_+(\Upsilon_0) &= n_+(\Upsilon_0^1) + n_+(\Upsilon_0^2) = dimX, \\ n_-(\Upsilon_0) &= n_-(\Upsilon_0^1) + n_-(\Upsilon_0^2) = dimX. \end{aligned}$$

□

Consequently, the operator Υ_0 has a maximal dissipative extension (see [3]). In order to describe all maximal dissipative extensions of Υ_0, it is necessary to construct a space of boundary values for it.

Definition 3. [3] *Let $\mathfrak{X}$ be any Hilbert space and $S : D(S) \subset \mathfrak{X} \to \mathfrak{X}$ be a closed densely defined symmetric operator on the Hilbert space having equal finite or infinite deficiency indices. A triplet $(\mathbf{X}, \beta_1, \beta_2)$, where $\mathbf{X}$ is a Hilbert space, β_1 and β_2 are linear mappings from $D(S^*)$ into $\mathbf{X}$, is called a space of boundary values for the operator S, if for any $\eta, \kappa \in D(S^*)$*

$$(S^*\eta, \kappa)_{\mathfrak{X}} - (\eta, S^*\kappa)_{\mathfrak{X}} = (\beta_1(\eta), \beta_2(\kappa))_{\mathbf{X}} - (\beta_2(\eta), \beta_1(\kappa))_{\mathbf{X}}$$

while for any $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{X}$, there exists an element $\eta \in D(S^*)$ such that $\beta_1(\eta) = \mathcal{G}_1$ and $\beta_2(\eta) = \mathcal{G}_2$.

Lemma 4. *The triplet (X, β_1, β_2), where*

$$\begin{aligned} \beta_1 : D(\Upsilon) \to X,\ \beta_1(\nu) &= \frac{1}{\sqrt{2}}\left(\nu_1(a_1) - \nu_2(a_2)\right) \quad \textit{and} \\ \beta_2 : D(\Upsilon) \to X,\ \beta_2(\nu) &= \frac{1}{i\sqrt{2}}\left(\nu_1(a_1) + \nu_2(a_2)\right),\ \nu = (\nu_1, \nu_2) \in D(\Upsilon) \end{aligned}$$

is a space of boundary values of the minimal operator Υ_0 in $\mathcal{X}$.

Proof. For any $\nu = (\nu_1, \nu_2), \vartheta = (\vartheta_1, \vartheta_2)$ from $D(\Upsilon)$, one can easily check that

$$\begin{aligned} (\Upsilon\nu, \vartheta)_{\mathcal{X}} - (\nu, \Upsilon\vartheta)_{\mathcal{X}} &= (\Upsilon^1\nu_1, \vartheta_1)_{L^2(X,(-\infty,a_1))} + (\Upsilon^2\nu_2, \vartheta_2)_{L^2(X,(a_2,\infty))} \\ &- (\nu_1, \Upsilon^1\vartheta_1)_{L^2(X,(-\infty,a_1))} - (\nu_2, \Upsilon^2\vartheta_2)_{L^2(X,(a_2,\infty))} \\ &= \left[(i\nu_1' + \Omega_1\nu_1, \vartheta_1)_{L^2(X,(-\infty,a_1))} - (\nu_1, i\vartheta_1' + \Omega_1 v_1)_{L^2(X,(-\infty,a_1))}\right] \\ &+ \left[(i\nu_2' + \Omega_2\nu_2, \vartheta_2)_{L^2(X,(a_2,\infty))} - (\nu_2, i\vartheta_2' + \Omega_2\vartheta_2)_{L^2(X,(a_2,\infty))}\right] \\ &= \left[(i\nu_1', \vartheta_1)_{L^2(X,(-\infty,a_1))} - (\nu_1, i\vartheta_1')_{L^2(X,(-\infty,a_1))}\right] \\ &+ \left[(i\nu_2', \vartheta_2)_{L^2(X,(a_2,\infty))} - (\nu_2, i\vartheta_2')_{L^2(X,(a_2,\infty))}\right] \\ &= i\left[(\nu_1', \vartheta_1)_{L^2(X,(-\infty,a_1))} + (\nu_1, \vartheta_1')_{L^2(X,(-\infty,a_1))}\right] \\ &+ i\left[(\nu_2', \vartheta_2)_{L^2(X,(a_2,\infty))} + (\nu_2, \vartheta_2')_{L^2(X,(a_2,\infty))}\right] \\ &= i\left[(\nu_1(a_1), \vartheta_1(a_1))_X - (\nu_2(a_2), \vartheta_2(a_2))_X\right] \\ &= (\beta_1(\nu), \beta_2(\vartheta))_X - (\beta_2(\nu), \beta_1(\vartheta))_X. \end{aligned}$$

Now let $f_1,\ f_2 \in X$. Let us find the function $\nu = (\nu_1, \nu_2) \in D(\Upsilon)$ such that

$$\beta_1(\nu) = \frac{1}{\sqrt{2}}\,(\nu_1(a_1) - \nu_2(a_2)) = f_1$$

and

$$\beta_2(\nu) = \frac{1}{i\sqrt{2}}\,(\nu_1(a_1) + \nu_2(a_2)) = f_2.$$

Hence, we can obtain

$$(\nu_1)(a_1) = (if_2 + f_1)/\sqrt{2}, \quad (\nu_2)(a_2) = (if_2 - f_1)/\sqrt{2}.$$

If we choose the functions $\nu_1(\,\cdot\,)$ and $\nu_2(\,\cdot\,)$ as

$$\begin{aligned} \nu_1(\tau) &= e^{\tau - a_1}(if_2 + f_1)/\sqrt{2},\ \tau < a_1 \text{ and} \\ \nu_2(\tau) &= e^{a_2 - \tau}(if_2 - f_1)/\sqrt{2},\ \tau > a_2, \end{aligned}$$

then we have $\nu = (\nu_1, \nu_2) \in D(\Upsilon)$ and $\beta_1(\nu) = f_1,\ \beta_2(\nu) = f_2$. □

With the use of the Calkin-Gorbachuk method [3], we obtain the following:

Theorem 5. *If $\widetilde{\Upsilon}$ is a maximal dissipative extension of Υ_0 in $\mathcal{X}$, then it is generated by the differential operator expression $l(\,\cdot\,)$ and the boundary condition*

$$\nu_2(a_2) = K\nu_1(a_1),$$

where $K : X \to X$ is a contraction operator. Moreover, the contraction operator K in X is uniquely determined by the extension $\widetilde{\Upsilon}$, i.e. $\widetilde{\Upsilon} = \Upsilon_K$, and vice versa.

Proof. Each maximal dissipative extension $\widetilde{\Upsilon}$ of Υ_0 is described by the differential operator expression $l(\,\cdot\,)$ with the boundary condition

$$(C - I)\beta_1(\nu) + i(C + I)\beta_2(\nu) = 0,\ \nu \in D(\Upsilon),$$

where $C : X \to X$ is a contraction operator and I is identity operator in corresponding space. Therefore, from Lemma 4, we obtain

$$(C - E)\,(\nu_1(a_1) - \nu_2(a_2)) + (C + E)\,(\nu_1(a_1) + \nu_2(a_2)) = 0,\ \nu = (\nu_1, \nu_2) \in D(\widetilde{\Upsilon}).$$

Hence it is obtained that

$$\nu_2(a_2) = -C\nu_1(a_1).$$

Choosing $K = -C$ in the last boundary condition we have

$$\nu_2(a_2) = K\nu_1(a_1).$$

□

4. The spectrum of the maximal dissipative extensions

In this section, we will investigate the structure of the spectrum of the maximal dissipative extensions Υ_K of the minimal operator Υ_0 in $\mathcal{X}$.

Theorem 6. *The point spectrum $\sigma_p(\Upsilon_K)$ of any maximal dissipative extension Υ_K is of the form:*
(1) If $KerK \neq \{0\}$, then $\sigma_p(\Upsilon_K) \supset H_+$, where $H_+ = \{\lambda \in \mathbb{C} : Im\lambda > 0\}$;
(2) If $KerK = \{0\}$, then $\sigma_p(\Upsilon_K) = \emptyset$.

Proof. Let us consider the following eigenvalue problem defined by

$$l(\nu) = \lambda\nu,\ \lambda = \lambda_r + i\lambda_i,\ \nu \in \mathcal{X}, \lambda \in H_+,$$

with the boundary condition

$$\nu_2(a_2) = K\nu_1(a_1).$$

Then, we have

$$\begin{aligned} &\nu_1'(\xi) = i(\Omega_1 - \lambda)\nu_1,\ \xi < a_1, \\ &\nu_2'(\xi) = i(\Omega_2 - \lambda)\nu_2,\ \xi > a_2, \\ &\nu_2(a_2) = K\nu_1(a_1). \end{aligned}$$

The general solutions of these differential equations are as follows:

$$\begin{aligned} \nu_1(\xi;\lambda) &= exp\left(i(\Omega_1 - \lambda)(\xi - a_1)\right) f_1,\ \xi < a_1, f_1 \in X, \\ \nu_2(\xi;\lambda) &= exp\left(i(\Omega_2 - \lambda)(\xi - a_2)\right) f_2,\ \xi > a_2, f_2 \in X \end{aligned}$$

with the boundary condition

$$\nu_2(a_2;\lambda) = K\nu_1(a_1;\lambda).$$

Moreover, $f_1 = \nu_1(a_1;\lambda),\ f_2 = \nu_2(a_2;\lambda)$.
It is clear that for any $f_1 \in X$, we can write

$$\nu_1(\xi;\lambda) = exp\left(i(\Omega_1 - \lambda_r)(\xi - a_1)\right) exp\left(\lambda_i(\xi - a_1)\right) f_1 \in L^2(X, (-\infty, a_1))$$

and for $f_2 \in X$ such that $f_2 \neq 0$, we get

$$\nu_2(\xi;\lambda) = exp\left(i(\Omega_2 - \lambda_r)(\xi - a_2)\right) exp\left(\lambda_i(\xi - a_2)\right) f_2 \notin L^2(X, (a_2, \infty)).$$

(1) If we choose the function $\nu \in \mathcal{X}$ of the following special form

$$\nu^*(\xi;\lambda) = \left(exp\left(i(\Omega_1 - \lambda_r)(\xi - a_1)\right) exp\left(\lambda_i(\xi - a_1)\right) f, 0\right),\ f \in KerK,$$

then we obtain $\Upsilon_K\nu^*(\xi;\lambda) = \lambda\nu^*(\xi;\lambda)$ and $\nu_2^*(a_2;\lambda) = K\nu_1^*(a_1;\lambda)$, for any $\lambda \in H_+$.
(2) If $KerK = \{0\}$, then from the boundary condition $0 = K\nu_1(a_1;\lambda)$ we have $\nu_1(a_1;\lambda) = f_1 = 0$. Hence, the boundary value problem $\Upsilon_K\nu = \lambda\nu,\ \lambda \in H_+,\ \nu \in \mathcal{X}$ have a zero solution once.

Now, let us consider the eigenvalue problem defined by

$$\Upsilon_K\nu = \lambda\nu,\ \nu \in \mathcal{X},\ \lambda \in \mathbb{R}.$$

Then we have

$$\nu_1'(\xi) = i(\Omega_1 - \lambda)\nu_1,\ \xi < a_1,$$
$$\nu_2'(\xi) = i(\Omega_2 - \lambda)\nu_2,\ \xi > a_2,$$
$$\nu_2(a_2) = K\nu_1(a_1).$$

The general solutions of these differential equations are as follows:

$$\begin{aligned} \nu_1(\xi;\lambda) &= exp\,(i(\Omega_1 - \lambda)(\xi - a_1))\, f_1 \notin L^2(X, (-\infty, a_1)),\ f_1 \in X \\ \nu_2(\xi;\lambda) &= exp\,(i(\Omega_2 - \lambda)(\xi - a_2))\, f_2 \notin L^2(X, (a_2, \infty)),\ f_2 \in X. \end{aligned}$$

Consequently, for $KerK \neq \{0\}$ we have

$$\sigma_p(\Upsilon_K) \supset H_+$$

and for $KerK = \{0\}$ we get

$$\sigma_p(\Upsilon_K) = \emptyset.$$

□

Theorem 7. *The residual spectrum* $\sigma_r(\Upsilon_K)$ *of any maximal dissipative extension* Υ_K *is empty, i.e.*

$$\sigma_r(\Upsilon_K) = \emptyset.$$

Proof. From Theorem 6 we get $\sigma_r(\Upsilon_K) \subset \mathbb{R}$ for $KerK \neq \{0\}$, and $\sigma_r(\Upsilon_K) \subset \mathbb{R} \cap H_+$ for $KerK = \{0\}$. In order to prove this theorem we will investigate the point spectrum of the adjoint operator Υ_K^* of Υ_K in $\mathcal{X}$. Let us consider the eigenvalue problem defined by

$$\Upsilon_K^* \vartheta = \lambda\vartheta,\ \lambda \in \mathbb{R},\ \vartheta = (\vartheta_1, \vartheta_2) \in \mathcal{X}.$$

In this case, we have

$$\begin{aligned} i\vartheta_1'(\xi) + \Omega_1\vartheta_1(\xi) &= \lambda\vartheta_1(\xi),\ \xi < a_1, \\ i\vartheta_2'(\xi) + \Omega_2\vartheta_2(\xi) &= \lambda\vartheta_2(\xi),\ \xi > a_2 \end{aligned}$$

with the boundary condition

$$\vartheta_1(a_1) = K^*\vartheta_2(a_2).$$

Hence, it is obtained

$$\begin{aligned} \vartheta_1(\xi;\lambda) &= exp\,(i(\Omega_1 - \lambda)(\xi - a_1))\, g_1,\ \xi < a_1 \\ \vartheta_2(\xi;\lambda) &= exp\,(i(\Omega_2 - \lambda)(\xi - a_2))\, g_2,\ \xi > a_2,\ g_1, g_2 \in X. \end{aligned}$$

Therefore for any $g_1, g_2 \in X$ and for each $\lambda \in \mathbb{R}$, we get

$$\begin{aligned} \vartheta_1(\,\cdot\,;\lambda) &\notin L^2(X, (-\infty, a_1)), \\ \vartheta_2(\,\cdot\,;\lambda) &\notin L^2(X, (a_2, \infty)). \end{aligned}$$

Now, let us consider the residual spectrum of Υ_K, namely,

$$\Upsilon_K^* \vartheta = \lambda\vartheta,\ \lambda \in \mathbb{C},\ \lambda_i = Im\lambda > 0,\ \vartheta = (\vartheta_1, \vartheta_2) \in \mathcal{X}.$$

We have

$$\begin{aligned}\vartheta_1(\xi;\lambda) &= exp\left((i\Omega_1 - i\lambda_r + \lambda_i)(\xi - a_1)\right) g_1,\ \xi < a_1\\ \vartheta_2(\xi;\lambda) &= exp\left((i\Omega_2 - i\lambda_r + \lambda_i)(\xi - a_2)\right) g_2,\ \xi > a_2.\end{aligned}$$

As a result, we get $\vartheta_1(\ \cdot\ ;\lambda) \in L^2(X,(-\infty,a_1))$ and $\nu_2(\ \cdot\ ;\lambda) \notin L^2(X,(a_2,\infty))$ for any $g_2 = \vartheta_2(a_2) \neq 0$.

The necessary and sufficient condition for $\vartheta_2(\ \cdot\ ;\lambda) \in L^2(X,(a_2,\infty))$ is $g_2 = \vartheta_2(a_2) = 0$. From the boundary condition we get

$$\vartheta_1(a_1) = K^*\vartheta_2(a_2)$$

which implies $\vartheta_1(a_1) = 0$. Then, $Ker(\Upsilon_K^*) = \{0\}$.
Consequently, we have $\lambda \notin \sigma_r(\Upsilon_K)$ for any $\lambda \in \mathbb{C}$ with $Im\lambda > 0$. □

By the general theory of linear closed operators in a Hilbert spaces and Theorem 6-Theorem 7, one can immediately obtain the following:

Theorem 8. *If $KerK \neq \{0\}$, then the continuous spectrum $\sigma_c(\Upsilon_K)$ of any maximal dissipative extension Υ_K in $\mathcal{X}$ coincides with $\mathbb{R}$, i.e.*

$$\sigma_c(\Upsilon_K) = \mathbb{R}.$$

Moreover, $\sigma(\Upsilon_K) = \{\lambda \in \mathbb{C} : Im\lambda \geq 0\}$.

With the use of Theorem 6-Theorem 8, the following result can be obtained.

Corollary 9. *If $KerK \neq \{0\}$, then the point spectrum $\sigma_p(\Upsilon_K)$ of any maximal dissipative extension Υ_K in $\mathcal{X}$ is of the form $\sigma_p(\Upsilon_K) = \{\lambda \in \mathbb{C} : Im\lambda > 0\}$.*

Theorem 10. *If $KerK = \{0\}$, then the spectrum of any maximal dissipative extension Υ_K in $\mathcal{X}$ is of the form*

$$\sigma(\Upsilon_K) = \sigma_c(\Upsilon_K) = \mathbb{R}.$$

Proof. Let us consider the following spectrum problem defined by

$$\Upsilon_K\nu = \lambda\nu + f,\ \lambda \in \mathbb{C},\ Im\lambda = \lambda_i > 0,\ \nu = (\nu_1,\nu_2),\ f = (f_1,f_2) \in \mathcal{X}.$$

Then, we have

$$\begin{aligned}i\nu_1'(\xi) + \Omega_1\nu_1(\xi) &= \lambda\nu_1(\xi) + f_1(\xi),\ \xi < a_1,\\ i\nu_2'(\xi) + \Omega_2\nu_2(\xi) &= \lambda\nu_2(\xi) + f_2(\xi),\ \xi > a_2,\\ \nu_2(a_2) &= K\nu_1(a_1).\end{aligned}$$

Hence, the general solutions of the following differential equations

$$\begin{aligned}\nu_1'(\xi) &= i(\Omega_1 - \lambda E)\nu_1(\xi) - if_1(\xi),\ \xi < a_1,\\ \nu_2'(\xi) &= i(\Omega_2 - \lambda E)\nu_2(\xi) - if_2(\xi),\ \xi > a_2\end{aligned}$$

are of the forms

$$\nu_1(\xi;\lambda) = exp\left(i(\Omega_1 - \lambda E)(\xi - a_1)\right) f_\lambda + i\int_\xi^{a_1} exp\left(i(\Omega_1 - \lambda E)(\xi - \tau)\right) f_1(\tau)d\tau,$$

$$\xi < a_1,\ f_\lambda \in X,$$

$$\nu_2(\xi;\lambda) = i\int_\xi^{\infty} exp\left(i(\Omega_2 - \lambda E)(\xi - \tau)\right) f_2(\tau)d\tau,\ \xi > a_2.$$

Additionally, from the boundary condition we have

$$\int_{a_2}^{\infty} exp\left(i(\Omega_2 - \lambda E)(a_2 - \tau)\right) f_2(\tau)d\tau = Kf_\lambda.$$

Consequently, the solution of above considered spectrum problem can be expressed by

$$\begin{aligned} \nu_1(\xi;\lambda) &= exp\left(i(\Omega_1 - \lambda E)(\xi - a_1)\right)\left(K^{-1}\int_{a_2}^{\infty} exp\left(i(\Omega_2 - \lambda E)(a_2 - \tau)\right) f_2(\tau)d\tau\right) \\ &\quad + i\int_\xi^{a_1} exp\left(i(\Omega_1 - \lambda E)(\xi - \tau)\right) f_1(\tau)d\tau,\ \xi < a_1, \\ \nu_2(\xi;\lambda) &= i\int_\xi^{\infty} exp\left(i(\Omega_2 - \lambda E)(\xi - \tau)\right) f_2(\tau)d\tau,\ \xi > a_2 \end{aligned}$$

in the spaces $L^2(X,(-\infty,a_1))$ and $L^2(X,(a_2,\infty))$, respectively.
As a result, we have $H_+ \subset \rho(\Upsilon_K)$. Since for $\lambda \in \mathbb{R}$ the problem

$$\Upsilon_K \nu = \lambda\nu,\ \nu \in \mathcal{X}$$

has zero solution once, $\sigma_p(\Upsilon_K) = \emptyset$ in case that $KerK = \{0\}$.

For $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0$ and $f = (f_1, f_2) \in \mathcal{X}$ the resolvent operator $R_\lambda(\Upsilon_K)$ in $\mathcal{X}$ can be written in the form

$$\|R_\lambda(\Upsilon_K))f(\xi)\|^2_{\mathcal{X}} \geq \|i\int_\xi^{\infty} \exp\left(i(\Omega_2 - \lambda E)(\xi - \tau)\right) f_2(\tau)d\tau\|^2_{L^2(X,(a_2,\infty))}.$$

The vector functions $f^*(\xi;\lambda)$ have the form $f^*(\xi,\lambda) = \left(0, exp\left(i(\Omega_2 - \lambda E)\xi\right) f\right)$, $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0,\ f \in X$ belong to $\mathcal{X}$. Indeed,

$$\|f^*(\xi,\lambda)\|^2_{L^2(X,(a_2,\infty))} = \int_{a_2}^{\infty} \|exp\left(i(\Omega_2 - \lambda E)\xi\right) f\|^2_X d\xi$$

$$= \int_{a_2}^{\infty} exp\,(-2\lambda_i\xi)\,d\xi\|f\|_X^2$$
$$= \frac{1}{2\lambda_i} exp\,(-2\lambda_i a_2)\,\|f\|_X^2 < \infty.$$

For the such functions $f^*(\lambda;\ \cdot\ )$, we have

$$\begin{aligned}
\|R_\lambda(\Upsilon_K)f^*(\ \cdot\ ;\lambda)\|_{\mathcal{X}}^2 &\geq \|i\int_{\xi}^{\infty} exp\,(i(\Omega_2-\lambda)(\xi-\tau)-i(\lambda-\Omega_2)\tau)\,fd\tau\|_{L^2(X,(a_2,\infty))}^2 \\
&= \|\int_{\xi}^{\infty} exp\,(-i\lambda\xi)\,exp\,(-2\lambda_i\tau)\,exp\,(i\Omega_2\xi)\,fd\tau\|_{L^2(X,(a_2,\infty))}^2 \\
&= \|exp\,(-i\lambda\xi)\,exp\,(i\Omega_2\xi)\int_{\xi}^{\infty} exp\,(-2\lambda_i\tau)\,fd\tau\|_{L^2(X,(a_2,\infty))}^2 \\
&= \|exp\,(-i\lambda\xi)\int_{\xi}^{\infty} exp\,(-2\lambda_i\tau)\,d\tau\|_{L^2(X,(a_2,\infty))}^2\|f\|_X^2 \\
&= \frac{1}{4\lambda_i^2}\int_{a_2}^{\infty} exp\,(-2\lambda_i\tau)\,d\tau\|f\|_X^2 \\
&= \frac{1}{8\lambda_i^3} exp\,(-2\lambda_i a_2)\,\|f\|_X^2.
\end{aligned}$$

Using the above inequality we get

$$\|R_\lambda(\Upsilon_K)f^*(\ \cdot\ ;\lambda)\|_{\mathcal{X}} \geq \frac{exp\,(\lambda_i a_2)}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}}\|f\|_X^2 = \frac{1}{2\lambda_i}\|f^*(\xi;\lambda)\|_{L^2(X,(a_2,\infty))},$$

i.e., for $\lambda_i = Im\lambda > 0$ and $f \neq 0$ we can write

$$\frac{\|R_\lambda(\Upsilon_K)f^*(\lambda;\ \cdot\ )\|_{\mathcal{X}}}{\|f^*(\lambda;\xi)\|_{\mathcal{X}}} \geq \frac{1}{2\lambda_i}$$

and it is also obvious that

$$\|R_\lambda(\Upsilon_K)\| \geq \frac{\|R_\lambda(\Upsilon_K)f^*(\ \cdot\ ;\lambda)\|_{\mathcal{X}}}{\|f^*(\xi;\lambda)\|_{\mathcal{X}}},\ f \neq 0.$$

As a consequence, we get

$$\|R_\lambda(\Upsilon_K)\| \geq \frac{1}{2\lambda_i} \text{ for } \lambda \in \mathbb{C},\ \lambda_i = Im\lambda > 0,$$

which shows that every $\lambda \in \mathbb{R}$ belongs to the continuous spectrum of the extension Υ_K. □

5. Examples

Example 11. *Let us consider the following linear multipoint differential operator expression for first order of the form*

$$l((\nu,\vartheta)) = \left(i\nu'(\tau,\varsigma) + \varsigma\nu(\tau,\varsigma), i\vartheta'(\tau,\varsigma) + \varsigma\vartheta(\tau,\varsigma)\right)$$

in the Hilbert space

$$\mathcal{X} = L^2\left((-\infty,-1)\times\mathbb{R}\right)\oplus L^2\left((1,\infty)\times\mathbb{R}\right).$$

Let $\widetilde{L}$ be a maximal dissipative extension of the minimal operator generated by above differential expression. Then, $\widetilde{L}$ is generated by the differential operator expression $l(\,\cdot\,)$ and the following boundary condition

$$\vartheta(1,\varsigma) = \nu(-1,\varsigma)$$

in $\mathcal{X}$.

By Corollary 9, Theorem 8 and Theorem 7, the point, continuous and residual spectrum of the maximal dissipative extension $\widetilde{L}$ in $\mathcal{X}$ are of the forms

$$\sigma_p\left(\widetilde{L}\right) = \{\lambda\in\mathbb{C} : Im\lambda > 0\},$$
$$\sigma_c\left(\widetilde{L}\right) = \mathbb{R},$$
$$\sigma_r\left(\widetilde{L}\right) = \emptyset,$$

respectively.

Consequently, the spectrum of the maximal dissipative extension $\widetilde{L}$ in $\mathcal{X}$ is of the form

$$\sigma\left(\widetilde{L}\right) = \{\lambda\in\mathbb{C} : Im\lambda \geq 0\}.$$

Remark 12. *In special case the representation of selfadjoint extensions of corresponding mentioned above minimal operator and their spectral analysis have been surveyed in* [1] *and* [5].

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