Available online: May 24, 2020

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 969–980 (2020)
DOI: 10.31801/cfsuasmas.671237
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr

APPROXIMATION BY SAMPLING TYPE DISCRETE OPERATORS

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Abstract. In this paper, we deal with discrete operators of sampling type. It is known that this type of operators are related to generalized sampling series and they have important applications. In this work, using bounded and uniformly continuous functions we get general estimations under usual supremum norm with the help of summability method. We also study the degree of approximation with respect to suitable Lipschitz class of continuous functions. Finally, we give specific kernels which verify our kernel assumptions.

1. Introduction

Sampling type discrete operators have significant applications in speech processing, medicine, economic forecasting, geophysics and etc. (see [2, 11, 12, 13, 14, 15, 16, 25]). In this paper, we mainly inspired from the paper [1], where Angeloni and Vinti had some convergence results using discrete operators. The authors utilized from convergence in φ-variation to get some convergence results in that work. Now, our aim is to get some approximations under usual supremum norm by generalizing them using Bell-type summability method. In this process, we use bounded and uniformly continuous functions on $\mathbb{R}$. Furthermore, we study the rate of approximation for our main theorem using suitable Lipschitz class. Then, taking some appropriate kernels we also get more general case of generalized sampling series. Finally, we illustrate the kernels $l_{k,w}$ which satisfy our kernel assumptions.

Some notations and definitions are given below.

- $\|\cdot\|_{l^1}$ denotes the l^1 norm, i.e., for a given $u_k : \mathbb{Z} \to \mathbb{R}$, $\|u_k\|_{l^1} = \sum_{k\in\mathbb{Z}} |u_k|$.
- By $\|\cdot\|$, we mean the usual supremum norm on $\mathbb{R}$.
- The space of bounded and uniformly continuous functions on $\mathbb{R}$ is shown by $BUC(\mathbb{R})$.
- Let $\mathcal{A} = \{A^v\}_{v\in\mathbb{N}} = \{[a^v_{nw}]\}_{v\in\mathbb{N}}$ $(n, w \in \mathbb{N})$ be a family of infinite matrices of real or complex numbers. Then, for a given sequence $x = (x_k)$ the

Received by the editors: January 06, 2020; Accepted: May 03, 2020.

2010 *Mathematics Subject Classification.* 41A25, 41A35, 40A25, 40C05.

Key words and phrases. Sampling type operators, rate of approximation, summability process.

following double sequence $(\mathcal{A}x)_n^{\upsilon}$

$$(\mathcal{A}x)_n^{\upsilon} := \left\{ \sum_{w=1}^{\infty} a_{nw}^{\upsilon} x_w \right\} \quad (n, \upsilon \in \mathbb{N})$$

is called by $\mathcal{A}$-transform of x, if the series is convergent for all $n, \upsilon \in \mathbb{N}$. Moreover, if

$$\lim_{n\to\infty} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} x_w = L \text{ uniformly in } \upsilon$$

holds, we call "x is $\mathcal{A}$-summable to L" and denote by

$$\mathcal{A} - \lim x = L$$

(see [9]).

- $\mathcal{A}$ is called regular if for any $\lim_k x_k = L$ implies that $\mathcal{A}-\lim x = L$ ([9, 10]).
- A characterization for the regularity of the given method $\mathcal{A}$ is found by Bell in [10] such that

$$\mathcal{A} \text{ is regular } \Leftrightarrow \quad \begin{array}{l} \cdot \text{ for each } w \in \mathbb{N},\ \lim_{n\to\infty} a_{nw}^{\upsilon} = 0 \text{ (uniformly in } \upsilon), \\ \cdot\ \lim_{n\to\infty} \sum\limits_{w=1}^{\infty} a_{nw}^{\upsilon} = 1 \text{ (uniformly in } \upsilon), \\ \cdot \text{ for all } n, \upsilon \in \mathbb{N},\ \sum\limits_{w=1}^{\infty} |a_{nw}^{\upsilon}| < \infty \text{ and there exist integers} \\ \quad N \text{ and } M \text{ such that } \sup\limits_{n\geq N, \upsilon\in\mathbb{N}} \sum\limits_{w=1}^{\infty} |a_{nw}^{\upsilon}| \leq M. \end{array}$$

- Throughout the paper, we will assume that $\mathcal{A}$ is regular with nonnegative real entries.

We should note that Bell-type summability method consists many well-known methods such as Cesàro summability [18], almost convergence [23], order summability [19, 20] and etc. It also allows us to increase the speed of convergence [21, 27, 29]. Some applications of Bell-type summability method are given in [3, 4, 5, 6, 7, 8, 17, 22, 24, 28].

Now, we can define our operator as follows:

$$\mathcal{T}_{n,\upsilon}(f; x) = \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} f\left(x - \frac{k}{w}\right) l_{k,w} \quad (x\in\mathbb{R},\ n, \upsilon \in \mathbb{N}) \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is bounded and $l_{k,w} \in l^1(\mathbb{Z})$ is a family of discrete kernels for all $w \in \mathbb{N}$.

Our aim is to prove the following general convergence result

$$\lim_{n\to\infty} \|\mathcal{T}_{n,\upsilon}(f) - f\| = 0 \text{ (uniformly in } \upsilon \in \mathbb{N})$$

for all $f \in BUC(\mathbb{R})$. It is not hard to see that operator (1.1) coincides with the following operator

$$T_w(f; x) = \sum_{k\in\mathbb{Z}} f\left(x - \frac{k}{w}\right) l_{k,w}$$

when $\mathcal{A} = \{A^{\upsilon}\} = \{I\}$ (identity matrix). Furthermore, we will indicate that operator (1.1) contains the $\mathcal{A}$-transform of generalized sampling series, defined by

$$\mathcal{S}_{n,\upsilon}(f;x) = \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} f\left(\tfrac{k}{w}\right)\chi(wx-k) \quad (x\in\mathbb{R},\ n,\upsilon\in\mathbb{N}) \tag{1.2}$$

where $f, \chi : \mathbb{R} \to \mathbb{R}$ and generalized sampling series

$$S_w(f;x) = \sum_{k\in\mathbb{Z}} f\left(\tfrac{k}{w}\right)\chi(wx-k)$$

is a special case of (1.2).

2. Approximation in Usual Supremum Norm

In this section, we will prove our main approximation theorem. For this, we need the following conditions on the kernel of the corresponding operator.

(l_1) There exists a constant $A > 0$ such that $\sup\limits_{n,\upsilon\in\mathbb{N}} \sum\limits_{w=1}^{\infty} a_{nw}^{\upsilon} \| l_{k,w}\|_{l^1} = A < \infty$,

(l_2) $\mathcal{A} - \lim \left(\sum\limits_{k\in\mathbb{Z}} l_{k,w}\right) = 1$,

(l_3) there exists $r > 0$ such that $\mathcal{A} - \lim \left(\sum\limits_{|k|\geq r} |l_{k,w}|\right) = 0$.

Here, when $\mathcal{A}$ is taken the identity matrix, conditions $(l_1) - (l_3)$ reduce to the approximate identities given in [1].

The following lemma shows that (1.1) is well defined for all bounded functions.

Lemma 2.1. *If f is bounded on $\mathbb{R}$ and (l_1) holds, then $\|\mathcal{T}_{n,\upsilon}(f)\| < \infty$ for every $n,\upsilon \in \mathbb{N}$. Moreover, if $f \in L^1(\mathbb{R})$, then $\mathcal{T}_{n,\upsilon}(f) \in L^1(\mathbb{R})$.*

Proof. Since f is bounded, there exists a positive number M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Considering this with (l_1), we get

$$\begin{aligned} |\mathcal{T}_{n,\upsilon}(f;x)| &\leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} \left|f\left(x-\tfrac{k}{w}\right)\right| |l_{k,w}| \\ &\leq MA \end{aligned}$$

and having supremum over $x\in\mathbb{R}$, we have

$$\|\mathcal{T}_{n,\upsilon}(f)\| \leq MA < \infty$$

for all $n,\upsilon \in \mathbb{N}$, which shows that $\mathcal{T}_{n,\upsilon}$ maps from the space of bounded functions into itself.

For the second part of the theorem, assume that $f \in L^1(\mathbb{R})$. Then, it is possible to write that

$$\int_{\mathbb{R}} |\mathcal{T}_{n,\upsilon}(f;x)|\,dx \leq \int_{\mathbb{R}} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left|f\left(x-\tfrac{k}{w}\right)\right| dx$$

and from a theorem of integration by series (see [26]),

$$\int_{\mathbb{R}} |\mathcal{T}_{n,\upsilon}(f;x)|\,dx \leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right)\right\|_{L^1}$$

holds for all $n,\upsilon \in \mathbb{N}$. Since $\left\| f\left(\cdot - \frac{k}{w}\right)\right\|_{L^1} = \|f\|_{L^1}$, then

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{T}_{n,\upsilon}(f;x)|\,dx &\leq \|f\|_{L^1} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \\ &\leq A\,\|f\|_{L^1}\end{aligned}$$

is obtained, where $\|f\|_{L^1}$ is the classical L^1 norm, i.e., $\|f\|_{L^1} = \int_{\mathbb{R}} |f(x)|\,dx$. □

Lemma 2.2. *Assume that* (l_1) *holds. If* $f \in BUC(\mathbb{R})$, *then* $\mathcal{T}_{n,\upsilon}(f) \in BUC(\mathbb{R})$ *for all* $n,\upsilon \in \mathbb{N}$.

Proof. By the previous lemma it is clear that if f is bounded, then $\mathcal{T}_{n,\upsilon}(f)$ is too. Now, let $\varepsilon > 0$ be given and let $|x-y| < \delta$ where δ corresponds to given ε and f. Then,

$$|\mathcal{T}_{n,\upsilon}(f;x) - \mathcal{T}_{n,\upsilon}(f;y)| \leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left| f\left(x - \tfrac{k}{w}\right) - f\left(y - \tfrac{k}{w}\right)\right|$$

holds. Since $\left| x - \frac{k}{w} - \left(y - \frac{k}{w}\right)\right| = |x-y| < \delta$, from (l_1)

$$|\mathcal{T}_{n,\upsilon}(f;x) - \mathcal{T}_{n,\upsilon}(f;y)| \leq A\varepsilon$$

for all $n,\upsilon \in \mathbb{N}$. □

The main approximation theorem is given below.

Theorem 2.3. *Assume that* $(l_1) - (l_3)$ *hold. Then, for all* $f \in BUC(\mathbb{R})$ *we have*

$$\lim_{n\to\infty} \|\mathcal{T}_{n,\upsilon}(f) - f\| = 0 \text{ uniformly in } \upsilon.$$

Proof. From triangle inequality, it is possible to write that

$$\begin{aligned}|\mathcal{T}_{n,\upsilon}(f;x) - f(x)| &= \Bigg| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} \left(f\left(x - \tfrac{k}{w}\right) - f(x)\right) \\ &\quad + f(x)\left(\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} - 1\right)\Bigg| \\ &\leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right\| \\ &\quad + \|f\| \left| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| \\ &:= A_1 + A_2\end{aligned}$$

holds. In A_1, we concentrate on the continuity of f. Since f is uniformly continuous, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \tag{2.1}$$

whenever $|x-y| < \delta$. Then, for a fixed $\bar{r}$ it is easy to find a number w_1 satisfying

$$\left|\tfrac{\bar{r}}{w}\right| < \delta$$

for all $w > w_1$.Now, if we divide A_1 as follows

$$\begin{aligned} A_1 = & \sum_{w=1}^{w_1} a_{nw}^{\upsilon} \sum_{|k|<\bar{r}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right\| \\ & + \sum_{w=w_1+1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<\bar{r}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right\| \\ & + \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|\geq\bar{r}} |l_{k,w}| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right\| \\ & := A_1^1 + A_1^2 + A_1^3 \end{aligned}$$

from (2.1) and (l_1)

$$A_1^2 \leq A\varepsilon$$

holds, since $\left|x - \frac{k}{w} - x\right| = \left|\frac{k}{w}\right| < \frac{\bar{r}}{w} < \delta$.

For A_1^1, from the regularity of $\mathcal{A}$, one can find a number $n_1 = n_1(\varepsilon)$ such that

$$A_1^1 < D' w_1 \varepsilon$$

where $D' := \max_{1\leq w\leq w_1} \left\{\sum_{|k|<\bar{r}} |l_{k,w}| \left\| f\left(\cdot - \frac{k}{w}\right) - f(\cdot)\right\|\right\}$. And from (l_3), we see that

$$A_1^3 < 2\|f\|\varepsilon$$

for sufficiently large $n \in \mathbb{N}$.

Finally, it follows from (l_2)

$$A_2 < \|f\|\varepsilon$$

yields for sufficiently large $n \in \mathbb{N}$. Hence, having supremum over $x\in\mathbb{R}$ in the first inequality, we complete the proof. $\square$

3. Rate of Convergence

In this section we investigate the rate of approximation, and therefore we need the following Lipschitz class.

For any given $\alpha > 0$, define $Lip(\alpha)$ as follows:

$$Lip(\alpha) = \{f \in BUC(\mathbb{R}) : \|f(\cdot - t) - f(\cdot)\| = O(|t|^{\alpha}) \text{ as } t \to 0\}$$

where $f(t) = O(g(t))$ as $t \to 0$ means that, there exist $\delta, N > 0$ such that $|f(t)| \leq N|g(t)|$ for $|t| < \delta$. Let Ψ be family of all functions $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, such that $\xi(0) = 0$, $\xi(t) > 0$ for $t > 0$ and ξ be continuous at $t = 0$. Now, for any fixed $\alpha > 0$ and $\xi \in \Psi$, consider the following conditions:

$$\left(\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} - 1\right) = O\left(\xi\left(1/n\right)\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon), \tag{3.1}$$

there exists a constant $r_0 > 0$ such that

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<r_0} \frac{|l_{k,w}|}{w^{\alpha}} = O\left(\xi\left(1/n\right)\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon), \tag{3.2}$$

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|\geq r_0} |l_{k,w}| = O\left(\xi\left(1/n\right)\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon) \tag{3.3}$$

and for a given $\mathcal{A} = \{[a_{nw}^{\upsilon}]\}_{\upsilon\in\mathbb{N}}$

$$\text{for each } w \in \mathbb{N},\ a_{nw}^{\upsilon} = O\left(\xi\left(1/n\right)\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon). \tag{3.4}$$

We obtain the following rates of approximations.

Theorem 3.1. *Suppose that for any fixed $\xi \in \Psi$ and $\alpha > 0$, (3.1)-(3.4) and (l_1) hold. Then, for all $f \in Lip\left(\alpha\right)$*

$$\left\|\mathcal{T}_{n,\upsilon}\left(f\right) - f\right\| = O\left(\xi\left(1/n\right)\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon).$$

Proof. From the proof of Theorem 2.3, we observe that

$$\begin{aligned}
\left\|\mathcal{T}_{n,\upsilon}\left(f\right) - f\right\| &\leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} |l_{k,w}| \left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \\
&+ \|f\| \left|\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| \\
&:= B_1 + B_2
\end{aligned}$$

holds. In B_1 for some fixed $r_0 > 0$, we can find a number w_2 such that for all $w > w_2$, $\left|x - \frac{k}{w} - x\right| = \left|\frac{k}{w}\right| < \frac{r_0}{w} < \delta$ and since $f \in Lip\left(\alpha\right)$, there exists a constant $N > 0$ such that

$$\left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \leq N \left|\tfrac{k}{w}\right|^{\alpha}$$

hold. Then, we get

$$\begin{aligned}
B_1 &= \sum_{w=1}^{w_2} a_{nw}^{\upsilon} \sum_{|k|<r_0} |l_{k,w}| \left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \\
&+ \sum_{w=w_2+1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<r_0} |l_{k,w}| \left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \\
&+ \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|\geq r_0} |l_{k,w}| \left\|f\left(\cdot - \tfrac{k}{w}\right) - f\left(\cdot\right)\right\| \\
&\leq D'' w_2 \max_{1\leq w\leq w_2} a_{nw}^{\upsilon} \\
&+ N \sum_{w=w_2+1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<r_0} |l_{k,w}| \left(\tfrac{r_0}{w}\right)^{\alpha}
\end{aligned}$$

$$+2\|f\|\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{|k|\geq r_0}|l_{k,w}|$$
$$:=B_1^1+B_1^2+B_1^3$$

where $D'':=\max_{1\leq w\leq w_2}\left\{\sum_{|k|<r_0}|l_{k,w}|\left\|f\left(\cdot-\frac{k}{w}\right)-f(\cdot)\right\|\right\}$. From (3.4), (3.2) and (3.3) it is clear that

$$B_1^1,B_1^2,B_1^3=O\left(\xi\left(1/n\right)\right)\text{ as }n\to\infty\text{ (uniformly in }\upsilon),$$

yields.

Finally, from (3.1) we conclude that

$$B_2=O\left(\xi\left(1/n\right)\right)\text{ as }n\to\infty\text{ (uniformly in }\upsilon).$$

□

Notice that, it is possible to find regular methods such that (3.4) is satisfied, for instance, $\{C_1\}$ (Cesàro Matrix) and $\mathcal{F}$ (almost convergence matrix) which are given in Corollary 4.3.

4. Conclusions and Applications

In the present section, we give some applications of the operators of type (1.1).

Let $f:\mathbb{R}\to\mathbb{R}$ be given, and suppose that $l_{k,w}\equiv\chi(k)$, that is, $l_{k,w}$ is not depending on w where $\chi:\mathbb{R}\to\mathbb{R}$. Then, (1.1) reduces to

$$\bar{\mathcal{T}}_{n,\upsilon}(f;x)=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}f\left(x-\frac{k}{w}\right)\chi(k),\ x\in\mathbb{R}$$

which is in some cases equal to $\mathcal{A}-$transform of generalized sampling series, namely

$$\mathcal{S}_{n,\upsilon}(f;x)=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}f\left(\frac{k}{w}\right)\chi(wx-k),\ x\in\mathbb{R}.$$

In this case (l_1) and (l_2) coincide with the following assumptions

(l_1') $\chi\in l^1(\mathbb{Z})$

(l_2') $\sum_{k\in\mathbb{Z}}\chi(k)=1$

where on the other hand, (l_3) is clearly not satisfied. But these two conditions are still enough to verify the following approximations (see also [1]).

Theorem 4.1. *Let $f\in BUC(\mathbb{R})$. If (l_1'), (l_2') hold, then*

$$\lim_{n\to\infty}\left\|\bar{\mathcal{T}}_{n,\upsilon}(f)-f\right\|=0\text{ (uniformly in }\upsilon\in\mathbb{N}).$$

Proof. Considering (l_2'), by the proof of the Theorem 2.3, we obtain the following inequalities

$$\left\|\bar{\mathcal{T}}_{n,\upsilon}(f)-f\right\|\leq\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}|\chi(k)|\left\|f\left(\cdot-\frac{k}{w}\right)-f(\cdot)\right\|+\|f\|\left|\sum_{w=1}^{\infty}a_{nw}^{\upsilon}-1\right|.$$

Since $\sum_{k\in\mathbb{Z}} |\chi(k)| < \infty$ from (l_1'), for all $\varepsilon > 0$ there exists a number $\breve{r} > 0$ such that

$$\sum_{|k|\geq\breve{r}} |\chi(k)| < \varepsilon$$

and hence, for sufficiently large $n \in \mathbb{N}$

$$\begin{aligned}\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|\geq\breve{r}} |\chi(k)| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right\| &< 2\|f\| \sum_{w=1}^{\infty} a_{nw}^{\upsilon}\varepsilon \\ &\leq 2M\|f\|\varepsilon\end{aligned}$$

holds where M comes from the regularity of $\mathcal{A}$. In a similar way with the proof of Theorem 2.3, it is possible to show

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|k|<\breve{r}} |\chi(k)| \left\| f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right\| < \varepsilon\left(\bar{D}\bar{w}_1 + \bar{A}M\right)$$

for sufficiently large $n \in \mathbb{N}$, where $\bar{D} := \max_{1\leq w\leq\bar{w}_1}\left\{\sum_{|k|<\breve{r}} |\chi(k)| \left\| f\left(\cdot - \frac{k}{w}\right) - f(\cdot)\right\|\right\}$. Finally, by the regularity of $\mathcal{A}$

$$\|f\| \left|\sum_{w=1}^{\infty} a_{nw}^{\upsilon} - 1\right| < \|f\|\varepsilon$$

for sufficiently large $n \in \mathbb{N}$. Since ε is arbitrary, the proof is completed. □

Although $\bar{\mathcal{T}}$ and $\mathcal{S}$ are similar, they are different in general. However, in some cases, they coincide (see [1]).

Corollary 4.2. *Let $f \in B_{\pi w}^1(\mathbb{R})$ (the Paley-Wiener Space $B_{\pi w}^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : |f(z)| \leq \exp(\pi w|z|)\|f\|$ for every $z \in \mathbb{C}\}$) for some $w > 0$ and $\chi \in B_{\pi}^{\infty}(\mathbb{R})$. If (l_1') and (l_2') hold, then*

$$\lim_{n\to\infty} \|\mathcal{S}_{n,\upsilon}(f) - f\| = 0 \text{ (uniformly in } \upsilon \in \mathbb{N}).$$

Proof. It is proved in [1] that $B_{\pi w}^1(\mathbb{R}) \subset Lip(\mathbb{R})$, and therefore bounded elements of $B_{\pi w}^1(\mathbb{R})$ are also elements of $BUC(\mathbb{R})$. On the other hand, using similar arguments in Lemma 4.2 in [1] we get

$$\mathcal{S}_{n,\upsilon}(f) = \bar{\mathcal{T}}_{n,\upsilon}(f)$$

for all $n, \upsilon \in \mathbb{N}$ and $f \in B_{\pi w}^1(\mathbb{R})$. Consequently, by the Theorem 4.1, the proof completes. □

Remark 4.1. *It may clearly be seen that, Corollary 4.2 holds for $f \in B_{\pi w}^p(\mathbb{R})$ where $1 \leq p \leq 2$. In this case, we need to assume $\chi \in B_{\pi}^q(\mathbb{R})$ to apply Lemma 4.2 in [1] where $1/p + 1/q = 1$. For some examples of χ which satisfy $\chi \in B_{\pi}^{\infty}(\mathbb{R})$, (l_1') and (l_2'), we refer to Example 4.5 in [1].*

It is clear that operator (1.1) can be written as

$$\mathcal{T}_{n,\upsilon}(f;x)=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}T_w(f;x) \tag{4.1}$$

where T_w is given by

$$T_w(f;x)=\sum_{k\in\mathbb{Z}}f\left(x-\tfrac{k}{w}\right)l_{k,w}\ \ (x\in\mathbb{R},\ w\in\mathbb{N}). \tag{4.2}$$

Considering (4.1) and (4.2), we get the following corollary.

Corollary 4.3. *Taking specific regular matrices, we observe the following estimations:*

- *Assume that $\mathcal{A}=\mathcal{F}=\{F^{\upsilon}\}=\{[a_{nw}^{\upsilon}]\}$ where $a_{nw}^{\upsilon}=1/n$ if $\upsilon\leq w\leq n+\upsilon-1$; $a_{nw}^{\upsilon}=0$ if otherwise. Assume further that $(l_1)-(l_3)$ hold for $\mathcal{A}=\mathcal{F}$ (almost convergence matrix). Then, for all $f\in BUC(\mathbb{R})$,*

$$\lim_{n\to\infty}\left\|\frac{T_{\upsilon}(f)+T_{\upsilon+1}(f)+\cdots+T_{n+\upsilon-1}(f)}{n}-f\right\|=0\quad \textit{(uniformly in } \upsilon)$$

  i.e., $T_n(f)$ is almost convergent to f,
- *Assume that $\mathcal{A}=\{C_1\}=\{[c_{nw}]\}$ where $c_{nw}=1/n$ if $1\leq w\leq n$; $c_{nw}=0$ if otherwise. Assume further that $(l_1)-(l_3)$ hold for $\mathcal{A}=\{C_1\}$ (Cesàro matrix). Then, for all $f\in BUC(\mathbb{R})$,*

$$\lim_{n\to\infty}\left\|\frac{T_1(f)+T_2(f)+\cdots+T_n(f)}{n}-f\right\|=0$$

  i.e., $T_n(f)$ is arithmetic mean convergent to f,
- *Suppose that $\mathcal{A}=\{I\}$ and $(l_1)-(l_3)$ hold. Then, for all $f\in BUC(\mathbb{R})$,*

$$\lim_{n\to\infty}\|T_n(f)-f\|=0$$

  i.e., $T_n(f)$ is uniformly convergent to f, where $T_n(f)$ is given in (4.2).

Similar corollaries also hold for generalized sampling series

$$S_w(f;x)=\sum_{k\in\mathbb{Z}}f\left(\tfrac{k}{w}\right)\chi(wx-k).$$

Now, we will give a specific kernel of $l_{k,w}$, which satisfies $(l_1)-(l_3)$ respectively. Take $\mathcal{A}=\{C_1\}$, and then define $l_{k,w}$ as follows:

$$l_{k,w}=\frac{(-1)^w+1}{2^{w(|k|)}}\left(\frac{2^w-1}{2^w+1}\right).$$

It is easy to see that (l_1) and (l_2) are satisfied from the following calculations:

$$\sup_{n\in\mathbb{N}}\sum_{w=1}^{n}\frac{1}{n}\sum_{k\in\mathbb{Z}}|l_{k,w}|\leq\sup_{n\in\mathbb{N}}\sum_{w=1}^{n}\frac{1}{n}\sum_{k\in\mathbb{Z}}\frac{2}{2^{w|k|}}$$

$$
\begin{aligned}
&= \sup_{n\in\mathbb{N}} \sum_{w=1}^{n} \frac{2}{n}\left(\frac{2^w+1}{2^w-1}\right) \\
&\leq \sup_{n\in\mathbb{N}} \sum_{w=1}^{n} \frac{6}{n} \\
&= 6
\end{aligned}
$$

and since $l_{k,w} > 0$, from the previous statement

$$\lim_{n\to\infty}\left|\sum_{w=1}^{n}\frac{1}{n}\sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| \leq \lim_{n\to\infty}\left|\sum_{w=1}^{n}\frac{(-1)^w}{n}\right| = 0.$$

On the other hand, for (l_3), for any integer $r \geq 1$, we get

$$\sum_{w=1}^{n}\frac{1}{n}\sum_{|k|\geq r}|l_{k,w}| = \sum_{w=1}^{n}\frac{1}{n}\left(\frac{(-1)^w+1}{2^w+1}\right)\frac{2^{w+1}}{2^{wr}}$$

where

$$\lim_{w\to\infty}\left(\frac{(-1)^w+1}{2^w+1}\right)\frac{2^{w+1}}{2^{wr}} = 0. \tag{4.3}$$

Then, since (4.3) is convergent to 0, its arithmetic mean is too, namely,

$$\lim_{n\to\infty}\sum_{w=1}^{n}\frac{1}{n}\left(\frac{(-1)^w+1}{2^w+1}\right)\frac{2^{w+1}}{2^{wr}} = 0,$$

which implies (l_3). For the behaviour of $l_{k,w}$, see Figure 1 ($k = 0, \cdots, 5$ and $w = 1, \cdots, 6$) which is symmetric for k. But in the classical sense, $l_{k,w}$ does not

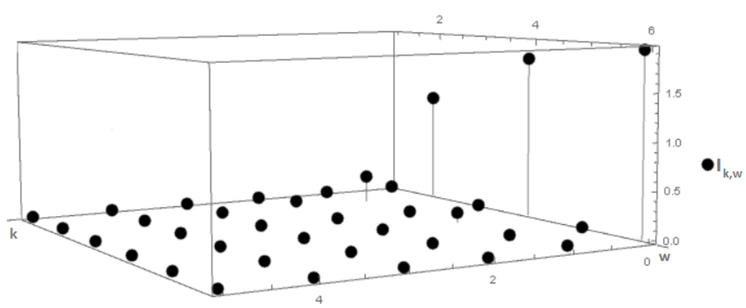

FIGURE 1. The kernel function $l_{k,w}$

satisfy the condition of (**A1**) since

$$\left|\sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| = (-1)^w + 1$$

is divergent. Therefore, our approximation is not trivial.

Acknowledgement. The author would like to thank to the reviewer(s) for reading the manuscript carefully.

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