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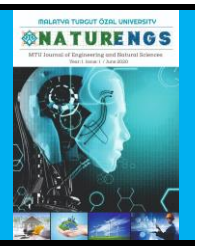

New Analytical Solutions for Space and Time Fractional Phi-4 Equation

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ABSTRACT: In this paper, the largest set in the literature of space, time and space-time conformable fractional Phi-4 equations is found by utilizing an analytical method based upon the Jacobi elliptic functions. These solutions are obtained in a general form including trigonometric, rational, complex and hyperbolic functions. Some problems are presented to illustrate the practical application of the proposed method and some of the solutions are also demonstrated with the two-dimensional and three-dimensional graphics.

Mathematics Subject Classification: 35G20, 35L05, 35R11, 33E05.

Keywords: Jacobi elliptic functions; Space-time fractional Phi-4 equation; Conformable fractional derivative; Analytic method.

1. INTRODUCTION

The Phi-4 equation which is a particular form of Klein-Gordon equation and has used in particle and nuclear physics is defined by

$$u_{tt} - au_{xx} - bu + \lambda u^3 = 0$$

where a, b and λ are constants. This equation has also been the subject of intensive investigation for classical and quantized field theory, and it has kink-like solutions which are not solitons [1]. Besides, the Phi-4 equation has been mostly examined as the easiest nontrivial relativistic invariant field theoretical model [2]. If $\lambda = 0$, the above equation turns into a linear differential equation.

Recently, various methods have been used to solve the integer order Phi-4 equation. The solution methods for this equation are sine-cosine [3], sine-cosine ansantz [4], tanh [4], generalized tanh [5], modified extended tanh-function [6], Weierstrass elliptic function [7], Jacobi-Gauss-Lobatto collocation [8], variational [9], modified simple equation [10,11], homotopy perturbation [12], homotopy analysis [12], Adomian decomposition [12], $(G'/G\,,1/G\,)$-expansion [13], trigonometric B-Spline collocation [14] generalized Kudyashov [15] and improved F-expansion [16].

In recent years, searching solutions of the fractional differential equations have drawn remarkable interest not only for applied mathematicians but also for the other scientists. So far, various solutions of the time fractional Phi-four equation have been founded by utilizing the tanh function [17], modified residual power series [18], modified Kudryashov [19], exponential

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function [19], extended direct algebraic [20], mapping [21], modified mapping [21], q-homotopy analysis transform (q-HATM) [22], and generalized Kudryashov [23] methods. The solutions of space-time conformable fractional Phi-four equation is also gained by using the $(G'/G\,,1/G\,)$–expansion [24] method. Among these methods, exponential function, the modified Kudryashov, extended direct algebraic, mapping, modified mapping and generalized Kudryashov methods contain the fractional derivatives with respect to time in the conformable sense, while the $(G'/G\,,1/G\,)$–expansion method includes the fractional derivatives with respect to space and time in the conformable sense. However, a solution method for space fractional Phi-4 equations is not yet available in the literature. On the other side, although there has not found any method involving the Jacobi elliptic functions for the solutions of the conformable space-time fractional Phi-4 equation, Jacobi elliptic functions have been used to obtain the exact solutions of different conformable fractional partial differential equations [25-34].

In this study, Jacobi elliptic functions have been used to improve an analytic method for the space, time, and space-time conformable fractional Phi-4 equations is presented. Our goal is to find the possible largest set of exact solutions of the space-time conformable fractional Phi-four equation in the form

$$D_t^{\alpha} D_t^{\alpha} u - a D_x^{\beta} D_x^{\beta} u - bu + \lambda u^3 = 0, \qquad 0 < \alpha, \beta \leq 1. \tag{1}$$

Here, D_x^{β} and D_t^{α} stand for the conformable fractional derivative of the unknown function $u(x,t)$ with respect to x and t, respectively.

The rest of the paper has been organized as follows: In the second section, conformable fractional derivative and some properties of Jacobi elliptic functions are presented. The exact solutions of the conformable space-time fractional Phi-four equation are obtained in terms of Jacobi elliptic functions in the third section. In the fourth section, to illustrate the practical application of the proposed method, four problems are presented and some solutions are demonstrated by the 2D and 3D graphics. Finally, the paper has been concluded in the fifth section.

2. PRELIMINARIES

Recently, Khalil et al. [35] has described the conformable fractional derivative. Since it is similar to the definition of the usual derivative this definition of the fractional derivative is the simplest way compared to the other fractional derivatives. Thus, the fractional Phi-4 equation is examined in the conformable sense in this paper. The definition and some of the properties of the mentioned conformable fractional derivative can be given as follows and other properties of it can be seen in [35] and [36].

Definition [35]: Let $f:[0,\infty) \to \mathbb{R}$ be a function. The conformable fractional derivative of α-the order of the function f is described by

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad \alpha \in (0,1), \quad t > 0.$$

If the function f is α-differentiable in conformable sense and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then

$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

Theorem 1 [35]: Let the functions f and g be α-differentiable in conformable sense for $t > 0$ such that $\alpha \in (0,1]$. Then, the following properties are satisfied:

1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g); \quad \forall\, a, b \in \mathbb{R}.$
2) $T_\alpha(t^p) = pt^{p-\alpha}\,; \quad \forall\, p \in \mathbb{R}.$
3) $T_\alpha(\lambda) = 0,$ where λ is a constant.
4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f).$
5) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}.$
6) If f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t).$

Theorem 2 [36]: Assume that the functions $f, g : (0, \infty) \to \mathbb{R}$ be α-differentiable in conformable sense and $0 < \alpha \leq 1$. Suppose that $h(t) = f(g(t))$, then the composite function $h(t)$ is α-differentiable in conformable sense also. For non-zero t and $g(t) \neq 0$, we obtain

$$T_\alpha(h)(t) = T_\alpha(f)\big(g(t)\big).T_\alpha(g)(t).\,g(t)^{\alpha-1}.$$

If $t = 0$, we get

$$T_\alpha(h)(0) = \lim_{t\to 0} T_\alpha(f)\big(g(t)\big).T_\alpha(g)(t).\,g(t)^{\alpha-1}\,.$$

The basic Jacobi elliptic functions are

$$\mathrm{sn}\xi = \mathrm{sn}(\xi|m^2), \quad \mathrm{cn}\xi = \mathrm{cn}(\xi|m^2), \quad \mathrm{dn}\xi = \mathrm{dn}(\xi|m^2)$$

where m is a complex number and it represents the modulus of the elliptic function. If this modulus is a real number, it can always be set up as $0 < m^2 < 1$. Besides these three well-known elliptic functions, there are nine other elliptic functions named as ns, nc, nd, sc, sd, cd, cs, ds and dc which are determined by taking reciprocals and quotients [37]. Twelve Jacobi elliptic functions are examined in four groups, and the notations can be clear from Table 1.

Table 1. Jacobi elliptic functions

1	**snξ**	**cnξ**	**dnξ**
2	$\mathrm{sd}\xi = \dfrac{\mathrm{sn}\xi}{\mathrm{dn}\xi}$	$\mathrm{cd}\xi = \dfrac{\mathrm{cn}\xi}{\mathrm{dn}\xi}$	$\mathrm{nd}\xi = \dfrac{1}{\mathrm{dn}\xi}$
3	$\mathrm{sc}\xi = \dfrac{\mathrm{sn}\xi}{\mathrm{cn}\xi}$	$\mathrm{nc}\xi = \dfrac{1}{\mathrm{cn}\xi}$	$\mathrm{dc}\xi = \dfrac{\mathrm{dn}\xi}{\mathrm{cn}\xi}$
4	$\mathrm{ns}\xi = \dfrac{1}{\mathrm{sn}\xi}$	$\mathrm{cs}\xi = \dfrac{\mathrm{cn}\xi}{\mathrm{sn}\xi}$	$\mathrm{ds}\xi = \dfrac{\mathrm{dn}\xi}{\mathrm{sn}\xi}$

These double periodic functions satisfy the properties in Table 2. Moreover, when $m = 0$ and $m = 1$, Jacobi elliptic functions transform trigonometric and hyperbolic functions, respectively. The derivatives and the other properties of these functions can be seen in Ref. [38].

Table 2. The relations of Jacobi elliptic functions

1	$\mathbf{sn}^2\xi + \mathbf{cn}^2\xi = 1$	$\mathbf{dn}^2\xi + m^2\mathbf{sn}^2\xi = 1$	$\mathbf{dn}^2\xi - m^2\mathbf{cn}^2\xi = 1 - m^2$	$\mathbf{cn}^2\xi + (1 - m^2)\mathbf{sn}^2\xi = \mathbf{dn}^2\xi$
2	$\mathrm{nd}^2\xi - m^2\mathrm{sd}^2\xi = 1$	$\mathrm{cd}^2\xi + (1 - m^2)\mathrm{sd}^2\xi = 1$	$m^2\mathrm{cd}^2\xi + (1 - m^2)\mathrm{nd}^2\xi = 1$	$\mathrm{cd}^2\xi + \mathrm{sd}^2\xi = \mathrm{nd}^2\xi$
3	$\mathrm{nc}^2\xi - \mathrm{sc}^2\xi = 1$	$\mathrm{dc}^2\xi - (1 - m^2)\mathrm{sc}^2\xi = 1$	$\mathrm{dc}^2\xi - (1 - m^2)\mathrm{nc}^2\xi = m^2$	$\mathrm{nc}^2\xi - m^2\mathrm{sc}^2\xi = \mathrm{dc}^2\xi$
4	$\mathrm{ns}^2\xi - \mathrm{cs}^2\xi = 1$	$\mathrm{ns}^2\xi - \mathrm{ds}^2\xi = m^2$	$\mathrm{ds}^2\xi - \mathrm{cs}^2\xi = 1 - m^2$	$m^2\mathrm{cs}^2\xi + (1 - m^2)\mathrm{ns}^2\xi = \mathrm{ds}^2\xi$

3. EXACT SOLUTIONS OF THE SPACE-TIME FRACTIONAL PHI-4 EQUATION

In this part of the paper, the space and time conformable fractional Phi-four equation (1) is considered. Utilizing the following change of the variables

$$\xi = k\frac{t^\alpha}{\alpha} + l\frac{x^\beta}{\beta}$$

such that k and l are arbitrary constants and using the chain rule, Eq. (1) becomes an ordinary differential equation in the form

$$(k^2 - al^2)\frac{d^2u}{d\xi^2} - bu + \lambda u^3 = 0 \tag{2}$$

where $k^2 - al^2 \neq 0$. When $\lambda = 0$, Eq. (2) becomes a linear ordinary differential equation, and the solution can be founded easily. In this study, the solutions are investigated for the nonlinear case.

The main purpose of this analytical method is to gain the solutions $u(\xi)$ in the form

$$u(\xi) = \sum_{j=0}^{N} c_j F^j\,(\xi).$$

Here, c_j are the coefficients to be determined and $F(\xi)$ are the solutions of the auxiliary nonlinear ordinary differential equation given by

$$(dF/d\xi)^2 = PF^4(\xi) + QF^2(\xi) + R \tag{3}$$

where P, Q and R are constants. The solutions of this auxiliary equation are in terms of Jacobi elliptic functions. Depending on the selected values of the constants P, Q and R, the set of these Jacobi elliptic function solutions of Eq. (3) are given in Table 3. Some of them can be seen also in Ref. [31-34].

Table 3. The solution F for P, Q and R

	P	**Q**	**R**	**F**
1	m^2	$-(m^2+1)$	1	$\pm\mathrm{sn}\xi,\quad \pm\,\mathrm{cd}\xi$
2	1	$-(m^2+1)$	m^2	$\pm\mathrm{ns}\xi,\quad \pm\,\mathrm{dc}\xi$
3	$-m^2$	$-(m^2+1)$	-1	$\pm i\mathrm{sn}\xi,\quad \pm\, i\mathrm{cd}\xi$
4	-1	$-(1+m^2)$	$-m^2$	$\pm i\mathrm{ns}\xi,\quad \pm\, i\mathrm{dc}\xi$
5	1	$-m^2+2$	$1-m^2$	$\pm\mathrm{cs}\xi,\quad \pm\, i\mathrm{dn}\xi$
6	$-m^2+1$	$-m^2+2$	1	$\pm\mathrm{sc}\xi,\quad \pm\, i\mathrm{nd}\xi$
7	-1	$2-m^2$	m^2-1	$\pm i\mathrm{cs}\xi,\quad \pm\,\mathrm{dn}\xi$
8	$-1+m^2$	$-m^2+2$	-1	$\pm i\mathrm{sc}\xi,\quad \pm\,\mathrm{nd}\xi$
9	$1-m^2$	$2m^2-1$	$-m^2$	$\pm\mathrm{nc}\xi,\quad \pm\, im\mathrm{sd}\xi$
10	$-m^2$	$2m^2-1$	$1-m^2$	$\pm\mathrm{cn}\xi,\quad \pm\frac{i}{m}\mathrm{ds}\xi$
11	m^2-1	$2m^2-1$	m^2	$\pm i\mathrm{nc}\xi,\quad \pm\, m\mathrm{sd}\xi$
12	m^2	$2m^2-1$	m^2-1	$\pm i\mathrm{cn}\xi,\quad \pm\frac{1}{m}\mathrm{ds}\xi$
13	m^4-m^2	$2m^2-1$	1	$\pm\frac{i}{m}\mathrm{nc}\xi,\quad \pm\,\mathrm{sd}\xi$
14	1	$2m^2-1$	m^4-m^2	$\pm im\mathrm{cn}\xi,\quad \pm\,\mathrm{ds}\xi$
15	$-m^4+m^2$	$2m^2-1$	-1	$\pm\frac{1}{m}\mathrm{nc}\xi\,,\quad \pm\, i\mathrm{sd}\xi$
16	-1	$2m^2-1$	$-m^4+m^2$	$\pm m\mathrm{cn}\xi,\quad \pm\, i\mathrm{ds}\xi$
17	$\frac{1}{4}$	$\frac{1+m^2}{2}$	$\frac{(1-m^2)^2}{4}$	$\mathrm{ds}\xi\pm\mathrm{cs}\xi,\quad -\mathrm{ds}\xi\mp\mathrm{cs}\xi,$ $i(m\mathrm{cn}\xi\pm\mathrm{dn}\xi),\ -i(m\mathrm{cn}\xi\pm\mathrm{dn}\xi)$
18	$\frac{(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1}{4}$	$\frac{1}{1-m^2}(\mathrm{ds}\xi\pm\mathrm{cs}\xi),\quad \frac{-1}{1-m^2}(\mathrm{ds}\xi\pm\mathrm{cs}\xi),$ $\frac{i}{1-m^2}(m\mathrm{cn}\xi\pm\mathrm{dn}\xi),\quad \frac{-i}{1-m^2}(m\mathrm{cn}\xi\pm\mathrm{dn}\xi)$
19	$-\frac{1}{4}$	$\frac{1+m^2}{2}$	$-\frac{(1-m^2)^2}{4}$	$i(\mathrm{ds}\xi\pm\mathrm{cs}\xi),\ -i(\mathrm{ds}\xi\pm\mathrm{cs}\xi),$ $m\mathrm{cn}\xi\pm\mathrm{dn}\xi,\ -m\mathrm{cn}\xi\mp\mathrm{dn}\xi$
20	$-\frac{(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$-\frac{1}{4}$	$\frac{i}{1-m^2}(\mathrm{ds}\xi\pm\mathrm{cs}\xi),\quad \frac{-i}{1-m^2}(\mathrm{ds}\xi\pm\mathrm{cs}\xi),$ $\frac{1}{1-m^2}(m\mathrm{cn}\xi\pm\mathrm{dn}\xi),\quad \frac{-1}{1-m^2}(m\mathrm{cn}\xi\pm\mathrm{dn}\xi)$
21	$\frac{-m^2+1}{4}$	$\frac{m^2+1}{2}$	$\frac{-m^2+1}{4}$	$\mathrm{nc}\xi\pm\mathrm{sc}\xi,\ -\mathrm{nc}\xi\mp\mathrm{sc}\xi,$ $i(m\mathrm{sd}\xi\pm\mathrm{nd}\xi),\ -i(m\mathrm{sd}\xi\pm\mathrm{nd}\xi),$
22	$\frac{-1+m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{-1+m^2}{4}$	$i(\mathrm{nc}\xi\pm\mathrm{sc}\xi),\ -i(\mathrm{nc}\xi\pm\mathrm{sc}\xi),$ $m\mathrm{sd}\xi\pm\mathrm{nd}\xi,\ -m\mathrm{sd}\xi\mp\mathrm{nd}\xi,$
23	$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^4}{4}$	$\mathrm{ns}\xi\pm\mathrm{ds}\xi,\ -\mathrm{ns}\xi\mp\mathrm{ds}\xi,$ $\mathrm{dc}\xi\pm\sqrt{1-m^2}\mathrm{nc}\xi,\ -\mathrm{dc}\xi\mp\sqrt{1-m^2}\mathrm{nc}\xi$

24	$\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{1}{m^2}(\mathrm{ns}\xi \pm \mathrm{ds}\xi),\ \frac{-1}{m^2}(\mathrm{ns}\xi \pm \mathrm{ds}\xi),$ $\frac{1}{m^2}\left(\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{nc}\xi\right),\ \frac{-1}{m^2}\left(\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{nc}\xi\right)$
25	$-\frac{1}{4}$	$\frac{m^2-2}{2}$	$-\frac{m^4}{4}$	$i(\mathrm{ns}\xi \pm \mathrm{ds}\xi),\ -i(\mathrm{ns}\xi \pm \mathrm{ds}\xi),$ $i\left(\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{nc}\xi\right), -i\left(\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{nc}\xi\right)$
26	$-\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$-\frac{1}{4}$	$\frac{i}{m^2}(\mathrm{ns}\xi \pm \mathrm{ds}\xi),\ \frac{-i}{m^2}(\mathrm{ns}\xi \pm \mathrm{ds}\xi),$ $\frac{i}{m^2}\left(\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{nc}\xi\right),\ \frac{-i}{m^2}\left(\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{nc}\xi\right)$
27	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{sn}\xi \pm i\mathrm{cn}\xi,\ -\mathrm{sn}\xi \mp i\mathrm{cn}\xi,$ $\mathrm{cd}\xi \pm i\sqrt{1-m^2}\mathrm{sd}\xi,\ -\mathrm{cd}\xi \mp i\sqrt{1-m^2}\mathrm{sd}\xi$
28	$-\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$-\frac{m^2}{4}$	$\mathrm{cn}\xi \pm i\mathrm{sn}\xi,\ -\mathrm{cn}\xi \mp i\mathrm{sn}\xi,$ $\sqrt{1-m^2}\mathrm{sd}\xi \pm i\mathrm{cd}\xi,\ -\sqrt{1-m^2}\mathrm{sd}\xi \mp i\mathrm{cd}\xi$
29	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}\xi \pm \mathrm{cs}\xi,\ -\mathrm{ns}\xi \mp \mathrm{cs}\xi,$ $m\mathrm{sn}\xi \pm i\mathrm{dn}\xi,\ -m\mathrm{sn}\xi \mp i\mathrm{dn}\xi,$ $\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{sc}\xi,\ -\mathrm{dc}\xi \mp \sqrt{1-m^2}\mathrm{sc}\xi,$ $m\mathrm{cd}\xi \pm i\sqrt{1-m^2}\mathrm{nd}\xi,\ -m\mathrm{cd}\xi \mp i\sqrt{1-m^2}\mathrm{nd}\xi$
30	$-\frac{1}{4}$	$\frac{1-2m^2}{2}$	$-\frac{1}{4}$	$i(\mathrm{ns}\xi \pm \mathrm{cs}\xi),\ -i(\mathrm{ns}\xi \mp \mathrm{cs}\xi),$ $\mathrm{dn}\xi \pm im\mathrm{sn}\xi,\ -\mathrm{dn}\xi \mp im\mathrm{sn}\xi,$ $i\left(\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{sc}\xi\right),\ -i\left(\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{sc}\xi\right),$ $\sqrt{1-m^2}\mathrm{nd}\xi \pm im\mathrm{cd}\xi,\ -\sqrt{1-m^2}\mathrm{nd}\xi \mp im\mathrm{cd}\xi$

By applying a balancing procedure to the highest orders of nonlinear and linear terms, we obtain the number $N = 1$. Hence, the solution of Eq. (2) can be stated as

$$u(\xi) = c_0 + c_1 F.$$

Since the ordinary differential equation (2) is a second order differential equation, we differentiate this solution two times and using the derives form of Eq. (3), we have

$$u''(\xi) = c_1 QF + 2c_1 PF^3. \tag{4}$$

Substituting Eq. (4) into Eq. (2), a third order polynomial in F is gained. After that, setting its coefficients to be zero, the following equations system arises

$$-bc_0 + \lambda c_0^3 = 0$$

$$k^2 Qc_1 - al^2 Qc_1 + 3\lambda c_0^2 c_1 - bc_1 = 0$$

$$3\lambda c_0 c_1^2 = 0$$

$$2k^2 Pc_1 - 2al^2 Pc_1 + \lambda c_1^3 = 0.$$

Solving this system, we get $c_0 = 0,\ c_1 = \mp\sqrt{-2bP/(\lambda Q)}$ and $c_0 = \mp\sqrt{b/\lambda}$, $c_1 = 0$ such that

$$Q = b/(k^2 - al^2). \tag{5}$$

If we take $A = \sqrt{b/\lambda}$, the solutions of Eq. (2) become

$$u = \pm A\sqrt{\frac{-2P}{Q}}F, \qquad u = \pm A.$$

Substituting the function F from Table 3 into the above solution and taking inverse transformation, the solutions of Eq. (1) are obtained. Moreover, the elementary function solutions of Eq. (2) can be found by utilizing the Jacobi elliptic functions for $m = 0$ and $m = 1$. Some of these solutions are constant such that $u = 0$ and $u = \pm A$, the other solutions are listed in Table 4.

Table 4. Nonconstant solutions of Eq. (2) when $m = 0$ and $m = 1$

$\boldsymbol{m = 0}$	$\boldsymbol{m = 1}$
$u = \pm\sqrt{2}A\sec\xi$	$u = \pm\sqrt{2}A\operatorname{sech}\xi$
$u = \pm\sqrt{2}A\csc\xi$	$u = \pm i\sqrt{2}A\operatorname{csch}\xi$
$u = \pm iA\tan\xi$	$u = \pm A\tanh\xi$
$u = \pm iA\cot\xi$	$u = \pm A\coth\xi$
$u = iA(\sec\xi \pm \tan\xi)$	$u = A(i\operatorname{sech}\xi \pm \tanh\xi)$
$u = -iA(\sec\xi \pm \tan\xi)$	$u = -A(i\operatorname{sech}\xi \pm \tanh\xi)$
$u = iA\left(\frac{\sin\xi}{1 \pm \cos\xi}\right) = iA(\csc\xi \mp \cot\xi)$	$u = A\left(\frac{\sinh\xi}{\cosh\xi \pm 1}\right) = A(\coth\xi \mp \operatorname{csch}\xi)$
$u = -iA\left(\frac{\sin\xi}{1 \pm \cos\xi}\right) = -iA(\csc\xi \mp \cot\xi)$	$u = -A\left(\frac{\sinh\xi}{\cosh\xi \pm 1}\right) = -A(\coth\xi \pm \operatorname{csch}\xi)$

4. APPLICATIONS

In that part of the paper, four different types of examples are considered. These examples are composed of time, space and space-time fractional types of Phi-4 Eq. (1). The solutions of the given examples are also demonstrated by graphics. In all figures, the solutions are drawn by the Mathematica 11.3.

Example 1. Consider the conformable space-time fractional Phi-4 Eq. (1) for $a = 1$, $b = 1$, $\lambda = 1$ and $\alpha = 0.5$, $\beta = 1$; that is

$$D_t^{1/2}D_t^{1/2}u - u_{xx} - u + u^3 = 0. \tag{6}$$

This equation is called the time fractional Phi-4 equation. The solutions of Eq. (6) are

$$u = \pm\sqrt{\frac{-2P}{Q}}F, \qquad u = \pm 1$$

such that $A = 1$. When $m = 1$, condition (5) is satisfied for $k = 1/2$ and $l = \sqrt{3}/2$, then transformation becomes $\xi = \sqrt{t} + (\sqrt{3}/2)x$. In Table 4, when $m = 1$, the solutions in case 1 and 3 become $u = \pm\tanh(\sqrt{t} + (\sqrt{3}/2)x)$. These solutions are illustrated for $-20 \leq x \leq 20$ and $0 \leq t \leq 20$ in Figure 1 and Figure 2. The $(\pm)$ signs correspond to localized soliton solutions that move with opposite screw senses. They have also named a kink soliton and an antikink soliton, respectively [39]. Thus, Figure 1 represents the kink type travelling wave solution, while Figure 2 represents the antikink type travelling wave solution. Besides, Figure 3 and Figure 4 illustrate the same solutions with a two-dimensional plot for $-20 \leq x \leq 20$ at time $t = 2$.

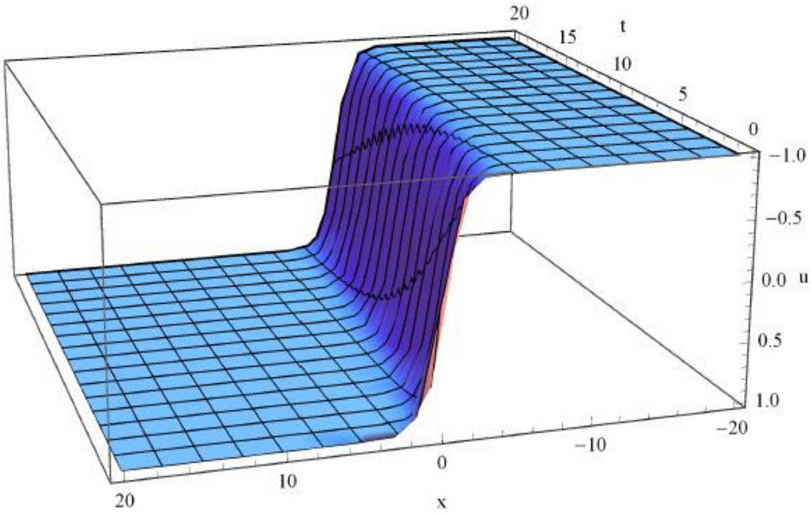

Figure 1. 3D plot of solution $u(x,t) = \tanh(\sqrt{t} + (\sqrt{3}/2)x)$

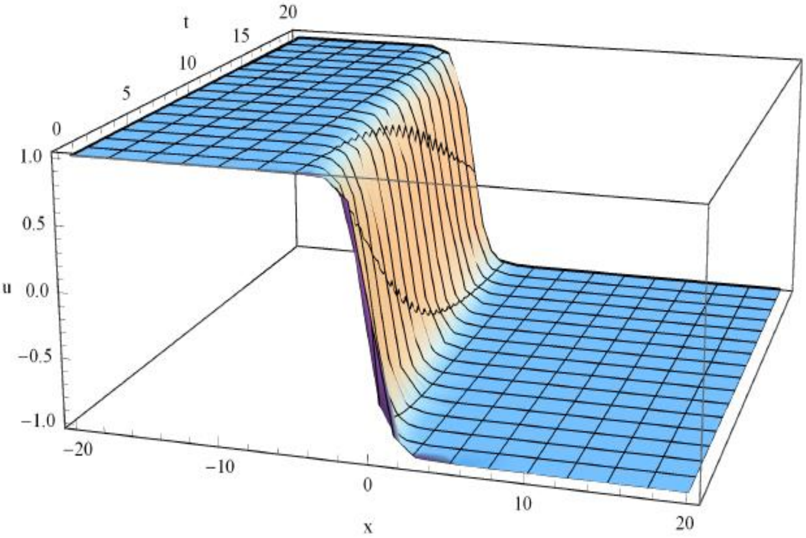

Figure 2. 3D plot of solution $u(x,t) = -\tanh(\sqrt{t} + (\sqrt{3}/2)x)$

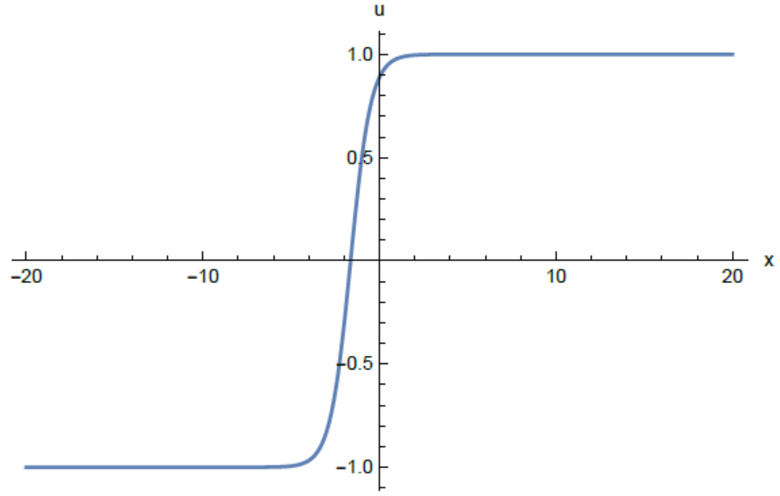

Figure 3. 2D plot of $u(x,t) = \tanh\left(\sqrt{2} + (\sqrt{3}/2)x\right)$

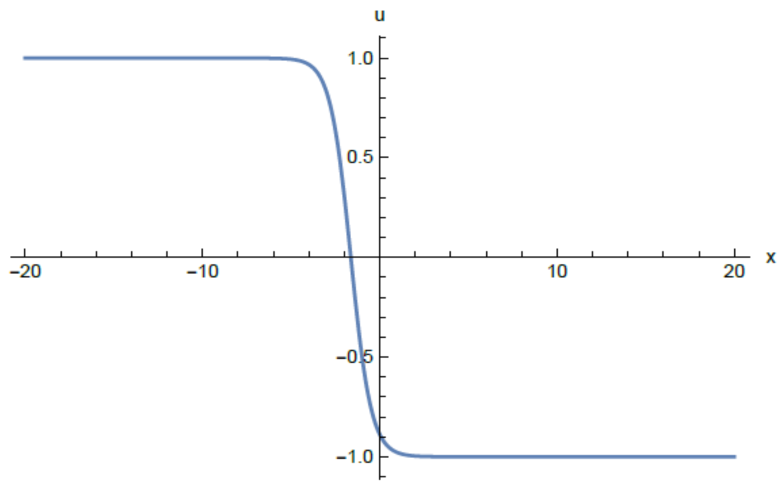

Figure 4. 2D plot of $u(x,t) = -\tanh\left(\sqrt{2} + (\sqrt{3}/2)x\right)$

Example 2. Consider the conformable space-time fractional Phi-4 Eq. (1) for $a = 5$, $b = -2$, $\lambda = 2$ and $\alpha = 0.5$, $\beta = 1$; that is

$$D_t^{1/2} D_t^{1/2} u - 5u_{xx} + 2u + 2u^3 = 0. \tag{7}$$

This equation is called the time fractional Phi-4 equation. The solutions of Eq. (7) are

$$u = \pm i\sqrt{\frac{-2P}{Q}}F, \qquad u = \pm i$$

such that $A = i$. When $m = 0$, condition (5) is satisfied for $k = l = 1$, then transformation becomes $\xi = x + 2\sqrt{t}$. In Table 4, when $m = 0$, the solutions in case 17 and 19 become $u = \mp\left(\csc\left(x + 2\sqrt{t}\right) \pm \cot\left(x + 2\sqrt{t}\right)\right)$. Here, there are four different solutions, but we only consider two of them. Figure 5 and Figure 6 demonstrate the following solutions

$$u = \mp\left(\csc\left(x + 2\sqrt{t}\right) + \cot\left(x + 2\sqrt{t}\right)\right)$$

in the region $-10 \leq x \leq 10$ and $1 \leq t \leq 2$. Figure 5 represents the kink type travelling wave solution, while Figure 6 represents the antikink type travelling wave solution. Besides, Figure 7 and Figure 8 illustrate the same solutions with a 2D plot for $-10 \leq x \leq 10$ at $t = 2$. Furthermore, the graphics of the solutions $u = -\csc(x + 2\sqrt{t}) + \cot(x + 2\sqrt{t})$ and $u = \csc(x + 2\sqrt{t}) - \cot(x + 2\sqrt{t})$ are similar to $u = \csc(x + 2\sqrt{t}) + \cot(x + 2\sqrt{t})$ and $u = -\csc(x + 2\sqrt{t}) - \cot(x + 2\sqrt{t})$, the only difference between the graphics is that 3.5 units are shifted to the right. Therefore,

$$u = \mp\csc(x + 2\sqrt{t}) - \cot(x + 2\sqrt{t})$$

are called antikink type solutions while

$$u = \mp\csc(x + 2\sqrt{t}) + \cot(x + 2\sqrt{t})$$

are called kink type solutions.

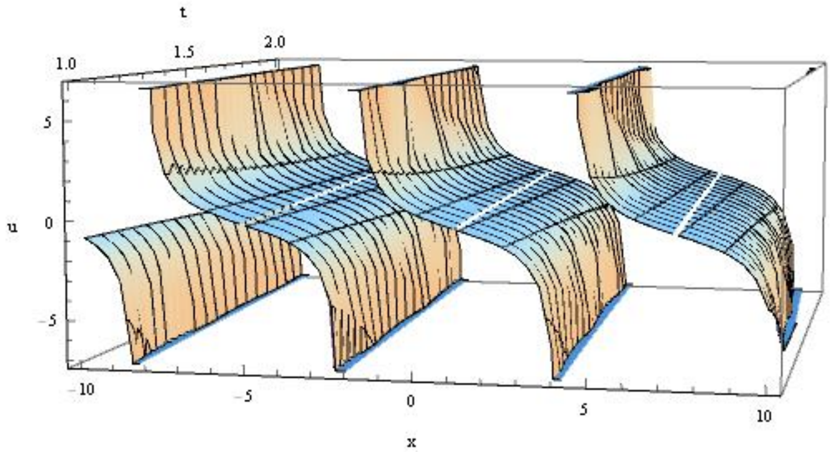

Figure 5. 3D plot of solution $u(x,t) = \csc(x + 2\sqrt{t}) + \cot(x + 2\sqrt{t})$

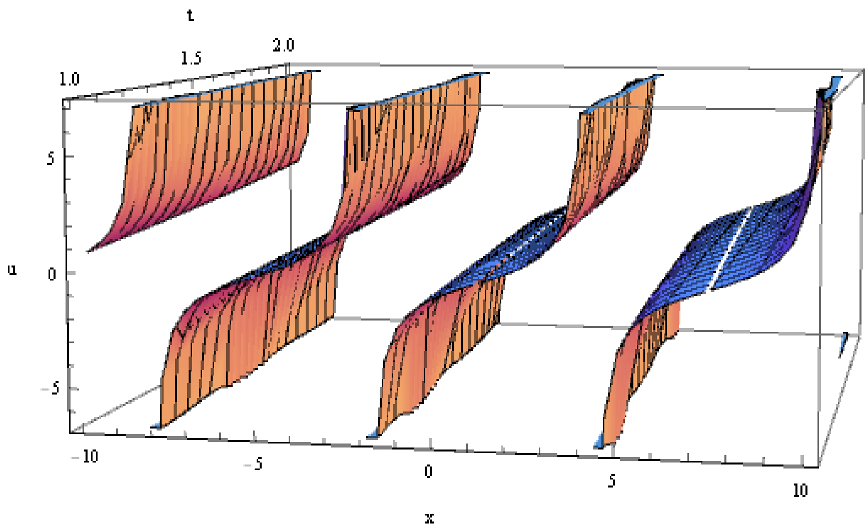

Figure 6. 3D plot of solution $u(x,t) = -\csc(x + 2\sqrt{t}) - \cot(x + 2\sqrt{t})$

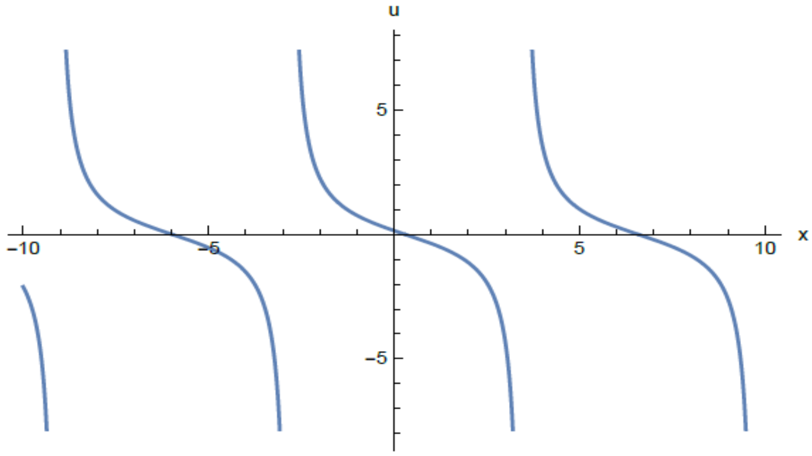

Figure 7. 2D plot of $u(x,t) = \csc(x + 2\sqrt{2}) + \cot(x + 2\sqrt{2})$

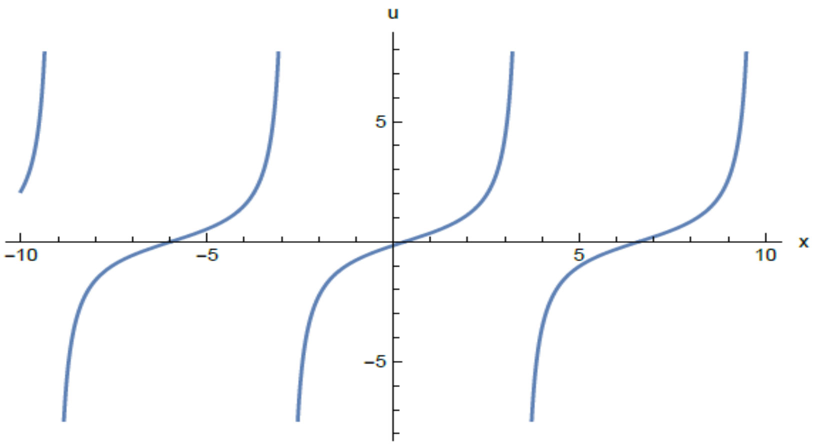

Figure 8. 2D plot of $u(x,t) = -\csc(x + 2\sqrt{2}) - \cot(x + 2\sqrt{2})$

Example 3. Consider the conformable space-time fractional Phi-4 Eq. (1) for $a = -3$, $b = 4$, $\lambda = 2$ and $\alpha = 1$, $\beta = 0.5$; that is

$$u_{tt} + 3D_x^{1/2}D_x^{1/2}u - 4u + 2u^3 = 0. \tag{8}$$

This equation is called the space fractional Phi-4 equation. The solutions of Eq. (8) are

$$u = \pm\sqrt{\frac{-4P}{Q}}F, \qquad u = \pm\sqrt{2}$$

such that $A = \sqrt{2}$. When $m = 1$, condition (5) is satisfied for $k = l = 1$, then transformation becomes $\xi = t + 2\sqrt{x}$. In Table 4, when $m = 1$, the solutions in case 10, 12, 14 and 16 become $u = \pm 2\operatorname{sech}(t + 2\sqrt{x})$. These solutions are illustrated for $5 \le x \le 10$ and $0 \le t \le 20$ in Figure 9 and Figure 10. Furthermore, Figure 11 and Figure 12 demonstrate the same solutions with the two-dimensional plot for $0 \le x \le 20$ at $t = 3$.

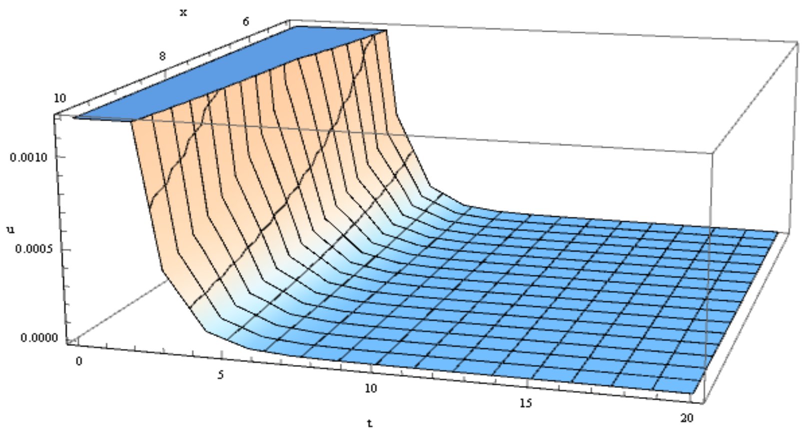

Figure 9. 3D plot of solution $u(x,t) = 2\text{sech}\left(t + 2\sqrt{x}\right)$

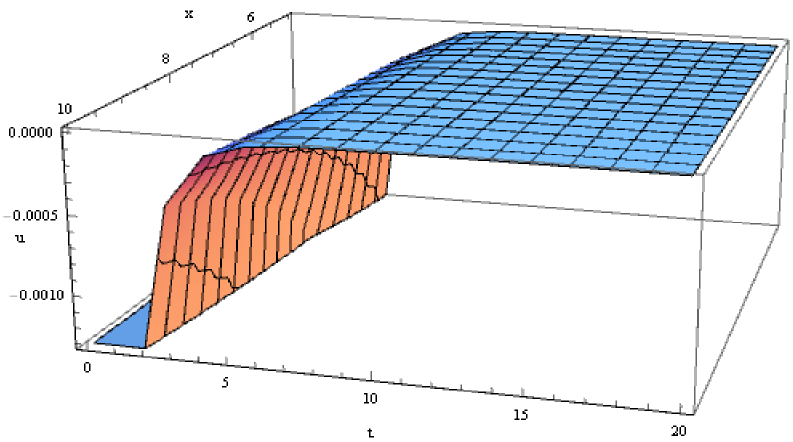

Figure 10. 3D plot of solution $u(x,t) = -2\text{sech}\left(t + 2\sqrt{x}\right)$

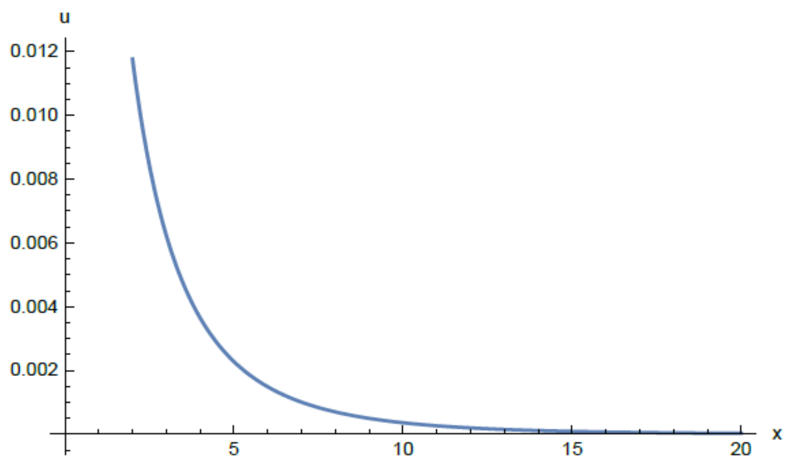

Figure 11. 2D plot of $u(x,t) = 2\text{sech}\left(3 + 2\sqrt{x}\right)$

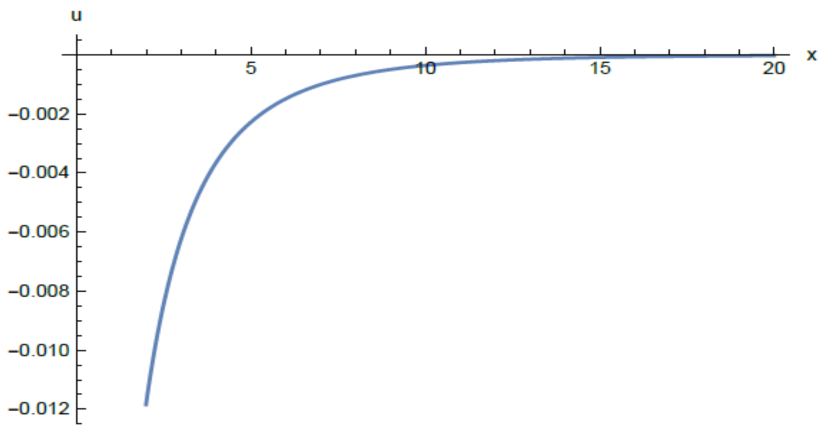

Figure 12. 2D plot of $u(x,t) = -2\text{sech}\left(3 + 2\sqrt{x}\right)$

Example 4. Consider the conformable space-time fractional Phi-4 Eq. (1) for $a = 1$, $b = 16$, $\lambda = -16$ and $\alpha = 0.2$, $\beta = 0.2$; that is

$$D_t^{1/5} D_t^{1/5} u - D_x^{1/5} D_x^{1/5} u - 16u - 16u^3 = 0. \tag{9}$$

This equation is called the space-time fractional Phi-4 equation. The solutions of Eq. (9) are

$$u = \pm i\sqrt{\frac{-2P}{Q}}F, \qquad u = \pm i$$

such that $A = i$. When $m = 0$, condition (5) is satisfied for $k = 3$ and $l = 1$, then transformation becomes $\xi = 15\sqrt[5]{t} + 5\sqrt[5]{x}$ In Table 4, when $m = 0$, the solutions in case 6 and 8 become $u = \mp\tan\left(15\sqrt[5]{t} + 5\sqrt[5]{x}\right)$. These solutions are illustrated for $0 \leq x \leq 20$ and $0 \leq t \leq 10$ in Figure 13 and Figure 14. Moreover, Figure 15 and Figure 16 demonstrate the same solutions with 2D plot for $0 \leq x \leq 50000$ at $t = 5$. From Figure 15 and Figure 16, we can see that the wave frequency increases as x approaches to zero.

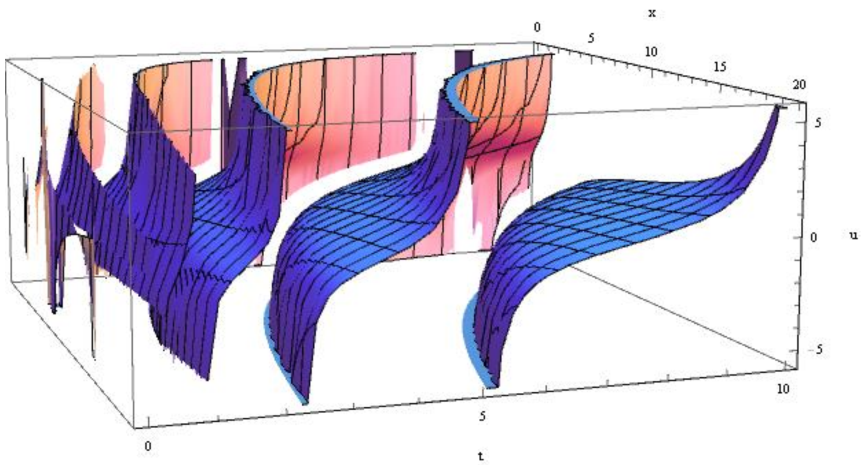

Figure 13. 3D plot of solution $u(x,t) = \tan\left(15\sqrt[5]{t} + 5\sqrt[5]{x}\right)$

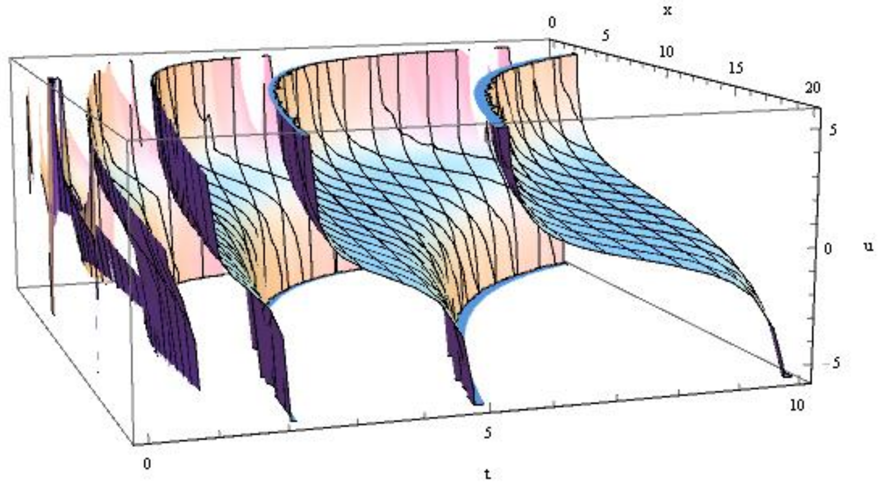

Figure 14. 3D plot of solution $u(x,t) = -\tan\left(15\sqrt[5]{t} + 5\sqrt[5]{x}\right)$

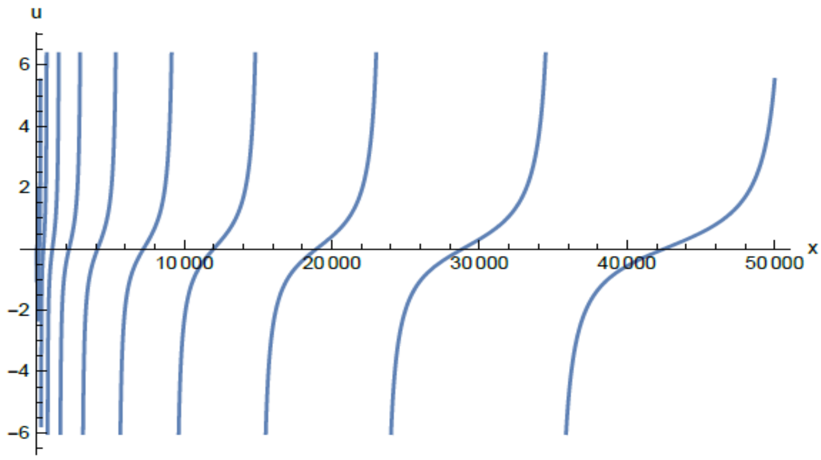

Figure 15. 2D plot of $u(x,t) = \tan\left(15\sqrt[5]{5} + 5\sqrt[5]{x}\right)$

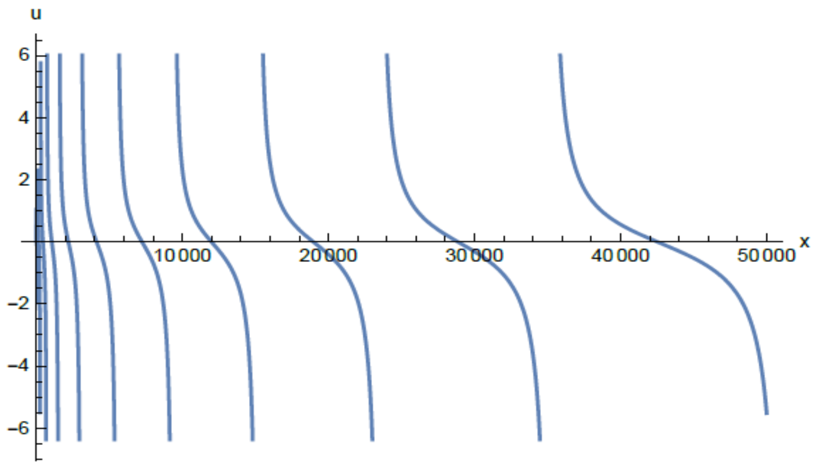

Figure 16. 2D plot of $u(x,t) = -\tan\left(15\sqrt[5]{5} + 5\sqrt[5]{x}\right)$

5. CONCLUSIONS

In this study, the exact solutions of the all of the time, space and space-time conformable fractional Phi-four equations, an analytic method has been developed using the Jacobi elliptic functions. This method is the first method in the literature and also this method is direct, quick and simple. The suggested method does not also need perturbation, linearization, boundary and initial conditions. Besides, by this method the solutions are found in a general form containing the hyperbolic, complex, rational and trigonometric functions, since the solutions include twelve Jacobi elliptic functions. Some of these solutions are solitary waves, such as kink like solutions illustrated in figures. Moreover, the solutions of the various methods such as sine-cosine ansatz and tanh methods are covered by this method.

In the literature, the modified Kudryashov [19], exponential function [19], extended direct algebraic [20], mapping [21], modified mapping [21] and generalized Kudryashov [23] methods include the conformable derivatives with respect to time, while the $(G'/G, 1/G)$-expansion method [24] contains the conformable derivatives with respect to space and time. When compared with these methods, it is seen that more solutions are obtained by our method. Because 10 solutions containing rational, trigonometric and hyperbolic functions with the exponential function method, 4 solutions containing logarithmic function with the modified Kudryashov method, 33 solutions containing rational, trigonometric and hyperbolic functions with the extended direct algebraic method, 13 solutions containing trigonometric, hyperbolic functions with the mapping methods, 4 solutions containing rational function with the generalized Kudryashov method and 10 solutions aining rational, trigonometric and hyperbolic functions with the $(G'/G, 1/G)$-expansion method are obtained. However, there exist 192 type solutions for 30 different cases in our suggested method and also infinitely many solutions can be determined depending on the parameters P, Q, R and m.

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