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Research Article

Linear algebra of the Lucas matrix

Nuretttin Irmak[1], Cahit Köme*[2]

[1] *Konya Technical University, Faculty of Engineering and Natural Science, Department of Fundamental Science, Konya, Turkey*
[2] *Nevşehir Hacı Bektaş Veli University, Department of Information Technology, Turkey*

Abstract

In this paper, we give the factorizations of the Lucas and inverse Lucas matrices. We also investigate the Cholesky factorization of the symmetric Lucas matrix. Moreover, we obtain the upper and lower bounds for the eigenvalues of the symmetric Lucas matrix by using some majorization techniques.

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1. Introduction

The Fibonacci and Lucas numbers play an important role in various areas such as mathematics, physics, computer science and related fields. For $n \geqslant 0$, the Fibonacci and Lucas numbers are defined by following recurrence relations

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \tag{1.1}$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1, \tag{1.2}$$

respectively. For more information about these numbers and their properties, we refer to the book [6].

Matrix factorizations provide considerable convenience while performing some difficult computations. Recently, several researchers have studied the factorizations of some matrices whose elements are the Fibonacci numbers. For example, Lee et al. investigated the factorizations and eigenvalues of the Fibonacci and symmetric Fibonacci matrices which are defined by

$$\mathscr{F}_n = [f_{ij}] = \begin{cases} F_{i-j+1}, & i-j+1 \geqslant 0 \\ 0, & i-j+1 < 0 \end{cases} \tag{1.3}$$

and

$$\mathscr{Q}_n = [q_{ij}] = [q_{ji}] = \begin{cases} \sum_{k=1}^{i} F_k^2, & i = j \\ q_{i,j-2} + q_{i,j-1}, & i+1 \leqslant j \end{cases}, \tag{1.4}$$

respectively [7]. Kılıç and Taşçı, presented factorizations and eigenvalues of the Pell matrix and symmetric Pell matrix whose elements are the Pell numbers [5]. They also

*Corresponding Author.
Email addresses: nirmak@ohu.edu.tr (N. Irmak), cahit@nevsehir.edu.tr (C. Köme)
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give some bounds for the eigenvalues of the symmetric Pell matrix. Motivated by the above works, several authors studied factorization of the lower triangular matrices and investigated some special type matrices (see [1, 2, 5, 8, 10]).

Besides these works, in this paper, we define the Lucas matrix and the symmetric Lucas matrix as follows:

$$\mathscr{L}_n = [l_{ij}] = \begin{cases} L_{i-j+1}, & i-j+1 \geqslant 0 \\ 0, & i-j+1 < 0 \end{cases}, \tag{1.5}$$

and

$$\mathscr{R}_n = [r_{ij}] = [r_{ji}] = \begin{cases} \sum_{k=1}^{i} L_k^2, & i = j \\ r_{i,j-2} + r_{i,j-1} + 2, & i+1 = j \\ r_{i,j-2} + r_{i,j-1}, & i+2 \leqslant j \end{cases}, \tag{1.6}$$

where $r_{1,0} = 0$. For example,

$$\mathscr{L}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 3 & 1 & 0 & 0 \\ 7 & 4 & 3 & 1 & 0 \\ 11 & 7 & 4 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathscr{R}_5 = \begin{bmatrix} 1 & 3 & 4 & 7 & 11 \\ 3 & 10 & 15 & 25 & 40 \\ 4 & 15 & 26 & 43 & 69 \\ 7 & 25 & 43 & 75 & 120 \\ 11 & 40 & 69 & 120 & 196 \end{bmatrix}.$$

From (1.5) and (1.6), we can see that $r_{1,j} = r_{j,1} = L_j$.

Here, we note that the definition of the Lucas matrix and its inverse were given directly in [11]. However, the factorization of this matrix was not given by using the $(0, 1, 2)$ matrix whose entries are 0, 1 and 2.

In this paper, we fulfill this gap.

The set of all $n-$square matrices is denoted by $\mathscr{M}_n$. Let $B \in \mathscr{M}_n$. If the matrix B can be written as $B = CC^T$ or $B = C^T C$, where C is lower triangular matrix with nonnegative diagonal entries, then this factorization is called as Cholesky factorization. It is known that if B is nonsingular, then this factorization is unique.

A matrix $A \in \mathscr{M}_n$ of the form

$$A = \begin{bmatrix} A_{11} & 0 & & \\ 0 & A_{22} & & 0 \\ & & \ddots & \\ & 0 & & A_{kk} \end{bmatrix}$$

in which $A_{ii} \in \mathscr{M}_{n_i}$, $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} n_i = n$, is called block diagonal. This type of matrix is indicated as $A = A_{11} \oplus A_{22} \oplus \ldots \oplus A_{kk}$.

2. Factorization of the Lucas matrix

In [7], the authors gave the Cholesky factorization of the Fibonacci matrix. Motivated by this paper, we find the factorization of the Lucas matrix.

Let I_n be the $n \times n$ identity matrix. We define the matrices $S_n, \overline{\mathscr{L}_n}$ and G_k as

$$S_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; S_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and $S_k = S_0 \oplus I_k$, $k = 1, 2, \ldots$, $\overline{\mathscr{L}_k} = [1] \oplus \mathscr{L}_{n-1}$, $G_1 = I_n$, $G_2 = I_{n-3} \oplus S_{-1}$, and $G_k = I_{n-k} \oplus S_{k-3}$ for $k \geqslant 3$.

Now, we define factor matrix as

$$W_n = [w_{ij}] = \begin{cases} 1, & i = j \\ 2, & i = j+1 \\ 0, & \text{otherwise} \end{cases}.$$

By using the matrices G_k and W_n, we have the following theorem.

Theorem 2.1. *The Lucas matrix $\mathscr{L}_n$ can be factored by the G_k's and W_n as follows:*

$$\begin{aligned}\mathscr{L}_n &= G_1G_2\cdots G_nW_n\\ &= W_nG_1G_2\cdots G_n.\end{aligned}$$

For example,

$$\begin{aligned}\mathscr{L}_5 &= W_5G_1G_2G_3G_4G_5\\
&= \begin{bmatrix}1&0&0&0&0\\2&1&0&0&0\\0&2&1&0&0\\0&0&2&1&0\\0&0&0&2&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&1&1&0\\0&0&1&0&1\end{bmatrix}\\
&\quad \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&1&1&0&0\\0&1&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\\
&= \begin{bmatrix}1&0&0&0&0\\3&1&0&0&0\\4&3&1&0&0\\7&4&3&1&0\\11&7&4&3&1\end{bmatrix}.\end{aligned}$$

Now, we give other factorization of $\mathscr{L}_n$. Define $n\times n$ matrix $C_n=[c_{ij}]$ by

$$c_{ij}=\begin{bmatrix}L_1&0&\cdots&0\\L_2&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\L_n&0&\cdots&1\end{bmatrix}. \tag{2.1}$$

Theorem 2.2. *For $n\geqslant 2$, the Lucas matrix $\mathscr{L}_n$ can be factored by the C_n's as*

$$\mathscr{L}_n=C_n\left(I_1\oplus C_{n-1}\right)\left(I_2\oplus C_{n-2}\right)\cdots\left(I_{n-2}\oplus C_2\right). \tag{2.2}$$

In order to find factorization of the inverse Lucas matrix, we need inverse of the factor matrix. So, the following lemma explains inverse of the factor matrix W_n.

Lemma 2.3. *Let k be the non-negative integer and $W_n^{-1}=\left[w'_{ij}\right]$ be the inverse of the matrix W_n. Then*

$$w'_{ij}=\begin{cases}0, & i<j,\\ (-2)^k & i=j+k\end{cases}$$

holds.

For example,

$$W_5^{-1}=\begin{bmatrix}1&0&0&0&0\\-2&1&0&0&0\\4&-2&1&0&0\\-8&4&-2&1&0\\16&-8&4&-2&1\end{bmatrix}.$$

Proof. Let $q_{ij}=\sum_{k=1}^{n}w_{ik}w'_{kj}$. Obviously, $q_{ii}=1$ and $q_{ij}=0$ for $i<j$. For $i>j$,

$$q_{ij}=2\left(-2\right)^k+1\left(-2\right)^{k+1}=0$$

follows. This proves the lemma. □

The inverses of the matrices S_0 and S_1 are given in [7] as follows:

$$S_0^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S_{-1}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

We know that $S_k^{-1} = S_0^{-1} \oplus I_k$. Define $H_k = G_k^{-1}$. Then

$$H_1 = G_1^{-1} = I_n, \ H_2 = G_2^{-1} = I_{n-3} \oplus S_{-1}^{-1} \text{ and } H_n = S_{n-3}^{-1}.$$

We also know that

$$C_n^{-1} = \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ -L_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -L_n & 0 & \cdots & 1 \end{bmatrix} \text{ and } (I_k \oplus C_{n-k})^{-1} = I_k \oplus C_{n-k}^{-1}.$$

We know that inverse of the Lucas matrix is given directly by (see [11] p. 459, Theorem 2.2).

$$\mathscr{L}_n^{-1} = \left[l'_{ij}\right] = \begin{cases} 1, & \text{if } i = j \\ -3, & \text{if } i = j+1 \\ 5\,(-1)^{i-j}\,2^{i-j-2}, & \text{if } i \geqslant j+2 \\ 0, & \text{otherwise} \end{cases}.$$

Here, we find inverse of the Lucas matrix by using the matrices G_k^{-1} and W_n^{-1}. Thus, the following theorem explains the factorization of the inverse Lucas matrix.

Theorem 2.4. *The inverse of the Lucas matrix $\mathscr{L}_n^{-1}$ can be factored by the G_k^{-1}'s and W_n^{-1} as*

$$\begin{aligned} \mathscr{L}_n^{-1} &= G_n^{-1} G_{n-1}^{-1} \cdots G_2^{-1} G_1^{-1} W_n^{-1} \\ &= H_n H_{n-1} \cdots H_2 H_1 W_n^{-1} \\ &= (I_{n-2} \oplus C_2)^{-1} \cdots (I_1 \oplus C_{n-1})^{-1} C_n^{-1}. \end{aligned}$$

For example, we have

$$\begin{aligned} \mathscr{L}_5^{-1} \quad &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \\ &\quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 \\ -8 & 4 & -2 & 1 & 0 \\ 16 & -8 & 4 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ 5 & -3 & 1 & 0 & 0 \\ -10 & 5 & -3 & 1 & 0 \\ 20 & -10 & 5 & -3 & 1 \end{bmatrix}. \end{aligned}$$

Now, we give the following lemma by using the definition of the symmetric Lucas matrix.

Lemma 2.5. *Each entries of the matrix $\mathscr{R}_n = [r_{i,j}]$ can be expressed by*

$$r_{i,j} = \begin{cases} L_{i+j+1} - L_{j-i+1+\xi(i)}, & \text{if } j \geqslant i \\ L_{i+j+1} - L_{i-j+1+\xi(j)}, & \text{if } j < i \end{cases}$$

where $\xi(k) = k - 2\lfloor \frac{k}{2} \rfloor$.

Proof. One can use the induction method together with the definition of the Lucas matrix. □

Before giving the Cholesky factorization of the matrix $\mathscr{R}_n$, we need to give the following main theorem.

Theorem 2.6. *If* $1 \leqslant i \leqslant j$, *then*

$$5\sum_{t=1}^{i-2}(-2)^{i-2-t}\,r_{t,j} - 3r_{i-1,j} + r_{i,j} = L_{j-i+1} \tag{2.3}$$

holds. Otherwise, we have

$$5\sum_{t=1}^{i-2}(-2)^{i-2-t}\,r_{t,j} - 3r_{i-1,j} + r_{i,j} = 0$$

Proof. Assume that $1 \leqslant i \leqslant j$. We use the induction method in order to prove the theorem. From the definition of (1.6), we observe that $r_{i,j+2} = r_{i,j+1} + r_{i,j}$. This finishes the induction on j.

Now, we focus on the induction on i. Since $5\sum_{t=1}^{i-2}(-2)^{i-2-t}\,r_{t,j} = 0$, the equation (2.3) is true for $i = 1$, 2. For $i = 3$ in the equation (2.3),

$$\begin{aligned}5r_{1,j} - 3r_{2,j} + r_{3,j} &= 5\left(L_{j+2} - L_{j+1}\right) - 3\left(L_{j+3} - L_{j-1}\right) + L_{j+4} - L_{j-1}\\ &= L_{j+4} - 3L_{j+3} + 5L_j + 2L_{j-1}\\ &= L_{j-2}\end{aligned}$$

holds as claimed. Now, suppose that it is true for all integer $i \geqslant 4$. By using the equation (2.3) and induction hypothesis, we have

$$\begin{aligned}5\sum_{t=1}^{i-1}(-2)^{i-1-t}\,r_{t,j} - 3r_{i,j} + r_{i+1,j} &= 5\left(r_{i-1,j} - 2\sum_{t=1}^{i-2}(-2)^{i-2-t}\,r_{t,j}\right) - 3r_{i,j} + r_{i+1,j}\\ &= 5r_{i-1,j} - 2\left(L_{j-i+1} + 3r_{i-1,j} - r_{i,j}\right) - 3r_{i,j} + r_{i+1,j}\\ &= -r_{i-1,j} + r_{i+1,j} - r_{i,j} - 2L_{j-i+1}.\end{aligned}$$

From Lemma 2.5, we know that $r_{i,j} = L_{i+j+1} - L_{j-i+1+\xi(i)}$. Therefore we have

$$\begin{aligned}5\sum_{t=1}^{i-1}(-2)^{i-1-t}\,r_{t,j} - 3r_{i,j} + r_{i+1,j} &= -r_{i-1,j} + r_{i+1,j} - r_{i,j} - 2L_{j-i+1}\\ &= -L_{i+j} - L_{i+j+1} + L_{i+j+2} - 2L_{j-i+1}\\ &\quad + L_{j-i+2+\xi(i-1)} + L_{j-i+1+\xi(i)} - L_{j-i+\xi(i+1)}\\ &= L_{j-i+1+\xi(i+1)} + L_{j-i+1+\xi(i)} - 2L_{j-i+1}\\ &= L_{j-i+3} - 2L_{j-i+1}\\ &= L_{j-i}.\end{aligned}$$

The other case $j < i$ can be proven similarly. Therefore, we omit the details. So, the proof is completed. □

Theorem 2.7. *For* $n \geqslant 1$ *positive integer,* $H_nH_{n-1}\dots H_1W_n^{-1}\mathscr{R}_n = \mathscr{L}_n^T$ *and the Cholesky factorization is given by* $\mathscr{R}_n = \mathscr{L}_n\mathscr{L}_n^T$.

Proof. Together with the facts $H_nH_{n-1}\dots H_1W_n^{-1} = \mathscr{L}_n^{-1}$ and $\mathscr{L}_n^{-1}\mathscr{R}_n = \mathscr{L}_n^T$, we have $\mathscr{R}_n = \mathscr{L}_n\mathscr{L}_n^T$. This gives the Cholesky factorization of the matrix $\mathscr{R}_n$. □

In particular, with the help of Theorem 2.7, we can give the Cholesky factorization of the inverse symmetric Lucas matrix as $\mathscr{R}_n^{-1} = \left(\mathscr{L}_n^T\right)^{-1} \mathscr{L}_n^{-1} = \left(\mathscr{L}_n^{-1}\right)^T \mathscr{L}_n^{-1}$. For example,

$$\mathscr{R}_7^{-1} = \begin{bmatrix} 8535 & -4268 & 2135 & -1070 & 540 & -280 & 80 \\ -4268 & 2135 & -1068 & 535 & -270 & 140 & -40 \\ 2135 & -1068 & 535 & -268 & 135 & -70 & 20 \\ -1070 & 535 & -268 & 135 & -68 & 35 & -10 \\ 540 & -270 & 135 & -68 & 35 & -18 & 5 \\ -280 & 140 & -70 & 35 & -18 & 10 & -3 \\ 80 & -40 & 20 & -10 & 5 & -3 & 1 \end{bmatrix}. \tag{2.4}$$

3. Eigenvalues of $\mathscr{R}_n$

In this section, we consider the eigenvalues of the symmetric Lucas matrix $\mathscr{R}_n$.

Let Ω be an $n \times n$ matrix. The authors, in [4], stated that if Ω is an $n \times n$ Hermitian matrix then it is positive definite if and only if $\det \Omega > 0$. For $n \geqslant 2$, we observe that $\mathscr{R}_n$ is Hermitian and by Theorem 2.7, we have $\det \mathscr{R}_n = \det\left(\mathscr{L}_n \mathscr{L}_n^T\right) = 1$. Hence, $\mathscr{R}_n$ is a positive definite matrix and therefore the eigenvalues of $\mathscr{R}_n$ are all positive.

Let $\mathscr{D} = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n; x_1 \geqslant x_2 \geqslant \dots \geqslant x_n\}$. For $x, y \in \mathscr{D}$,

$$x \prec y, \quad \text{if} \quad \begin{cases} \sum_{j=1}^{i} x_j \leqslant \sum_{j=1}^{i} y_j, & i = 1, 2, \dots, n-1, \\ \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j \end{cases}.$$

When $x \prec y$, x is said to be majorized by y or y is said to majorize x. On the other hand, the condition for majorization can be rewritten as follows:

$$x \prec y, \quad \text{if} \quad \begin{cases} \sum_{j=0}^{i} x_{n-j} \geqslant \sum_{j=0}^{i} y_{n-j}, & i = 0, 1, \dots, n-2, \\ \sum_{j=0}^{n-1} x_{n-j} = \sum_{j=0}^{n-1} y_{n-j}. \end{cases}.$$

Note that, there is an interesting simple fact as follows:

$$(\overline{x}, \overline{x}, \dots, \overline{x}) \prec (x_1, x_2, \dots, x_n), \tag{3.1}$$

where $\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$. For more information about majorizations, we refer to the book [9].

For $i, j = 1, 2, \dots, n$, an $n \times n$ matrix $\Phi_n = [\phi_{ij}]$ is a doubly stochastic matrix if $\phi_{ij} \geqslant 0$, $\sum_{i=1}^{n} \phi_{ij} = 1$ and $\sum_{j=1}^{n} \phi_{ij} = 1$. Hardy, Littlewood and Pólya [3], stated that a necessary and sufficient condition that $x \prec y$ is that there exists a doubly stochastic matrix Φ_n such that $x = y\Phi_n$.

Note that $\det \mathscr{L}_n = 1$ and $\det \mathscr{R}_n = 1$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $\mathscr{R}_n$. We know that $\mathscr{R}_n = \mathscr{L}_n \mathscr{L}_n^T$ and $\sum_{i=1}^{k} L_i^2 = (L_{k+1} L_k - 2)$, all of the eigenvalues of $\mathscr{R}_n$ are positive and

$$(L_{n+1} L_n - 2, L_n L_{n-1} - 2, \dots, L_2 L_1 - 2) \prec (\lambda_1, \lambda_2, \dots, \lambda_n). \tag{3.2}$$

So, we have the following corollaries.

Corollary 3.1. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $\mathscr{R}_n$. Then*

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = L_{2n+2} - 2n - 3 - \xi(n). \tag{3.3}$$

Proof. By virtue of $(L_{n+1} L_n - 2, L_n L_{n-1} - 2, \dots, L_2 L_1 - 2) \prec (\lambda_1, \lambda_2, \dots, \lambda_n)$, we get

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = \begin{cases} (L_{n+1})^2 - 2n - 1, & \text{if } n \text{ is even} \\ (L_{n+1})^2 - 2n - 6, & \text{if } n \text{ is odd} \end{cases} = L_{2n+2} - 2n - 3 - \xi(n).$$

□

Corollary 3.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\mathscr{R}_n$. Then*

$$n\lambda_n \leqslant L_{2n+2} - 2n - 3 - \xi(n) \leqslant n\lambda_1.$$

Proof. Let $s_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Since

$$\left(\frac{s_n}{n}, \frac{s_n}{n}, \ldots, \frac{s_n}{n}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n), \tag{3.4}$$

we get $\lambda_n \leqslant \frac{s_n}{n} \leqslant \lambda_1$. Hence, the proof is completed. □

From (2.4), we get

$$\left(10 + 5^2\left(\frac{2^{2n-4}-1}{3}\right), 10 + 5^2\left(\frac{2^{2n-6}-1}{3}\right), \ldots, 35, 10, 1\right) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right). \tag{3.5}$$

Hence, there exists a doubly stochastic matrix $\Phi_n = [\varphi_{ij}]$ such that

$$\left(10 + 5^2\left(\frac{2^{2n-4}-1}{3}\right), 10 + 5^2\left(\frac{2^{2n-6}-1}{3}\right), \ldots, 35, 10, 1\right)$$

$$= \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right) \begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{bmatrix}.$$

Namely, we have $\frac{1}{\lambda_n}\varphi_{1n} + \frac{1}{\lambda_{n-1}}\varphi_{2n} + \cdots + \frac{1}{\lambda_1}\varphi_{nn} = 1$ and $\varphi_{1n} + \varphi_{2n} + \cdots + \varphi_{nn} = 1$.

Lemma 3.3. *For each $i = 1, 2, \ldots, n$, we have $\varphi_{n-(i-1),n} \leqslant \frac{\lambda_i}{n-1}$.*

Proof. Suppose that $\varphi_{n-(i-1),n} > \frac{\lambda_i}{n-1}$. Then

$$\varphi_{1n} + \varphi_{2n} + \cdots + \varphi_{nn} > \frac{\lambda_1}{n-1} + \frac{\lambda_2}{n-1} + \cdots \frac{\lambda_n}{n-1} = \frac{1}{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n). \tag{3.6}$$

Since $\sum_{i=1}^{n} \varphi_{in} = 1$ and $\sum_{i=1}^{n} \lambda_i \geqslant n$, this yields a contradiction, so $\varphi_{n-(i-1),n} \leqslant \frac{\lambda_i}{n-1}$. □

Let

$$\tau = \frac{1}{n}\left(1 + 10 + 35 + \ldots + 10 + 5^2\left(\frac{2^{2n-4}-1}{3}\right)\right) = \frac{5^2 4^{n-1} + 15(n-2) - 1}{9n} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\lambda_i}.$$

Then, we have

$$(\tau, \tau, \ldots, \tau) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_1}\right). \tag{3.7}$$

The next theorem explains the majorization of the eigenvalues of $\mathscr{R}_n$.

Theorem 3.4. *For $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathscr{D}$, we have*

$$\left(\frac{1}{n-1}\left(s_n - \frac{1}{\tau}\right), \ldots, \frac{1}{n-1}\left(s_n - \frac{1}{\tau}\right), \frac{1}{\tau}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n). \tag{3.8}$$

Proof. Let Ψ_n be an $n \times n$ matrix as follows:

$$\Psi_n = \begin{bmatrix} \frac{1-\psi_{1n}}{n-1} & \frac{1-\psi_{1n}}{n-1} & \cdots & \frac{1-\psi_{1n}}{n-1} & \psi_{1n} \\ \frac{1-\psi_{2n}}{n-1} & \frac{1-\psi_{2n}}{n-1} & \cdots & \frac{1-\psi_{2n}}{n-1} & \psi_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1-\psi_{nn}}{n-1} & \frac{1-\psi_{nn}}{n-1} & \cdots & \frac{1-\psi_{nn}}{n-1} & \psi_{nn} \end{bmatrix}, \tag{3.9}$$

where

$$\psi_{ij} = \frac{1-\psi_{in}}{n-1}, \quad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,n-1$$

and

$$\psi_{in} = \frac{1}{n\tau\lambda_i}, \quad i = 1,2,\ldots,n.$$

Then, for $i = 1,2,\ldots,n$, we observe that $\psi_{in} \geqslant 0$ and

$$\psi_{1n} + \psi_{2n} + \cdots + \psi_{nn} = \frac{1}{n\tau\lambda_1} + \frac{1}{n\tau\lambda_2} + \cdots + \frac{1}{n\tau\lambda_n} = 1, \tag{3.10}$$

$$(n-1)\frac{1-\psi_{in}}{n-1} + \psi_{in} = 1 \tag{3.11}$$

and

$$\frac{1-\psi_{1n}}{n-1} + \frac{1-\psi_{2n}}{n-1} + \cdots + \frac{1-\psi_{nn}}{n-1} = \frac{1}{n-1}\left(n - (\psi_{1n} + \psi_{2n} + \cdots + \psi_{nn})\right) = 1.$$

Therefore, Ψ_n is a doubly stochastic matrix. Moreover, we have

$$\lambda_1\psi_{1n} + \lambda_2\psi_{2n} + \cdots + \lambda_n\psi_{nn} = \frac{1}{\tau} \tag{3.12}$$

and

$$\begin{aligned}\lambda_1\frac{1-\psi_{1n}}{n-1} + \lambda_2\frac{1-\psi_{2n}}{n-1} + \cdots + \lambda_n\frac{1-\psi_{nn}}{n-1} &= \frac{1}{n-1}\left(s_n - (\lambda_1\psi_{1n} + \lambda_2\psi_{2n} + \cdots + \lambda_n\psi_{nn})\right)\\ &= \frac{1}{n-1}\left(s_n - \frac{1}{\tau}\right).\end{aligned}$$

Hence, we get

$$\left(\frac{1}{n-1}\left(s_n - \frac{1}{\tau}\right), \ldots, \frac{1}{n-1}\left(s_n - \frac{1}{\tau}\right), \frac{1}{\tau}\right) = (\lambda_1, \lambda_2, \ldots, \lambda_n)\,\Psi_n.$$

As a result, we have

$$\left(\frac{1}{n-1}\left(s_n - \frac{1}{\tau}\right), \ldots, \frac{1}{n-1}\left(s_n - \frac{1}{\tau}\right), \frac{1}{\tau}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

□

By virtue of equation (2.4), we have the following lemma which explains the lower bounds of the eigenvalues of $\mathscr{R}_n$.

Lemma 3.5. *For $k = 2,3,\ldots,n$, we have*

$$\frac{1}{\gamma_k} \leqslant \lambda_k, \tag{3.13}$$

where $\gamma_k = \frac{5^2 4^{k-1} + 15(k-2) - 1}{9}$ is sum of the diagonal elements of $\mathscr{R}^{-1}$.

Proof. By virtue of (2.4), we have

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_k} \leqslant 1 + 10 + 35 + \ldots + 10 + 5^2\left(\frac{2^{2k-4}-1}{3}\right) = \gamma_k. \tag{3.14}$$

Hence,

$$\frac{1}{\lambda_k} \leqslant \gamma_k - \Bigg(\underbrace{\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_{k-1}}}_{\geqslant 0} \Bigg) \leqslant \gamma_k. \tag{3.15}$$

Therefore, we have $\frac{1}{\gamma_k} \leqslant \lambda_k$. □

In the following theorem, we give some upper and lower bounds for the eigenvalues of $\mathscr{R}_n$.

Theorem 3.6. *For $k = 1, 2, \ldots, n-2$, we have*

$$\frac{1}{\gamma_{n-k}} \leqslant \lambda_{n-k} \leqslant \frac{1}{n-1}\left(\frac{n-k-1}{\tau} + k s_n\right) - \sum_{i=0}^{k-1} \frac{1}{\gamma_{n-i}}. \tag{3.16}$$

In particular,

$$\frac{1}{n-1}\left(s_n - \frac{1}{\tau}\right) \leqslant \lambda_1 \leqslant \prod_{i=2}^{n} \gamma_i \tag{3.17}$$

and

$$\frac{1}{\gamma_n} \leqslant \lambda_n \leqslant \frac{1}{\tau}. \tag{3.18}$$

Proof. By virtue of Theorem 3.4, we observe that $\frac{1}{n-1}\left(s_n - \frac{1}{\tau}\right) \leqslant \lambda_1$ and $\lambda_n \leqslant \frac{1}{\tau}$. From Lemma 3.5, we have $\frac{1}{\gamma_n} \leqslant \lambda_n$. Since $\det \mathscr{R}_n = \det\left(\mathscr{L}_n \mathscr{L}_n^T\right) = 1 = \lambda_1 \lambda_2 \ldots \lambda_n$, by Lemma 3.5, we have

$$\lambda_1 \prod_{i=2}^{n} \frac{1}{\gamma_i} \leqslant \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n = 1.$$

Thus, we obtain $\lambda_1 \leqslant \prod_{i=2}^{n} \gamma_i$. From Theorem 3.4, we have

$$\begin{aligned} \lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-k} &\leqslant \frac{1}{\tau} + \frac{k}{n-1}\left(s_n - \frac{1}{\tau}\right) \\ &= \frac{1}{n-1}\left(\frac{n-k-1}{\tau} + k s_n\right). \end{aligned}$$

Thus, by using Lemma 3.5, we obtain

$$\begin{aligned} \lambda_{n-k} &\leqslant \frac{1}{n-1}\left(\frac{n-k-1}{\tau} + k s_n\right) - (\lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-k+1}) \\ &\leqslant \frac{1}{n-1}\left(\frac{n-k-1}{\tau} + k s_n\right) - \sum_{i=0}^{k-1} \frac{1}{\gamma_{n-i}}. \end{aligned}$$

Therefore, we obtain

$$\frac{1}{\gamma_{n-k}} \leqslant \lambda_{n-k} \leqslant \frac{1}{n-1}\left(\frac{n-k-1}{\tau} + k s_n\right) - \sum_{i=0}^{k-1} \frac{1}{\gamma_{n-i}}.$$

□

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