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On Quasi-Hemi-Slant Riemannian Maps

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Highlights

- This paper focuses on quasi-hemi-slant Riemannian maps.
- Distributions to be integrable and parallel investigated.
- A quasi-hemi-slant Riemannian map to be totally geodesic investigated.

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Abstract

In this paper, quasi-hemi-slant Riemannian maps from almost Hermitian manifolds onto Riemannian manifolds are introduced. The geometry of leaves of distributions that are involved in the definition of the submersion and quasi-hemi-slant Riemannian maps are studied. In addition, conditions for such distributions to be integrable and totally geodesic are obtained. Also, a necessary and sufficient condition for proper quasi-hemi-slant Riemannian maps to be totally geodesic is given. Moreover, structured concrete examples for this notion are given.

1. INTRODUCTION

A differentiable map F between Riemannian manifolds (N_1, g_1) and (N_2, g_2) is said to be a Riemannian map if

$$g_2(F_*Z_1, F_*Z_2) = g_1(Z_1, Z_2), \text{ for } Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}.$$

The theory of smooth maps between Riemannian manifolds plays a preeminent role in differential geometry and also in physics. It is useful for comparing geometric structures between the source manifolds and the target manifolds. A conspicuous property of Riemannian map provides the generalized eikonal equation $\| F_*\|^2 = \text{rank } F$ [1]. Since rank F is an integer value function and $\| F_* \|^2$ is continuous function on the Riemannian manifold. Since energy density $2e(F) = \| F_* \|^2 = \text{rank } F$, i.e. density is quantized to integer if the Riemannian manifold is connected. In addition, complex manifolds are very useful tools for studying spacetime geometry [2]. In fact, Calabi-Yau manifolds and Teichmuller spaces are two interesting classes of Kähler manifold, which have applications in superstring theory [3] and in general relativity [4, 5]. Thus, the notion of Riemannian maps deserves through study from different perspectives.

In addition, O'Neills [6] and Gray [7] studied Riemannian submersions. Watson introduced almost Hermitian submersions as follows: A Riemannian submersion $F : (N_1, g_1, J_{N_1}) \rightarrow (N_2, g_2, J_{N_2})$ is said to be an almost Hermitian submersion if $F_* J_{N_1} = J_{N_2} F_*$ [8]. Watson also showed that, in most cases [8] and [9], each fiber and base manifold have the same kind of structure as the total space.

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After that, several kinds of Riemannian submersions were introduced and studied, some of them are like: contact-submersions [10], semi-slant and generic submersions [11, 12], semi-invariant $\xi^{\perp}$-Riemannian submersions [13], hemi-slant submersions [14] etc. Sayar, Akyol and Prasad studied on bi slant submersions [15], and Prasad, Shukla and Kumar introduce quasi-bi slant submersions [16]. Recently, Longwap, Massamba and Homti introduce and study quasi-hemi slant Riemannian submersions which generalizes hemi-slant, semi-slant and semi-invariant Riemannian submersions [17]. It is well known that Riemannian submersion is a particular Riemannian map with $(\text{range } F_*)^{\perp} = \{0\}$, so we generalize the notion of quasi-hemi slant Riemannian submersions to quasi-hemi slant Riemannian maps in the present paper and study its geometry.

The notion of Riemannian map between Riemannian manifolds was introduced by Fischer [18]. Let $F : (N_1, g_1) \to (N_2, g_2)$ be a differentiable map with $0 < \text{rank } F_* < \min(m, n)$. If the kernal space of F_* is denoted by $\ker F_*$, and the orthogonal complementary space of $\ker F_*$ is denoted by $(\ker F_*)^{\perp}$ in TN_1, then

$$TN_1 = \ker F_* \oplus (\ker F_*)^{\perp}.$$

Also, if the range of F_* is denoted by range F_*, and for a point $q \in N_1$ the orthogonal complementary space of range $F_{*F(q)}$ is denoted by $(\text{range } F_{*F(q)})^{\perp}$ in $T_{F(q)}N_2$ then the tangent space $T_{F(q)}N_2$ has the following orthogonal decomposition:

$$T_{F(q)}N_2 = (\text{range} F_{*F(q)}) \oplus (\text{range } F_{*F(q)})^{\perp}.$$

A differentiable map $F: (N_1, g_1) \to (N_2, g_2)$ is called a Riemannian map at $q \in N_1$ if $F^{h}_{*q} : (\ker F_{*q})^{\perp} \to (\text{range } F_{*F(q)})$ is linear isometry.

In this paper, we study the quasi-hemi-slant Riemannian maps from an almost Hermitian manifolds to Riemannian manifolds. In section 3, quasi-hemi-slant Riemannian maps are defined, and the geometry of leaves of distributions that are involved in the definition of such maps is studied. In addition, a necessary and sufficient condition for quasi-hemi-slant Riemannian maps to be totally geodesic is given. Finally, concrete examples for this setting are provided.

2. PRELIMINARIES

If J is a (1, 1) tensor field on an even-dimensional differentiable manifold N_1 such that

$$J^2 = -I \tag{1}$$

then (N_1, J) is said to be an almost complex manifold where I is identity operator [19, 20]. Nijenhuis tensor N of J is described as:

$$N(X_1, X_2) = [JX_1, JX_2] - [X_1, X_2] - J[JX_1, X_2] - J[X_1, JX_2] \tag{2}$$

for all $X_1, X_2 \in \Gamma(TN_1)$. If $N = 0$, then N_1 is said to be a complex manifold. If g_1 is a Riemannian metric on N_1 such that

$$g_1(JX_1, JX_2) = g_1(X_1, X_2), \quad \text{for all } X_1, X_2 \in \Gamma(TN_1) \tag{3}$$

then (N_1, g_1, J) is said to be an almost Hermitian manifold, and if $(\nabla_{X_1} J) X_2 = 0$ for all $X_1, X_2 \in \Gamma(TN_1)$ then (N_1, g_1, J) is said to be a Kähler manifold where ∇ is the Levi-Civita connection on N_1.

O'Neill's tensors T and A are defined by

$$\mathcal{A}_{\mathcal{E}_1}\mathcal{E}_2 = \mathcal{H}\nabla_{\mathcal{H}\mathcal{E}_1}\mathcal{V}\mathcal{E}_2 + \mathcal{V}\nabla_{\mathcal{H}\mathcal{E}_1}\mathcal{H}\mathcal{E}_2, \tag{4}$$

$$\mathcal{T}_{\mathcal{E}_1}\mathcal{E}_2 = \mathcal{H}\nabla_{\mathcal{V}\mathcal{E}_1}\mathcal{V}\mathcal{E}_2 + \mathcal{V}\nabla_{\mathcal{V}\mathcal{E}_1}\mathcal{H}\mathcal{E}_2 \tag{5}$$

for any $\mathcal{E}_1, \mathcal{E}_2 \in \Gamma(TN_1)$. From Equations (4) and (5), we have

$$\nabla_{X_1}X_2 = \mathcal{T}_{X_1}X_2 + \mathcal{V}\nabla_{X_1}X_2,. \tag{6}$$

$$\nabla_{X_1}Z_1 = \mathcal{T}_{X_1}Z_1 + \mathcal{H}\nabla_{X_1}Z_1, \tag{7}$$

$$\nabla_{Z_1}X_1 = \mathcal{A}_{Z_1}X_1 + \mathcal{V}\nabla_{Z_1}X_1, \tag{8}$$

$$\nabla_{Z_1}Z_2 = \mathcal{H}\nabla_{Z_1}Z_2 + \mathcal{A}_{Z_1}Z_2, \tag{9}$$

for all $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$, where $H\nabla_{X_1}Z_1 = A_{Z_1}X_1$, if Z_1 is basic. For $q \in N_1$, $X_1 \in \mathcal{V}_q$ and $Z_1 \in H_q$ the linear operators

$$\mathcal{A}_{Z_1} \text{ and } \mathcal{T}_{X_1} : T_qN_1 \rightarrow T_qN_1$$

are skew-symmetric, that is

$$g_1(\mathcal{A}_{Z_1}\mathcal{E}_1, \mathcal{E}_2) = -\, g_1(\mathcal{E}_1, \mathcal{A}_{Z_1}\mathcal{E}_2) \text{ and } g_1(\mathcal{T}_{X_1}\mathcal{E}_1, \mathcal{E}_2) = -g_1(\mathcal{E}_1, \mathcal{T}_{X_1}\mathcal{E}_2)$$

for each $\mathcal{E}_1, \mathcal{E}_2 \in T_qN_1$.

Let $F : (N_1, g_1) \rightarrow (N_2, g_2)$ is a smooth map. F is said to be a totally geodesic if

$(\nabla F_*)(X_1, X_2) = 0$, for all $X_1, X_2 \in \Gamma(TN_1)$.

The differential map F_* of F can be observed a section of the bundle Hom $(TN_1, F^{-1}TN_2) \rightarrow N_1$, where $F^{-1}TN_2$ is the bundle which has fibers $(F^{-1}TN_2)_x = T_{F(x)}N_2$, has a connection ∇ induced from the Riemannian connection ∇^{N_1} and the pullback connection. In addition, the second fundamental form of F is given by

$$(\nabla F_*)(X_1, X_2) = \nabla^F_{X_1}F_*(X_2) - F_*(\nabla^{N_1}_{X_1}X_2) \tag{10}$$

for vector field $X_1, X_2 \in \Gamma(TN_1)$, where ∇^F is the pullback connection. Bi-harmonic Riemannian maps and the second fundamental form $(\nabla F_*)(U_1, U_2)$, for all $U_1, U_2 \in \Gamma(\ker F_*)^{\perp}$ of a Riemannian map has components in range F_* [21].

Lemma 1. Let $F : (N_1, g_1) \rightarrow (N_2, g_2)$ be a Riemannian map. Then $g_2((\nabla F_*)(U_1, U_2), F_*(U_3)) = 0$ for all $U_1, U_2, U_3 \in \Gamma(\ker F_*)^{\perp}$.

As a consequence of the above lemma, we get $(\nabla F_*)(U_1, U_2) \in \Gamma\,(\text{range } F_*)^{\perp}$, for all $U_1, U_2, \in \Gamma\,(\ker F_*)^{\perp}$.

Let $F: (N_1, g_1, J) \rightarrow (N_2, g_2)$ be Riemannian map from an almost Hermitian manifold onto a Riemannian manifold.

F is said to be a semi-invariant Riemannian map if there is a distribution $D_1 \subseteq \ker F_*$ such that

$\ker F_* = D_1 \oplus D_2$, $J(D_1) = D_1$,

where $D_1 \oplus D_2$ is an orthogonal decomposition of $\ker F_*$ [1]. The complementary orthogonal subbundle to $J(\ker F_*)$ in $(\ker F_*)^\perp$ is denoted by μ. Thus, we get $(\ker F_*)^\perp = J(D_2) \oplus \mu$. It is clear that μ is an invariant subbundle.

If $\operatorname{Ker} F_* = D^\theta \oplus D^\perp$ with D^θ is slant distribution and $D^\perp$ is anti-invariant distribution then an F is said to be a hemi-slant map, and θ is said to be the hemi-slant angle [14].

If $\operatorname{Ker} F_* = D \oplus D_1 \oplus D_2$, $J(D) = D$, $JD_2 \subseteq (\ker F_*)^\perp$ the angle θ between JZ and the space $(D_1)_p$ is constant for any non-zero vector Z in $(D_1)_p$ then F is said to be quasi-hemi-slant Riemannian map and the angle θ is said to be the quasi-hemi-slant angle of the map [17].

3. QUASI-HEMI-SLANT RIEMANNIAN MAPS

Let F be quasi-hemi-slant Riemannian map from an almost Hermitian manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2). Thus, we get

$TN_1 = \ker F_* \oplus (\ker F_*)^\perp$.

Let P, Q and R be projection morphisms of $\ker F_*$ onto D, D_1 and D_2 respectively. For any vector field $X_1 \in \Gamma(\ker F_*)$, we put

$$X_1 = PX_1 + QX_1 + RX_1. \tag{11}$$

For all $Z_1 \in \Gamma(\ker F_*)$, we get

$$JZ_1 = \phi Z_1 + \omega Z_1 \tag{12}$$

where $\phi Z_1 \in \Gamma(\ker F_*)$ and $\omega Z_1 \in \Gamma(\omega D_1 \oplus \omega D_2)$. The horizontal distribution $(\ker F_*)^\perp$ is decomposed as

$(\ker F_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mu$.

Here μ is an invariant distribution of $\omega D_1 \oplus \omega D_2$ in $(\ker F_*)^\perp$. From Equations (11) and (12), we have

$JX_1 = J(PX_1) + J(QX_1) + J(RX_1)$

$= \phi(PX_1) + \omega(PX_1) + \phi(QX_1) + \omega(QX_1) + \phi(RX_1) + \omega(RX_1)$.

Since JD = D, we have $\omega PX_1 = 0$ and $\phi(RX_1) = 0$. Thus, we get

$JX_1 = \phi(PX_1) + \phi QX_1 + \omega QX_1 + \omega RX_1$.

Hence we get the below decomposition

$J(\ker F_*) = D \oplus \phi(D_1) \oplus (\omega D_1 \oplus \omega D_2)$

where $\oplus$ denotes orthogonal direct sum. Further, let $X_1 \in \Gamma(D_1)$ and $X_2 \in \Gamma(D_2)$. Then

$g_1(X_1, X_2) = 0$.

From above equation, we have

$g_1(JX_1, X_2) = -g_1(X_1, JX_2) = 0.$

Now, consider

$g_1(\phi X_1, X_2) = g_1(JX_1 - \omega X_1, X_2) = g_1(JX_1, X_2).$

Similarly, we have $g_1(X_1, \phi X_2) = 0$.

Let $V_1 \in \Gamma(D)$ and $V_2 \in \Gamma(D_1)$. Then we have

$g_1(\phi V_1, V_2) = g_1(JV_1 - \omega V_1, V_2) = g_1(JV_1, V_2) = -g_1(V_1, JV_2) = 0$

as D is invariant i.e., $JV_1 \in \Gamma(D)$.

Similarly, for $Z_1 \in \Gamma(D)$ and $Z_2 \in \Gamma(D_2)$, we obtain $g_1(\phi Z_2, Z_1) = 0$. From above equations, we have

$g_1(\phi Y_1, \phi Y_2) = 0$ and $g_1(\omega Y_1, \omega Y_2) = 0$

for all $Y_1 \in \Gamma(D_1)$ and $Y_2 \in \Gamma(D_2)$. Since $\omega D_1 \subseteq (\ker F_*)^{\perp}$, $\omega D_2 \subseteq (\ker F_*)^{\perp}$. So we can write

$(\ker F_*)^{\perp} = \omega D_1 \oplus \omega D_2 \oplus \mathcal{V}$

where $\mathcal{V}$ is orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ in $(\ker F_*)^{\perp}$. For any $X_1 \in \Gamma(\ker F)^{\perp}$, we get

$$JX_1 = BX_1 + CX_1 . \tag{13}$$

where $BX_1 \in \Gamma(\ker F_*)$ and $CX_1 \in \Gamma(\mathcal{V})$.

Lemma 2. If F is a quasi-hemi-slant Riemannian map then we have

$\phi^2 V_1 + B\omega V_1 = -V_1,\ \omega\phi V_1 + C\omega V_1 = 0,$

$\omega B V_2 + C^2 V_2 = -V_2,\ \phi B V_2 + BCV_2 = 0$

for all $V_1 \in \Gamma(\ker F_*)$ and $V_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. The desired results are obtained by using Equations (1), (12) and (13).

Evidence of the following result is the same as given in [1], so we will skip the proof.

Lemma 3. If F is a quasi-hemi-slant Riemannian map then we have

i) $\phi^2 V_1 = -(\cos^2\theta_1)V_1,$

ii) $g_1(\phi V_1, \phi V_2) = \cos^2\theta_1 g_1(V_1, V_2),$

iii) $g_1(\omega V_1, \omega V_2) = \sin^2\theta_1 g_1(V_1, V_2),$

for all $V_1, V_2 \in \Gamma(D_1)$.

From now on we will denote a quasi-hemi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2) by F.

Lemma. 4. If F is a quasi-hemi-slant Riemannian map then, we have

$\mathcal{V}\nabla_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2 = B\mathcal{T}_{X_1}X_2 + \phi\mathcal{V}\nabla_{X_1}X_2,$

$\mathcal{T}_{X_1}\phi X_2 + \mathcal{H}\nabla_{X_1}\omega X_2 = C\mathcal{T}_{X_1}X_2 + \omega\mathcal{V}\nabla_{X_1}X_2,$

$\mathcal{V}\nabla_{X_1}BZ_1 + \mathcal{T}_{X_1}CZ_1 = \phi\mathcal{T}_{X_1}Z_1 + B\mathcal{H}\nabla_{X_1}Z_1,$

$\mathcal{T}_{X_1}BZ_1 + \mathcal{H}\nabla_{X_1}CZ_1 = \omega\mathcal{T}_{X_1}Z_1 + C\mathcal{H}\nabla_{X_1}Z_1.$

$\mathcal{V}\nabla_{Z_1}\phi X_1 + \mathcal{A}_{Z_1}\omega X_1 = B\mathcal{A}_{Z_1}X_1 + \phi\mathcal{V}\nabla_{Z_1}X_1,$

$\mathcal{A}_{Z_1}\phi X_1 + \mathcal{H}\nabla_{Z_1}\omega X_1 = \omega\mathcal{V}_{Z_1}X_1 + C\mathcal{A}_{Z_1}X_1,$

$\mathcal{V}\nabla_{Z_1}BZ_2 + \mathcal{A}_{Z_1}CZ_2 = B\mathcal{H}\nabla_{Z_1}Z_2 + \phi\mathcal{A}_{Z_1}Z_2,$

$\mathcal{A}_{Z_1}BZ_2 + \mathcal{H}\nabla_{Z_1}CZ_2 = \omega\mathcal{A}_{Z_1}Z_2 + C\mathcal{H}\nabla_{Z_1}Z_2,$

for any $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. Using Equations (3), (6), (7), (8), (9), (12) and (13), we get the lemma completely.

Now, we define

$(\nabla_{X_1}\phi)X_2 = \mathcal{V}\nabla_{X_1}\phi X_2 - \phi\mathcal{V}\nabla_{X_1}X_2,$

$(\nabla_{X_1}\omega)X_2 = \mathcal{H}\nabla_{X_1}\omega X_2 - \omega\mathcal{V}\nabla_{X_1}X_2,$

$(\nabla_{Z_1}C)Z_2 = \mathcal{H}\nabla_{Z_1}CZ_2 - C\mathcal{H}\nabla_{Z_1}Z_2,$

$(\nabla_{Z_1}B)Z_2 = \mathcal{V}\nabla_{Z_1}BZ_2 - B\mathcal{H}\nabla_{Z_1}Z_2$

for any $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$.

Lemma 5. If F is a quasi-hemi-slant Riemannian map then, we have

$(\nabla_{X_1}\phi)X_2 = B\mathcal{T}_{X_1}X_2 - \mathcal{T}_{X_1}\omega X_2,$

$(\nabla_{X_1}\omega)X_2 = C\mathcal{T}_{X_1}X_2 - \mathcal{T}_{X_1}\phi X_2,$

$(\nabla_{Z_1}C)Z_2 = \omega\mathcal{A}_{Z_1}Z_2 - \mathcal{A}_{Z_1}BZ_2,$

$(\nabla_{Z_1}B)Z_2 = \phi\mathcal{A}_{Z_1}Z_2 - \mathcal{A}_{Z_1}CZ_2,$

for any vectors $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. The proof is straightforward, so we omit its proof.

If ϕ and ω are parallel with respect to ∇ on N_1 respectively, then

$B\mathcal{T}_{X_1}X_2 = \mathcal{T}_{X_1}\omega X_2$ and $C\mathcal{T}_{X_1}X_2 = \mathcal{T}_{X_1}\phi X_2$

for any $X_1, X_2 \in \Gamma(TN_1)$.

Theorem 1. D is integrable if and only if

$g_1(\mathcal{T}_{X_2}JX_1 - \mathcal{T}_{X_1}JX_2, \omega QZ_1 + \omega RZ_1) = g_1(\mathcal{V}\nabla_{X_1}JX_2 - \mathcal{V}\nabla_{X_2}JX_1, \phi QZ_1)$

for all $X_1, X_2 \in \Gamma(D)$ and $Z_1 \in \Gamma(D_1 \oplus D_2)$.

Proof. For all $X_1, X_2 \in \Gamma(D)$, $Z_1 \in \Gamma(D_1 \oplus D_2)$ and $Z_2 \in (\ker F_*)^{\perp}$, since $[X_1, X_2] \in (\ker F_*)$, we have $g_1([X_1, X_2], Z_2) = 0$. Thus D is integrable $\Leftrightarrow g_1([X_1, X_2], Z_1) = 0$. Now, using Equations (2), (3), (6), (7), (11), (12) and (13), we have

$g_1([X_1, X_2], Z_1) = g_1(J\nabla_{X_1}X_2, JZ_1) - g_1(J\nabla_{X_2}X_1, JZ_1)$

$= g_1(\nabla_{X_1}JX_2, JZ_1) - g_1(\nabla_{X_2}JX_1, JZ_1)$

$= g_1(\mathcal{T}_{X_1}JX_2 - \mathcal{T}_{X_2}JX_1, \omega QZ_1 + \omega RZ_1) - g_1(\mathcal{V}\nabla_{X_1}JX_2 - \mathcal{V}\nabla_{X_2}JX_1, QZ_1)$.

Theorem 2. D_1 is integrable if and only if

$g_1(\mathcal{T}_{Z_1}\omega\phi Z_2 - \mathcal{T}_{Z_2}\omega\phi Z_1, V_1) = g_1(\mathcal{T}_{Z_1}\omega Z_2 - \mathcal{T}_{Z_2}\omega Z_1, \phi PV_1) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2 - \mathcal{H}\nabla_{Z_2}\omega Z_1, \omega RV_1)$

for all $Z_1, Z_2 \in \Gamma(D_1)$ and $V_1 \in \Gamma(D_1 \oplus D_2)$.

Proof. For all $Z_1, Z_2 \in \Gamma(D)$ and $V_1 \in \Gamma(D_1 \oplus D_2)$ and $V_2 \in (\ker F_*)^{\perp}$, since $[Z_1, Z_2] \in (\ker F_*)$, we have $g_1([Z_1, Z_2], V_2) = 0$. Thus D_1 is integrable $\Leftrightarrow g_1([Z_1, Z_2], V_1) = 0$. Using Equations (2), (3), (6), (7), (11), (12), (13) and the Lemma 4, we have

$g_1([Z_1, Z_2], V_1) = g_1(\nabla_{Z_1} JZ_2, JV_1) - g_1(\nabla_{Z_2} JZ_1, JV_1)$

$= g_1(\nabla_{Z_1}\phi Z_2, JV_1) + g_1(\nabla_{Z_1}\omega Z_2, JV_1) - g_1(\nabla_{Z_2}\phi Z_1, JV_1) - g_1(\nabla_{Z_2}\omega Z_1, JV_1)$

$= \cos^2\theta_1 g_1(\nabla_{Z_1}Z_2, V_1) - \cos_2\theta_1 g_1(\nabla_{Z_2}Z_1, V_1) - g_1(\mathcal{T}_{Z_1}\omega\phi Z_2 - \mathcal{T}_{Z_2}\omega\phi Z_1, V_1)$

$+ g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2 + \mathcal{T}_{Z_1}\omega Z_2, JPV_1 + \omega RV_1) - g_1(\mathcal{H}\nabla_{Z_2}\omega Z_1 + \mathcal{T}_{Z_2}\omega Z_1, JPV_1 + \omega RV_1)$.

Now, we have

$\text{Sin}^2\theta_1 g_1([Z_1, Z_2], V_1) = g_1(\mathcal{T}_{Z_1}\omega Z_2 - \mathcal{T}_{Z_2}\omega Z_1, JPV_1) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2 - \mathcal{H}\nabla_{Z_2}\omega Z_1, \omega RV_1)$

$- g_1(\mathcal{T}_{Z_1}\omega\phi Z_2 - \mathcal{T}_{Z_2}\omega Z_1, V_1)$

which proofs the assertion.

Theorem 3. D_2 is always integrable.

Theorem 4. $(\ker F_*)^{\perp}$ is integrable if and only if

$$g_1(\mathcal{V}\nabla_{X_1}BX_2 - \mathcal{V}\nabla_{X_2}BX_1, \phi Z_1) = -g_2(F_*(CX_2), (\nabla F_*)(X_1, \phi Z_1)) + g_2(F_*(CX_1), (\nabla F_*)(X_2, \phi Z_1)),$$

$$g_1(\mathcal{A}_{X_1}BX_2 - \mathcal{A}_{X_2}BX_1, \omega QZ_2) = g_2((\nabla F_*)(X_1, CX_2), F_*(\omega QZ_2)) + g_2((\nabla F_*)(X_2, CX_1), F_*(\omega QZ_2)),$$

$$g_1(\mathcal{A}_{X_1}BX_2 - \mathcal{A}_{X_2}BX_1, \omega QZ_3) = g_2((\nabla F_*)(X_1, CX_2), F_*(\omega QZ_3)) + g_2((\nabla F_*)(X_2, CX_1), F_*(\omega QZ_3)),$$

for all $X_1, X_2 \in \Gamma(\ker F_*)^{\perp}$, $Z_1 \in \Gamma(D)$, $Z_2 \in \Gamma(D_1)$ and $Z_3 \in \Gamma(D_3)$.

Proof. For $X_1, X_2 \in \Gamma(\ker F_*)^{\perp}$, $Z_1 \in \Gamma(D)$, $Z_2 \in \Gamma(D_1)$ and $Z_3 \in \Gamma(D_3)$ and using Equations (2), (3), (8), (12) and (13), we have

$$g_1([X_1, X_2]), Z_1) = g_1(\nabla_{X_1}\phi X_2, \phi Z_1) - g_1(\nabla_{X_2}\phi X_1, \phi Z_1)$$

$$= g_1(\mathcal{V}\nabla_{X_1}BX_2 - \mathcal{V}\nabla_{X_2}BX_1, \phi Z_1) - g_1(CX_2, \nabla_{X_1}\phi Z_1) + g_1(CX_1, \nabla_{X_2}\phi Z_1).$$

Using Equation (10), we get

$$g_1([X_1, X_2]), Z_1) = g_1(\mathcal{V}\nabla_{X_1}BX_2 - \mathcal{V}\nabla_{X_2}BX_1, \phi Z_1) + g_2(F_*(CX_2), (\nabla F_*)(X_1, \phi Z_1))$$

$$- g_2(F_*(CX_1), (\nabla F_*)(X_2, \phi Z_1)).$$

From Equations (2), (3), (8), (9), (11), (12), (13) and the Lemma 4, we obtain

$$g_1([X_1, X_2]), Z_2) = g_1(\phi\nabla_{X_1}X_2, \phi QZ_2) + g_1(\phi\nabla_{X_1}X_2, \omega QZ_2) - g_1(\phi\nabla_{X_2}X_1, \phi QZ_2) - g_1(\phi\nabla_{X_2}X_1, \omega QZ_2)$$

$$= \cos^2\theta_1 g_1([X_1,X_2], Z_2) - g_1(\nabla_{X_1}X_2, \omega\phi QZ_2) + g_1(\nabla_{X_2}X_1, \omega\phi QZ_2) + g_1(\nabla_{X_1}BX_2, \omega QZ_2)$$

$$+ g_1(\nabla_{X_1}CX_2, \omega QZ_2) - g_1(\nabla_{X_2}BX_1, \omega QZ_2 - g_1(\nabla_{X_2}CX_1, \omega QZ_2).$$

Using Equation (10), we have

$$\sin^2\theta_1 g_1([X_1,X_2], Z_2) = g_1(\mathcal{A}_{X_1}BX_2 - \mathcal{A}_{X_2}BX_1, \omega QZ_2) - g_2((\nabla F_*)(X_1, CX_2), F_*(\omega QZ_2))$$

$$+ g_2((\nabla F_*)(X_2, CX_1), F_*(\omega QZ_2)).$$

Similarly, we get

$$\sin^2\theta_2 g_1([X_1, X_2], Z_3) = g_1(\mathcal{A}_{X_1}BX_2 - \mathcal{A}_{X_2}BX_1, \omega QZ_3) - g_2((\nabla F_*)(X_1, CX_2), F_*(\omega QZ_3))$$

$$+ g_2((\nabla F_*)(X_2, CX_1), F_*(\omega QZ_3)).$$

Theorem 5. $(\ker F_*)^{\perp}$ is totally geodesic if and only if

$$g_1(\mathcal{A}_{X_1}X_2, PZ_1 + \cos^2\theta_1 QZ_1) = g_1(\mathcal{H}\nabla_{X_1}X_2, \omega\phi PZ_1 + \omega\phi QZ_1) - g_1(\mathcal{A}_{X_1}BX_2 + \mathcal{H}\nabla_{X_1}CX_2, \omega QZ_1 + \omega RZ_1)$$

for all $X_1, X_2 \in \Gamma(\ker F_*)^{\perp}$ and $Z_1 \in \Gamma(\ker F_*)$.

Proof. For all $X_1, X_2 \in \Gamma(\ker F_*)^{\perp}$ and $Z_1 \in \Gamma(\ker F_*)$ and using Equations (2), (3), (8), (9), (11), (12), (13) and the Lemma 4, we have

$$g_1(\nabla_{X_1} X_2, Z_1) = g_1(J\nabla_{X_1} X_2, JZ_1)$$

$$= -g_1(\nabla_{X_1} X_2, \phi^2 PZ_1 + \omega\phi PZ_1 + \omega\phi QZ_1) + g_1(\nabla_{X_1} BX_2, \omega QZ_1 + \omega RZ_1) + g_1(\nabla_{X_1} CX_2, \omega QZ_1 + \omega RZ_1)$$

$$= g_1(\mathcal{A}_{X_1} X_2, PZ_1 + \cos^2\theta_1 QZ_1) - g_1(\mathcal{H}\nabla_{X_1} X_2, \omega\phi PZ_1 + \omega\phi QZ_1) + g_1(\mathcal{A}_{X_1} BX_2, \omega QZ_1 + \omega RZ_1)$$

$$+ g_1(\mathcal{H}\nabla_{X_1} CX_2, \omega QZ_1 + \omega RZ_1)$$

which shows our assertion.

Theorem 6. $\ker F_*$ is parallel if and only if

$$g_1(\mathcal{T}_{X_1} PX_2, X_3) + \cos^2\theta_1 g_1(\mathcal{T}_{X_1} Qx_2, X_3) = g_1(\mathcal{H}\nabla_{X_1} \omega\phi PX_2, X_3) + g_1(\mathcal{H}\nabla_{X_1} \omega\phi QX_2, X_3)$$

$$-g_1(\mathcal{H}\nabla_{X_1} \omega QX_2 + \mathcal{H}\nabla_{X_1} \omega RX_2, CX_3) + g_1(\mathcal{T}_{X_1} \omega QX_2 + \mathcal{T}_{X_1} \omega RX_2, BX_3)$$

for all $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1 \in \Gamma(\ker F_*)^{\perp}$.

Proof. For all $X_1, X_2 \in \Gamma(\ker F_*)$ and $X_3 \in \Gamma(\ker F_*)^{\perp}$, using Equations (2), (3), (8), (9), (11), (12), (13) and the Lemma 4, we have

$$g_1(\nabla_{X_1} X_2, X_3) = g_1(J\nabla_{X_1} X_2, JX_3)$$

$$= g_1(\nabla_{X_1} \phi PX_2, JX_3), + g_1(\nabla_{X_1} \phi QX_2, JX_3) + g_1(\nabla_{X_1} \omega QX_2, JX_3) + g_1(\nabla_{X_1} \omega RX_2, JX_3)$$

$$= g_1(\mathcal{T}_{X_1} PX_2, X_3) + \cos^2\theta_1 g_1(\mathcal{T}_{X_1} QX_2, X_3) - g_1(\mathcal{H}\nabla_{X_1} \omega\phi PX_2, X_3) - g_1(\mathcal{H}\nabla_{X_1} \omega\phi QX_2, X_3)$$

$$+ g_1(\mathcal{H}\nabla_{X_1} \omega QX_2 + \mathcal{H}\nabla_{x_1} \omega RX_2, CX_3) + g_1(\mathcal{T}_{X_1} \omega QX_2 + \mathcal{T}_{X_1} \omega RX_2, BX_3)$$

which completes the proof.

Theorem 7. D is parallel if and only if

$$g_1(\mathcal{T}_{X_1} JPX_2, \omega QZ_1 + \omega RZ_1) = -g_1(\mathcal{V}\nabla_{X_1} JPX_2, \phi Z_1)$$

and

$$g_1(\mathcal{T}_{X_1} JPX_2, CZ_2) = -g_1(\mathcal{V}\nabla_{X_1} JPX_2, BZ_2)$$

for all $X_1, X_2 \in \Gamma(D)$, $Z_1 \in \Gamma(D_1 \oplus D_2)^{\perp}$ and $Z_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. For all $X_1, X_2 \in \Gamma(D)$, $Z_1 \in \Gamma(D_1 \oplus D_2)^{\perp}$ and $Z_2 \in \Gamma(\ker F_*)^{\perp}$, using Equations (2), (3), (7), (11), (12) and (13), we have

$g_1(\nabla_{X_1} X_2, Z_1) = g_1(\nabla_{X_1} JX_2, JZ_1)$

$= g_1(\nabla_{X_1} JPX_2, JQZ_1 + JRZ_1)$

$= g_1(\mathcal{T}_{X_1} \phi PX_2, \omega QZ_1 + \omega RZ_1) + g_1(\mathcal{V}\nabla_{X_1} \phi PX_2, \phi QZ_1)$.

Using equations (2), (3), (7), (11) and (13), we obtain

$g_1(\nabla_{X_1} X_2, Z_2) = g_1(\nabla_{X_1} JX_2, JZ_2)$

$= g_1(\nabla_{X_1} JPX_2, BZ_2 + CZ_2)$

$= g_1(\mathcal{V}\nabla_{X_1} JPX_2, BZ_2) + g_1(\mathcal{T}_{X_1} JPX_2, CZ_2)$

which completes the assertion.

Theorem 8. D_1 is parallel if and only if

$g_1(\mathcal{T}_{Z_1} \omega\phi Z_2, X_1) = g_1(\mathcal{T}_{Z_1} \omega Z_2, \phi PX_1) + g_1(\mathcal{H}\nabla_{Z_1} \omega Z_2, \omega RX_1)$

and

$g_1(\mathcal{H}\nabla_{Z_1} \omega\phi Z_2, X_2) = g_1(\mathcal{H}\nabla_{Z_1} \omega Z_2, CX_2) + g_1(\mathcal{T}_{Z_1} \omega Z_2, BX_2)$

for all $Z_1, Z_2 \in \Gamma(D_1)$, $X_1 \in \Gamma(D \oplus D_2)$ and $X_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. For all $Z_1, Z_2 \in \Gamma(D_1)$, $X_1 \in \Gamma(D \oplus D_2)$ and $X_2 \in \Gamma(\ker F_*)^{\perp}$, using Equations (2), (3), (8), (11), (13) and the Lemma 4, we have

$g_1(\nabla_{Z_1} Z_2, X_1) = g_1(\nabla_{Z_1} JZ_2, JX_1)$

$= g_1(\nabla_{Z_1} \phi Z_2, JX_1) + g_1(\nabla_{Z_1} \omega Z_2, JX_1)$

$= \cos^2\theta_1 g_1(\nabla_{Z_1} Z_2, X_1) - g_1(\mathcal{T}_{Z_1} \omega\phi Z_2, X_1) + g_1(\mathcal{T}_{Z_1} \omega Z_2, \phi PX_1) + g_1(\mathcal{H}\nabla_{Z_1} \omega Z_2, \omega RX_1)$.

That is,

$\sin^2\theta_1 g_1(\nabla_{Z_1} Z_2, X_1) = -g_1(\mathcal{T}_{Z_1} \omega\phi Z_2, X_1) + g_1(\mathcal{T}_{Z_1} \omega Z_2, JPX_1) + g_1(\mathcal{H}\nabla_{Z_1} \omega Z_2, \omega RX_1)$.

From Equations (2), (3), (8), (12), (13) and the Lemma 4, we have

$g_1(\nabla_{Z_1} Z_2, X_2) = g_1(\nabla_{Z_1} JZ_2, JX_2) = g_1(\nabla_{Z_1} \phi Z_2, JX_2) + g_1(\nabla_{Z_1} \omega Z_2, JX_2)$

$= \cos^2\theta_1 g_1(\nabla_{Z_1} Z_2, X_2) - g_1(\mathcal{H}\nabla_{Z_1} \omega\phi Z_2, X_2) + g_1(\mathcal{H}\nabla_{Z_1} \omega Z_2, CX_2) + g_1(\mathcal{T}_{Z_1} \omega Z_2, BX_2)$.

So, we have

$\mathrm{Sin}^2\theta_1 g_1(\nabla_{Z_1} Z_2, X_2) = -g_1(\mathcal{H}\nabla_{Z_1} \omega\phi Z_2, X_2) + g_1(\mathcal{H}\nabla_{Z_1} \omega Z_2, CX_2) + g_1(\mathcal{T}_{Z_1} \omega Z_2, BX_2)$, which completes the proof.

Similarly as above, we get the following theorem:

Theorem 9. D_2 is parallel if and only if

$$g_1(\mathcal{H}\nabla_{X_1}\omega RX_2, \omega QZ_1) = -g_1(\mathcal{T}_{X_1}\omega RX_2, \phi PZ_1 + \phi QZ_1)$$

and

$$g_1(\mathcal{H}\nabla_{X_1}\omega RX_2, CZ_2) = -g_1(\mathcal{T}_{X_1}\omega RX_2, BZ_2)$$

for all $X_1, X_2 \in \Gamma(D_2)$, $Z_1 \in \Gamma(D \oplus D_1)$ and $Z_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. For all $X_1, X_2 \in \Gamma(D_2)$, $Z_1 \in \Gamma(D \oplus D_1)$ and $Z_2 \in \Gamma(\mathrm{Ker}\, F_*)^{\perp}$. Using Equations (2), (3), (8), (11) and (12), we have

$$g_1(\nabla_{X_1}X_2, Z_1) = g_1(\nabla_{X_1}JX_2, JZ_1)$$

$$= g_1(\nabla_{X_1}\omega RX_2, \phi PZ_1 + \phi QZ_1 + \omega QZ_1)$$

$$= g_1(\mathcal{T}_{X_1}\omega RX_2, \phi PZ_1 + \phi QZ_1) + g_1(\mathcal{H}\nabla_{X_1}\omega RX_2, \omega QZ_1).$$

Using Equations (2), (3), (8), (11) and (13), we have

$$g_1(\nabla_{X_1}X_2, Z_2) = g_1(\nabla_{X_1}JX_2, JZ_2)$$

$$= g_1(\nabla_{X_1}\omega RX_2, BZ_2 + CZ_2)$$

$$= g_1(\mathcal{T}_{X_1}\omega RX_2, BZ_2) + g_1(\mathcal{H}\nabla_{X_1}\omega RX_2, CZ_2)$$

which shows our assertion.

Theorem 10. F is a totally geodesic map if and only if

$$g_1(\mathcal{T}_{Z_1}PZ_2 + \cos^2\theta_1\mathcal{T}_{Z_1}QZ_2 - \mathcal{H}\nabla_{Z_1}\omega\phi PZ_2 - \mathcal{H}\nabla z_1\omega\phi QZ_2, V_1) = g_1(\mathcal{T}_{Z_1}\omega QZ_2 + \mathcal{T}_{Z_1}\omega RZ_2, BV_1)$$

$$+ g_1(\mathcal{H}\nabla_{Z_1}\omega\phi QZ_2 + \mathcal{H}\nabla_{Z_1}\omega\phi RZ_2, V_1)$$

and

$$g_1(\mathcal{A}_{V_1}PZ_1 + \cos^2\theta_1\mathcal{A}_{V_1}QZ_1 - \mathcal{H}\nabla_{V_1}\omega\phi PZ_1 - \mathcal{H}\nabla_{V_1}\omega\phi QZ_1, V_2) = g_1(\mathcal{A}_{V_1}\omega QZ_1 + \mathcal{A}_{V_1}\omega RZ_1, BV_2)$$

$$+ g_1(\mathcal{H}\nabla_{V_1}\omega QZ_1 + \mathcal{H}\nabla_{V_1}\omega RZ_1, CV_2)$$

for all $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $V_1, V_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. For F is a Riemannian map, we have

$$(\nabla F_*)(V_1, V_2) = 0$$

for all $V_1, V_2 \in \Gamma(\ker F_*)^{\perp}$. For all $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $V_1, V_2 \in \Gamma(\ker F_*)^{\perp}$, using Equations (2), (3), (7), (8), (10), (11), (12), (13) and the Lemma 4, we have

$g_2((\nabla F_*)(Z_1, Z_2), F_*(V_1)) = -g_1(\nabla_{Z_1} Z_2, V_1)$

$= -g_1(\nabla_{Z_1} JZ_2, JV_1)$

$= -g_1(\nabla_{Z_1} JPZ_2, JV_1) - g_1(\nabla_{Z_1} JQZ_2, JV_1) - g_1(\nabla_{Z_1} JRZ_2, JV_1)$

$= -g_1(\nabla_{Z_1} \phi PZ_2, JV_1) - g_1(\nabla_{Z_1} \phi QZ_2, JV_1) - g_1(\nabla_{Z_1} \omega QZ_2, JV_1) - g_1(\nabla_{Z_1} \omega RZ_2, JV_1)$

$= -g_1(\mathcal{T}_{Z_1} PZ_2 + \cos^2\theta_1 \mathcal{T}_{Z_1} QZ_2 - \mathcal{H}\nabla z_1 \omega\phi PZ_2, -\mathcal{H}\nabla z_1 \omega QZ_2, V_1) - g_1(\mathcal{T}_{Z_1} \omega QZ_2 + \mathcal{T}_{Z_1} \omega RZ_2, V_1)$

$-g_1(\mathcal{H}\nabla z_1 \omega\phi QZ_2 + \mathcal{H}\nabla z_1 \omega\phi RZ_2, V_1)$.

Similarly, from Equations (2), (3), (7), (8), (10), (11), (12), (13) and the Lemma 4, we get

$g_2((\nabla F_*)(V_1, Z_1), F_*(V_2)) = -g_1(\nabla v_1 Z_1, V_2)$

$= -g_1(\nabla v_1 JZ_1, JV_2)$

$= -g_1(\nabla_{V_1} JPZ_1 + JV_2) - g_1(\nabla_{V_1} JQZ_1, JV_2) - g_1(\nabla v_1 JRZ_1, JV_2)$

$= -g_1(\nabla_{V_1} \phi PZ_1, JV_2) - g_1(\nabla_{V_1} \phi QZ_1, JV_2) - g_1(\nabla_{V_1} \omega QZ_1, JV_2) - g_1(\nabla_{V_1} \omega RZ_1, JV_2)$

$= -g_1(\mathcal{A}_{V_1} PZ_1 + \cos^2\theta_1 \mathcal{A}_{V_1} QZ_1 - \mathcal{H}\nabla_{V_1} \omega\phi PZ_1 - \mathcal{H}\nabla_{V_1} \omega\phi QZ_1, V_2) - g_1(\mathcal{A}_{V_1} \omega QZ_1 + \mathcal{A}_{V_1} \omega RZ_1, BV_2)$

$-g_1(\mathcal{H}\nabla_{V_1} \omega QZ_1 + \mathcal{H}\nabla_{V_1} \omega RZ_1, CV_2)$

which completes the proof.

4. EXAMPLE

Let $(x_1, x_2, \ldots, x_{2n-1}, x_{2n})$ be coordinates on Euclidean space $\mathbb{R}^{2n}$ .An almost complex structure J on $\mathbb{R}^{2n}$ is defined by

$$J(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \ldots + a_{2n-1} \frac{\partial}{\partial x_{2n-1}} + a_{2n} \frac{\partial}{\partial x_{2n}})$$

$$= (-a_2 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} + \ldots - a_{2n} \frac{\partial}{\partial x_{2n-1}} + a_{2n-1} \frac{\partial}{\partial x_{2n}})$$

where $a_1, a_2, \ldots, a_{2n}$ are C^∞ functions defined on $\mathbb{R}^{2n}$. This notation will use throughout this section.

Example 1. Let $(\mathbb{R}^{14}, g_{14}, J)$ be an almost Hermitian manifold as defined above. $F: \mathbb{R}^{14} \to \mathbb{R}^8$ is defined by

$F(x_1, x_2, \ldots, x_{14}) = (x_3 \sin\alpha + x_5 \cos\alpha, x_6, x_7, x_{10}, a, b, x_{13}, x_{14})$

where $\theta_1 \in (0, \frac{\pi}{2})$ and $a, b \in \mathbb{R}$. Then F is a quasi-hemi-slant Riemannian map (where rank $F_* = 6$) such that

$X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2}, X_3 = \cos\alpha \frac{\partial}{\partial x_3} - \sin\alpha \frac{\partial}{\partial x_5}, X_4 = \frac{\partial}{\partial x_4}, X_5 = \frac{\partial}{\partial x_8}, X_6 = \frac{\partial}{\partial x_9}, X_7 = \frac{\partial}{\partial x_{11}}, X_8 = \frac{\partial}{\partial x_{12}},$

$\ker F_* = D \oplus D_1 \oplus D_2$

where

$D = < X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2}, X_7 = \frac{\partial}{\partial x_{11}}, X_8 = \frac{\partial}{\partial x_{12}} >,$

$D_1 = < X_3 = \cos\alpha \frac{\partial}{\partial x_3} - \sin\alpha \frac{\partial}{\partial x_5}, X_4 = \frac{\partial}{\partial x_4} >,$

$D_2 = < X_5 = \frac{\partial}{\partial x_8}, X_6 = \frac{\partial}{\partial x_9} >,$

and

$(\ker F_*)^{\perp} = < \frac{\partial}{\partial x_6}, \sin\alpha \frac{\partial}{\partial x_3} + \cos\alpha \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{14}} >$

which $D = \mathrm{Span}\{X_1, X_2, X_7, X_8\}$ is invariant, $D_1 = \mathrm{Span}\{X_3, X_4\}$ is slant with slant angle $\theta_1 = \alpha$ and $D_2 = \mathrm{Span}\{X_5, X_6\}$ is anti-invariant.

Example 2. Let $(\mathbb{R}^{12}, g_{12}, J)$ be an almost Hermitian manifold as defined above. $F: \mathbb{R}^{12} \rightarrow \mathbb{R}^8$ is defined by

$F(x_1, x_2,..,x_{12}) = (x_1, x_2, c, x_5, \frac{x_7 + \sqrt{3}x_9}{2}, x_{10}, d, x_{12})$

where $\theta_1 \in (0, \frac{\pi}{2})$ and $c, d \in \mathbb{R}$. Then F is a quasi-hemi-slant Riemannian map (where rank $F_* = 6$) such that

$X_1 = \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_4}, X_3 = \frac{\partial}{\partial x_6}, X_4 = \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_9}), X_5 = \frac{\partial}{\partial x_8}, X_6 = \frac{\partial}{\partial x_{11}},$

$\ker F_* = D \oplus D_1 \oplus D_2,$

where

$D = < X_1 = \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_4} >,$

$D_1 = < X_4 = \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_9}), X_5 = \frac{\partial}{\partial x_8} >,$

$D_2 = < X_3 = \frac{\partial}{\partial x_6}, X_6 = \frac{\partial}{\partial x_{11}} >$

and

$(\ker F_*)^{\perp} = < \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_5}, \frac{1}{2}(\frac{\partial}{\partial x_7} + \sqrt{3}\frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{12}} >$

which $D = \mathrm{span}\{X_1, X_2\}$ is invariant, $D_1 = \mathrm{Span}\{X_4, X_5\}$ is slant with slant angle $\theta_1 = \frac{\pi}{6}$ and $D_1 = \mathrm{Span}\{X_3, X_6\}$ is anti-invariant.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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