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A Note on Centres in a Chain of Circles

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

In this note, we study a chain of circles whose pairwise intersection points, taken in a certain order, also lie on two circles. We give a short elementary proof of the following fact. There exists a conic which touches each line connecting the centres of adjacent circles of such chain. In the case of six circles of the chain, this means that the centres of these circles form a Brianchon hexagon. We consider all cases of the possible radically distinct positions of the original chain of circles. In the case when the touching conic is unique, we find out its type.

Keywords: *Chain of circles, Dao's theorem, Miquel's theorem, Brianchon hexagon, inscribed conic.*
AMS Subject Classification (2020): *Primary: 51F99; Secondary: 97G40.*

1. Introduction

An interesting recent elementary statement on circles is Dao's theorem on six circles [3, 5, 6, 10]. This theorem states that if six triangles defined by the lines of three consecutive sides of a cyclic hexagon, then the circumcenters of these triangles are the vertices of a Brianchon hexagon [1, p. 47], that is, a hexagon whose main diagonals are concurrent (see Figure 1).

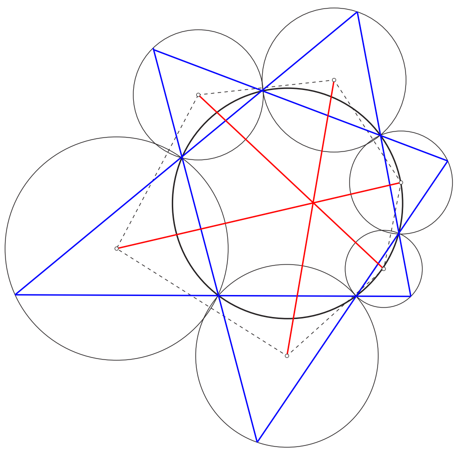

Figure 1. The configuration of Dao's Theorem

By Brianchon's theorem, the main diagonals of a hexagon circumscribed around a conic are concurrent. Hence the hexagons circumscribed around a conic are always Brianchon hexagons (see, for instance, [7, p. 36]). If we draw the six circles circumscribed the Dao's triangles, we get a closed chain of circles. Two consecutive

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circles of this chain have at least one common point. This point is a vertex of the original hexagon. If other common point of these circles exists, then we call it the "second point of intersection" of the two circles.

Miquel's Six-Circles theorem [11] can be formulated in the following way: If for cyclic quadrangles $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ the quadrangles $P_1Q_1Q_2P_2$, $P_2Q_2Q_3P_3$, $P_3Q_3Q_4P_4$ are cyclic, then the last quadrangle of this type $P_4Q_4Q_1P_1$ is also cyclic. The configuration of Miquel's theorem is shown in Figure 2.

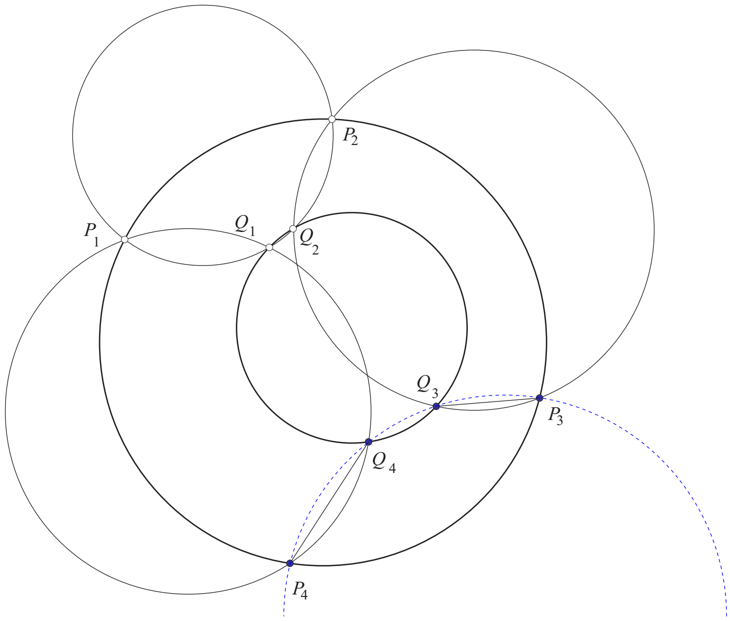

Figure 2. The configuration of Miquel's theorem

In symbols used for the configuration of Miquel's theorem, the circumcircles of the quadruples $P_1Q_1Q_2P_2$, $P_2Q_2Q_3P_3$, $P_3Q_3Q_4P_4$, and $P_4Q_4Q_1P_1$ form a closed chain of intersecting circles with the property that the points of intersection belong to two other circles transversal to each circle of the chain. The following extension of Miquel's theorem can be proved easily by induction [8], [9, Theorem 4.2].

Theorem 1.1. *Let α and β be two circles. Let $n > 2$ be an even number, and take the points $P_1, \dots, P_n$ on α and $Q_1, \dots, Q_n$ on β, such that each quadruple $P_1Q_1Q_2P_2$, $\dots$, $P_{n-1}Q_{n-1}Q_nP_n$ is concyclic. Then the quadruple $P_nQ_nQ_1P_1$ is also concyclic.*

In the case of $n = 6$ the obtained configuration of circles is very similar to the configuration in Dao's theorem. Hence it is not to surprising that L. Szilassi observed that the centers of the circles form Brianchon hexagon. But he did not provide proof. On the other hand this problem without solution was published earlier in Crux Mathematicorum by Dao [4]. It is interesting that, in a later volume of Crux Mathematicorum, we can find a correction for another problem (see [2]), which contains the key statement for simply solving the problem in question.

We note that the second points of intersection in a Dao's configuration are not concyclic in general. This implies that the problem we are discussing is independent of that of Dao.

Our short paper contains a simpler and shorter proof of that the hexagon in our question is a Brianchon hexagon. This proof allows us to generalize the statement to any number of circles in the chain.

2. The main theorem

Theorem 2.1. *Let $\omega(K)$ and $\omega(L)$ be two circles with respective centers K and L. Let n, $n > 2$, be a natural number. Assume that the points of the sequences $P_1, \dots P_n$ and $Q_1, \dots, Q_n$ belong to the circles $\omega(K)$ and $\omega(L)$, respectively, such that each of the quadrangles $P_1Q_1Q_2P_2$, $\dots$, $P_{n-1}Q_{n-1}Q_nP_n$ is cyclic. Denote the center of the circle $c_i(O_i)$ circumscribed the quadrangle $P_iQ_iQ_{i+1}P_{i+1}$, $i = 1, \dots, n-1$, by O_i and assume that none of the points K and L lies on the line O_jO_{j+1}, $j = 1, \dots, n-2$. Let N be the common point of some line O_jO_{j+1} and the line KL. Then there exists a conic γ which touches each line O_jO_{j+1}. Under the condition $n > 6$, the conic γ is uniquely defined, it is:*

- *an ellipse with foci K and L if $K \neq L$ and the point N does not belong to the segment KL;*
- *a hyperbola with foci K and L if $K \neq L$ and the point N lies between the points K and L;*
- *a circle with center L if $K = L$.*

Proof. We prove the theorem in three steps.

I. First assume that $n \leq 6$. Then $i \leq 5$ and $j \leq 4$, that is, there are no more than four fixed lines O_jO_{j+1}. Since the family of all conics in the plane depends on five parameters, the family of all conics touching d fixed lines, depends on $5 - d$ parameters. Consequently, in the case under consideration, the family of all conics touching the lines O_jO_{j+1} for $j = 1, 2, 3, 4$, depends on at least one parameter. Thus, when $n \leq 6$, the theorem holds.

II. Now assume that $n > 6$ and $K \neq L$.

For any five lines O_jO_{j+1}, there exists a unique conic, which touches each of these lines. Let us show that such a conic is common for all lines O_jO_{j+1}.

Let the point T_j be the reflected image of the point K with respect to the line O_jO_{j+1}. We prove that each point T_j belongs to the same circle $\sigma(L)$ with centre L. To this end we consider some cyclic quadrangle $P_{j+1}Q_{j+1}Q_{j+2}P_{j+2}$ and denote the perpendicular bisectors of the segments $Q_{j+1}P_{j+1}$, $P_{j+1}P_{j+2}$, and $P_{j+2}Q_{j+2}$ by f, g, and h, respectively. The configuration of the theorem for the quadrangle $P_2Q_2Q_3P_3$ in the case when the conic γ is an ellipse, is shown in Figure 3.

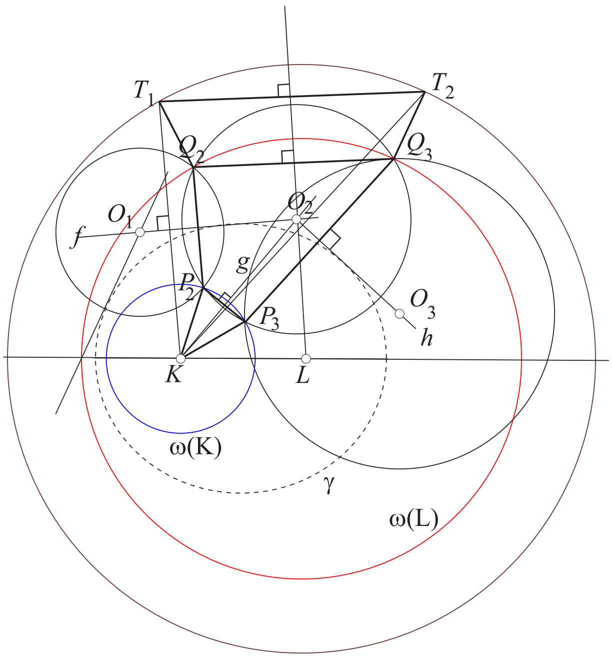

Figure 3. The configuration of the theorem in the case when the conic γ is an ellipse.

Denote the reflections in the lines f, g, and h by symbols S_f, S_g, and S_h, respectively. The lines f, g, and h meet at the point O_{j+1}, in particular, they are parallel if O_{j+1} lies at infinity. Therefore, the composition of the reflections S_f, S_g, and S_h is also a reflection. Since the conditions $S_f \circ S_g \circ S_h(Q_{j+2}) = S_f \circ S_g(P_{j+2}) = S_f(P_{j+1}) = Q_{j+1}$ hold, the composition $S_f \circ S_g \circ S_h$ is the reflection in the perpendicular bisector of the segment $Q_{j+1}Q_{j+2}$. At the same time $S_f \circ S_g \circ S_h(T_{j+1}) = S_f \circ S_g(K) = S_f(K) = T_j$. Hence $S_f \circ S_g \circ S_h$ is the reflection in the perpendicular bisector of the segment T_jT_{j+1}. Thus, the segments $Q_{j+1}Q_{j+2}$ and T_jT_{j+1} have the same perpendicular bisector, denote it by t. Since the point L lies on t, the distances of points T_j and T_{j+1} from L are equal to each other, that is, $|LT_j| = |LT_{j+1}|$. Consequently, for any j the point T_j lies in the circle $\sigma(L)$.

Consider the triangles KLT_j. Let $|LT_j| = 2a$ and $|KL| = 2c$, respectively. By the assumption of the theorem, the point L does not lie in the perpendicular bisector O_jO_{j+1} of the segment KT_j. Consequently, $a \neq c$ and by Pasch's axiom the line O_jO_{j+1} passes through a point of the segment KL, or through a point of the segment LT_j. Let N_j and M_j be the common points of the line O_jO_{j+1} with the lines KL and LT_j, respectively. Then the following assertions are equivalent.

1. The point N_j does not belong to the segment KL.
2. The point M_j belongs to the segment LT_j.

3. The inequality $a > c$ holds.

Since the numbers a and c are constant in our task, the equivalence of the assertions (1), (3) implies that each of the points N_j can be taken as the point N in the formulation of the theorem. Thus the type of the desired conic γ is not depend on the choice of the number j.

Based on the equality $|M_jK| = |M_jT_j|$ and the equivalence of the assertions (1) – (3), we obtain the following metric characteristics of the point M_j.

If $a > c$, that is, the point N_j does not belong to the segment KL, then for any j the point M_j satisfies the equality $|M_jK| + |M_jL| = 2a$.

If $a < c$, that is, the point N_j lies between the points K and L then for any j the point M_j satisfies the equality $||M_jK| - |M_jL|| = 2a$.

According to these characteristics and the classical definitions of conics, each point M_j lies in a conic ω with foci K, L and the orthotomic circle $\sigma(L; 2a)$ (see, for instance, [7, pp. 12, 33]). This conic is an ellipse or hyperbole if and only if $a > c$ or $a < c$, respectively. The line O_jO_{j+1} is the perpendicular bisector of the segment KT_j and passes through the point M_j of the line LT_j. Therefore, it forms equal angles with the focal segments M_jK and M_jL of the point M_j. Consequently, for any j the line O_jO_{j+1} is the tangent of the conic ω in the point M_j. Thus the conic ω is the desired conic γ. This proves the statement of the theorem in the case, when $n > 6$ and $K \neq L$.

III. Finally, assume that $n > 6$ and $K = L$.

In this case the circles $\omega(K)$ and $\omega(L)$ are concentric. Hence the perpendicular bisectors of the segments P_iP_{i+1} and Q_iQ_{i+1} coincide and go through the point $K = L$ (see Figure 4). Keeping the notations of the previous step, we find that the symmetries composition $S_f \circ S_g \circ S_h$ transferes the point T_{j+1} to the point T_j and leaves L invariant.

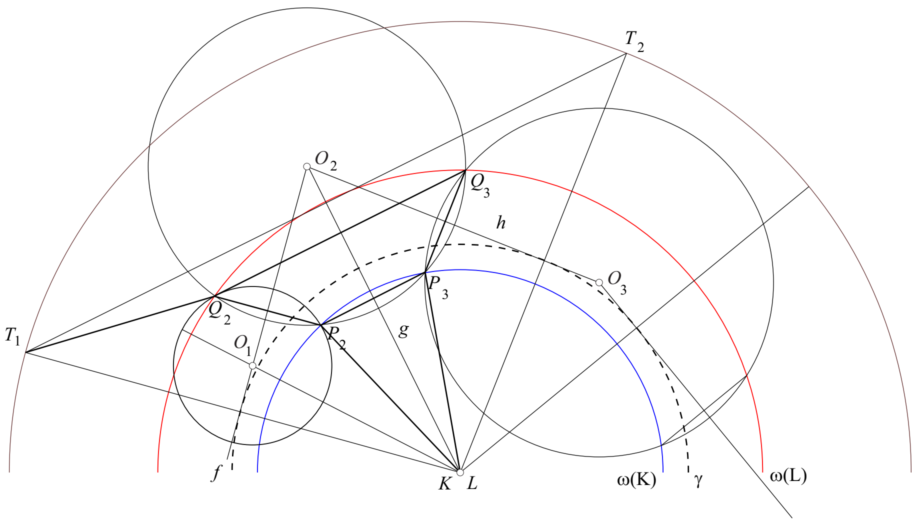

Figure 4. The configuration of the theorem in the case when $n = 6$ and $K = L$.

Consequently, $|LT_j|$ is a constant, that is, all points T_j lie in the same circle with center L. The investigated lines O_iO_{i+1} are perpendicular bisectors of the radiuses of this circle. Hence, they are tangents of the circle with center L and radius $|LT_j|/2$. Thus under the conditions $n > 6$ and $K = L$ the conic γ is a circle.

The proof of the theorem now is complete. □

Remark 2.1. We excluded in the theorem that case, when at least one of the points K and L lies on some line O_jO_{j+1}. Note that in this case the conic γ degenerates into a pair of lines and is defined ambiguously.

Remark 2.2. If the number n in Theorem 2 is even, then due to Theorem 1 the quadrangle $P_nQ_nQ_1P_1$ is also cyclic. Denote the center of the circle circumscribed this quadrangle by O_n. By Theorem 2 there exists a conic, wich is circumscribed of a polygon $O_1O_2 \ldots O_n$. Consequently, under the condition $n = 6$, this polygon is a Brianchon hexagon as we noted in the introduction.

Remark 2.3. Observe that in the case, when K is an inner point of the circle $\sigma(L)$ containing the points T_i, the conic γ is an ellipse. If the point K is an outer point of $\sigma(L)$, then the conic γ is a hyperbola. The case that the conic is a parabola occurs when one of the given circles $\omega_1(K)$ and $\omega_2(L)$ degenerates into a line. Here we present this statement without proof, showing it in Figure 5.

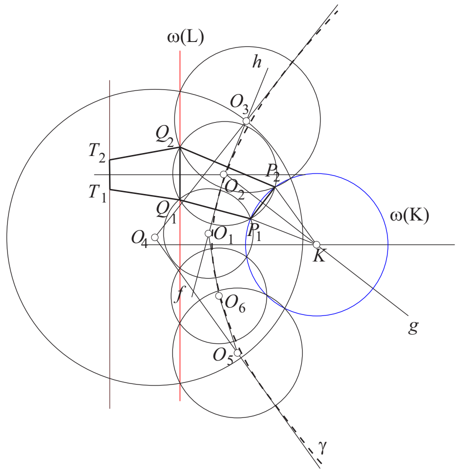

Figure 5. The configuration of the theorem in the case when the circle $\omega(L)$ degenerates into a line and the conic γ is parabola.

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