



A COMMON FIXED POINT THEOREM FOR MULTI-VALUED θ_δ CONTRACTIONS VIA SUBSEQUENTIAL CONTINUITY

Ahmed ALI¹, Saadia MAHIDEB², and Said BELOUL³

¹Laboratory of Analysis and Control of Differential Equations "ACED", Department of Mathematics, Faculty of Mathematics and Computer Sciences and Material Sciences, University of 8 may 1945 Guelma, ALGERIA

²Higher Normal School of Constantine, ALGERIA

³Department of Mathematics, Exact Sciences Faculty, University of El Oued, P. O.Box789, El Oued 39000, ALGERIA

ABSTRACT. The main objective of this paper is to present a common fixed point theorem for two pairs of single and set valued mappings via subsequential continuity and δ - compatibility. To illustrate the validity of our results, an example is provided and we give also an application for a system of integral inclusions of Volterra type.

1. INTRODUCTION

Fixed point theory is one of the important tools in the study of several problems in non linear analysis, physics, economics,.... Starting from Banach principle, some results and generalizations were given in this way. A common fixed point theorem generally involves conditions on commutativity, continuity and contractive condition of the given mappings, with completeness, or closedness of the underlying space or subspaces, along with conditions on suitable containment amongst the ranges of involved mappings. Sessa [20] has weakened the notion of commuting mappings to weakly commuting, later Jungck [14] introduced the concept of compatibility for a pair of self maps, which was extended to hybrid pair of mappings by Kaneko and Sessa [16]. Afterwards Jungck et al. [15] have furnished an extension to compatible mappings notion, called weak compatibility in the setting of single-valued and

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✉ alipromath@gmail.com; saadiamahideb@gmail.com; beloulsaid@gmail.com-Corresponding author

ORCID 0000-0002-4525-021X; 0000-0001-6207-2710; 0000-0002-2814-2161.

multi-valued mappings. Recently, Bouhadjera and Godet Tobie [7] introduced subsequential continuity which is weaker than the reciprocal continuity introduced by Pant [19]. In fact every non-vacuously pair of reciprocally continuous maps is naturally subsequentially continuous. However subsequentially continuous mappings are neither sequentially continuous nor reciprocally continuous. Quite recently, Beloul et al. [5] extended the notion of subsequential continuity to the context of set value maps in order to establish a common fixed point by using Hausdorff distance, while there is a function called δ -distance which defined by Fisher [10], although δ -distance is not a metric like the Hausdorff distance, but shares most of the properties of a metric, some results on δ -distance can be found in [1, 4, 6]. Common fixed point theorem commonly require commutativity, continuity, completeness together with a suitable condition on containment of ranges of involved maps beside an appropriate contraction condition. Thus, research in this field is aimed at weakening one or more of these conditions.

In this paper we will utilize a θ -contraction introduced by Jleli and Samet [12] and δ -distance to establish a strict coincidence and a strict common fixed point of a δ -compatible and subsequentially hybrid pair of mappings, without continuity or reciprocal continuity, weak reciprocal continuity, completeness and containment of ranges.

2. PRELIMINARIES

Let (X, d) be a metric space, $B(X)$ is the set of all non-empty bounded subsets of X . For all $A, B \in B(X)$ we define the two functions: $D, \delta : B(X) \times B(X) \rightarrow \mathbb{R}_+$ such that

$$D(A, B) = \inf\{d(a, b); a \in A, b \in B\},$$

$$\delta(A, B) = \sup\{d(a, b); a \in A, b \in B\}.$$

If A consists of a single point a , we write $\delta(A, B) = \delta(a, B)$ and $D(A, B) = D(a, B)$, also if $B = \{b\}$ is a singleton we write

$$\delta(A, B) = D(A, B) = d(a, b).$$

Clearly that δ satisfies the following properties:

$$\delta(A, B) = \delta(B, A) \geq 0,$$

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B),$$

$$\delta(A, A) = \text{diam } A,$$

$$\delta(A, B) = 0 \text{ implies } A = B = \{a\},$$

for all $A, B, C \in B(X)$.

Notice that for all $a \in A$ and $b \in B$ we have

$$D(A, B) \leq d(a, b) \leq \delta(A, B),$$

where $A, B \in B(X)$.

Definition 1. [20] Two mappings $S : X \rightarrow B(X)$ and $f : X \rightarrow X$ are to be weakly commuting on X if $fSx \in B(X)$ and for all $x \in X$:

$$\delta(Sfx, fSx) \leq \max\{\delta(fx, Sx), \text{diam}(fSx)\}.$$

Definition 2. [17] A hybrid pair of mappings (f, S) of a metric space (X, d) is δ -compatible if

$$\lim_{n \rightarrow \infty} \delta(Sfx_n, fSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $fSx_n \in B(X)$, $\lim_{n \rightarrow \infty} Sx_n = \{z\}$, and $\lim_{n \rightarrow \infty} fx_n = z$, for some $z \in X$.

Definition 3. [19] The pair of self mappings (f, g) on a metric space (X, d) is said to be reciprocally continuous if

$$\lim_{n \rightarrow \infty} fgx_n = ft$$

and

$$\lim_{n \rightarrow \infty} gfx_n = gt,$$

where $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$, for some t in X .

Later, Singh and Mishra [21] generalized the concept of reciprocal continuity to the setting of single and set-valued maps as follows.

Definition 4. [21] Two maps $f : X \rightarrow X$ and $S : X \rightarrow B(X)$ are reciprocally continuous on X (resp. at $t \in X$) if and only if $fSx \in B(X)$ for each $x \in X$ (resp. $fSt \in B(X)$) and

$$\lim_{n \rightarrow \infty} fSx_n = fM, \quad \lim_{n \rightarrow \infty} Sfx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = M \in B(X)$, $\lim_{n \rightarrow \infty} fx_n = t \in M$

Definition 5. [7] Two self-mappings f and g on a metric space (X, d) are said to be subcompatible if there exists a sequence $\{x_n\}$ such that:

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t \quad \text{and} \quad \lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0,$$

for some $t \in X$.

Definition 6. [7] The pair (f, g) of self mappings is said to be subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$, for some $z \in X$ and $\lim_{n \rightarrow \infty} fgx_n = fz$, $\lim_{n \rightarrow \infty} gfx_n = gz$.

Definition 7. [5] Let $f : X \rightarrow X$ and $S : X \rightarrow CB(X)$ two single and set-valued mappings respectively, the hybrid pair (f, S) is to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Sx_n = M \in CB(X) \quad \text{and} \quad \lim_{n \rightarrow \infty} fx_n = z \in M,$$

for some $z \in X$ and $\lim_{n \rightarrow \infty} fSx_n = fM$, $\lim_{n \rightarrow \infty} Sfx_n = Sz$.

Notice that continuity or reciprocal continuity implies subsequential continuity, but the converse may be not.

Example 8. Let $X = [0, 1]$ and d the euclidian metric, we define f, S by

$$fx = \begin{cases} 1-x, & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} < x \leq 1 \end{cases} \quad Sx = \begin{cases} [0, x], & 0 \leq x \leq \frac{1}{2} \\ [x - \frac{1}{2}, x], & \frac{1}{2} < x \leq 1 \end{cases}$$

We consider a sequence $\{x_n\}$ such that for each $n \geq 1$ we have: $x_n = \frac{1}{2} - \frac{1}{n+1}$, clearly that $\lim_{n \rightarrow \infty} fx_n = \frac{1}{2} \in [0, \frac{1}{2}]$ and $\lim_{n \rightarrow \infty} Sx_n = [0, \frac{1}{2}] \in B(X)$, also we have:

$$\lim_{n \rightarrow \infty} fSx_n = \lim_{n \rightarrow \infty} [\frac{1}{2} + \frac{1}{n}, 1] = [\frac{1}{2}, 1] = f([0, \frac{1}{2}]),$$

and

$$\lim_{n \rightarrow \infty} Sfx_n = \lim_{n \rightarrow \infty} [\frac{1}{n}, \frac{1}{2} + \frac{1}{n}] = [0, \frac{1}{2}] = S(\frac{1}{2}),$$

then (f, S) is subsequentially continuous.

On the other hand, consider a sequence $\{y_n\}$ which defined for all $n \geq 1$ by: $y_n = \frac{1}{2} + \frac{1}{n+1}$, we have

$$\lim_{n \rightarrow \infty} fx_n = \frac{1}{2} \in [0, 1], \quad \text{and} \quad \lim_{n \rightarrow \infty} Sx_n = [0, 1] \in B(X),$$

however

$$\lim_{n \rightarrow \infty} fSx_n = \lim_{n \rightarrow \infty} f([\frac{1}{n}, 1 + \frac{1}{n}]) \neq f([0, 1]),$$

then f and S are never reciprocally continuous.

Let Θ be the set of all functions $\theta : (0, +\infty) \rightarrow (1, +\infty)$ be a function satisfying:

- (θ_1) : θ is non decreasing,
- (θ_2) : for each sequence $\{t_n\}$ in $(0, +\infty)$, $\lim_{n \rightarrow \infty} t_n = 1$ if and only if $\lim_{n \rightarrow \infty} t_n = 0$,
- (θ_3) : there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t \rightarrow 0^+} \frac{\theta(t)-1}{t^r} = l$.

Example 9. For all $i \in \{1, 2, 3\}$, the following functions are elements of Θ .

- 1) $\theta_1(t) = e^t$.
- 2) $\theta_2(t) = e^{te^t}$.
- 3) $\theta_3(t) = e^{\sqrt{t}}$.
- 4) $\theta_4(t) = e^{\sqrt{t}e^t}$.

Definition 10. [12] Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping. For $\theta \in \Theta$, we say T is θ -contraction, if there exists $k \in [0, 1]$ such that for $x, y \in X$, $d(Tx, Ty) > 0$ implies $\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k$.

Theorem 11. [12] Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be an θ -contraction. Then T has a unique fixed point in X .

3. MAIN RESULTS

In this section, we introduce a multivalued θ_δ -contraction and prove a common fixed point theorem for hybrid pair mappings with δ -distance.

Definition 12. Let (X, d) be a metric space and $T : X \rightarrow B(X)$ be a mapping. For $\theta \in \Theta$, we say T is θ_δ -contraction, if there exists $k \in [0, 1]$ such that for $x, y \in X$, $\delta(Tx, Ty) > 0$ implies $\theta(\delta(Tx, Ty)) \leq [\theta(d(x, y))]^k$.

Definition 13. Let f be a self mapping on a metric space (X, d) and let $T : X \rightarrow B(X)$ be a multivalued mapping. Then T is called generalized multivalued (f, θ_δ) -contraction if for all $x, y \in X$ there exists $k \in [0, 1]$ such that,

$$\delta(Tx, Ty) > 0 \text{ implies } \theta(\delta(Tx, Ty)) \leq [\theta(R(x, y))]^k,$$

where $\theta \in \Theta$

$$R(x, y) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{1}{2}[D(fx, Ty) + D(fy, Tx)]\}.$$

Now we extend the last definition for two pairs of hybrid pair, in order to establish a common fixed point for set valued and single valued mapping in metric space, without continuity and completeness of space, we use only subsequential continuity with δ -compatibility.

Theorem 14. Let $f, g : X \rightarrow X$ be single valued mappings and $S, T : X \rightarrow B(X)$ be multi-valued mappings of metric space (X, d) . If the two pairs (f, S) and (g, T) are subsequentially continuous and δ -compatible. Then the pair (f, S) as well as (g, T) has a strict coincidence point. Moreover, f, g, S and T have a common strict fixed point provided that there exists $k \in (0, 1)$ such that for all x, y in X we have:

$$\delta(Sx, Ty) > 0 \text{ implies } \theta(\delta(Tx, Ty)) \leq [\theta(R(x, y))]^k, \tag{1}$$

where $\theta \in \Theta$. and

$$R(x, y) = \max\{d(fx, fy), D(fx, Tx), D(gy, Ty), \frac{1}{2}[D(fx, Ty) + D(fy, Tx)]\}.$$

Proof. Since (f, S) is subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\begin{aligned} \lim_{n \rightarrow \infty} Sx_n &= M \in B(X), & \lim_{n \rightarrow \infty} fx_n &= z \in M. \\ \lim_{n \rightarrow \infty} fSx_n &= fM, & \lim_{n \rightarrow \infty} Sfx_n &= Sz, \end{aligned}$$

Also, the pair (f, S) is δ -compatible implies that

$$\lim_{n \rightarrow \infty} \delta(fSx_n, Sfx_n) = \delta(fM, Sz) = 0,$$

which gives that $fM = Sz = \{fz\}$, and so z is a coincidence point of f and S . Similarly, for the pair (g, T) there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Ty_n = N \in B(X) \text{ and } \lim_{n \rightarrow \infty} gy_n = t \in N$$

and

$$\lim_{n \rightarrow \infty} gTy_n = gN, \quad \lim_{n \rightarrow \infty} Tgy_n = Tt.$$

The pair (g, T) is δ -compatible, implies that

$$\lim_{n \rightarrow \infty} \delta(gTy_n, Tgy_n) = \delta(gN, Tt) = 0.$$

Then $gN = Tt$ and Tt is a singleton, i.e, $Tt = \{gt\}$ and t is a strict coincidence point of g and T .

Now, we claim $fz = gt$, if not so $\delta(Sz, Tt) > 0$, otherwise

$$d(fz, gt) \leq \delta(Sz, Tt) = 0,$$

which is a contradiction. Then by using (1), we get:

$$\theta(\delta(Sz, Tt)) \leq [\theta(R(z, t))]^k.$$

Since $Sz = \{fz\}$ and $Tt = \{gt\}$, we get

$$D(fz, Sz) = D(gt, Tt) = 0,$$

$$D(fz, Tt) = d(fz, gt)$$

and $D(gt, Sz) = d(fz, gt)$. Hence

$$R(z, t) = \max\{d(fz, gt), D(fz, Sz), D(gt, Tt), \frac{1}{2}(D(fz, Tt) + D(gt, Sz))\}.$$

Subsisting in (1) we get

$$\theta(d(fz, gt)) = \theta(\delta(Sz, Tt)) \leq [\theta(d(fz, gt))]^k < \theta(d(fz, gt)),$$

which is a contradicts that $\theta(t) > 1$ for all $t \geq 0$. Then $fz = gt$.

Now we claim $z = fz$, if not by taking $x = x_n$ and $y = t$ in (1), $\delta(Sx_n, Tt) > 0$, otherwise letting $n \rightarrow \infty$, we get

$$d(z, fz) = d(z, gt) \leq \delta(M, Tt) = 0,$$

which contradicts that $z \neq fz$, and so we have

$$\theta(\delta(Sx_n, Tt)) \leq [\theta(R(x_n, t))]^k.$$

Letting $n \rightarrow \infty$, we get $M(x_n, t) \rightarrow d(z, fz)$ and so we have:

$$\theta(d(z, fz)) \leq \theta(\delta(M, Tt)) \leq [\theta(d(z, fz))]^k < \theta(d(z, fz)),$$

which is a contradiction. Hence z is a fixed point for f and S .

We will show $z = t$, if not by taking $x = x_n$ and $y = y_n$ in (1), $\delta(Sx_n, Ty_n) > 0$, if not letting $n \rightarrow \infty$, we get:

$$d(z, t) \leq \delta(M, N) = 0,$$

which is a contradiction, so we have:

$$\theta[\delta(Sx_n, Ty_n)] \leq \theta[R(x_n, y_n)]^k$$

Taking $n \rightarrow \infty$, we get:

$$\theta(d(z, t)) < \theta(\delta(M, N) \leq [\theta(d(z, t))]^k < \theta(d(z, t)),$$

which is a contradiction. Hence $z = t$ and consequently z is a common fixed point for f, g, S and T . For the uniqueness, let w be another fixed point, by and using (1), $\delta(Sz, Tw) > 0$, if not $d(z, t) \leq \delta(Sz, Tw) = 0$, which is a contradiction. Then we have:

$$\theta(d(z, w)) < \theta(\delta(Sz, Tw)) \leq [\theta(d(z, w))]^k < \theta(d(z, w)),$$

which is a contradiction. Then z is unique. □

If $f = g$ and $S = T$ we obtain the following corollary:

Corollary 15. *Let $f : X \rightarrow X$ be a single valued mapping and $S : X \rightarrow B(X)$ be a multi-valued mapping of metric space (X, d) . Suppose that the pair (f, S) is subsequentially continuous, as well is δ -compatible and there exist $\theta \in \Theta$ and $k \in [0, 1)$ such that for all x, y in X we have:*

$$\delta(Sx, Sy) > 0 \text{ implies } \theta(\delta(Sx, Sy)) \leq [\theta(M(x, y))]^k,$$

where

$$R(x, y) = \max\{d(fx, fy), D(fx, Sx), D(fy, Sy), \frac{1}{2}[D(fx, Sy) + D(fy, Sx)]\}.$$

Then f and S have a strict common fixed point.

If S and T are single valued maps, we get the following corollary:

Corollary 16. *Let (X, d) a metric space and let $f, g, S, T : X \rightarrow X$ be self mappings, if the pair (f, S) is subsequentially continuous and compatible as well as (g, T) . Then f and S have a coincidence point as well as g and T . Moreover, f, g, S and T have a common fixed point provided that there exists $k \in [0, 1)$ and $\theta \in \Theta$ such that for all x, y in X we have:*

$$d(Sx, Ty) > 0 : \text{ implies } : \theta(d(Tx, Ty)) \leq [\theta(R(x, y))]^k,$$

where

$$R(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)]\}.$$

Example 17. *Let $X = [0, 2]$, $d(x, y) = |x - y|$ and f, g, S and T defined by*

$$fx = gx = \begin{cases} \frac{x+1}{2}, & 0 \leq x \leq 1 \\ 0, & 1 < x \leq 2 \end{cases} \quad Fx = Tx = \begin{cases} \{1\}, & 0 \leq x \leq 1 \\ [\frac{5}{4}, 2], & 1 < x \leq 2 \end{cases}$$

Consider a sequence $\{x_n\}$ for all $n \geq 1$ such that $x_n = 1 - \frac{1}{n}$, it is clear that

$$\lim_{n \rightarrow \infty} fx_n = 1 \in \{1\}$$

and

$$\lim_{n \rightarrow \infty} Tx_n = \{1\},$$

which implies that the pair (f, T) is subsequentially continuous. On other hand, we have

$$\lim_{n \rightarrow \infty} \delta(fTx_n, Tfx_n) = \delta(\{1\}, \{1\}) = 0.$$

so (f, S) is δ -compatible.

For the inequality (1), by taking $\theta(t) = e^t$ and $k = \frac{9}{10}$, we discuss the following cases:

- (1) For $x, y \in [0, 1]$, we have $\delta(Tx, Ty) = 0$.
 (2) For $x \in [0, 1]$ and $y \in (1, 2]$, we have:

$$\delta(Tx, Ty) = 1 \leq \frac{3}{2} \leq \frac{9}{10} D(fy, Ty),$$

which implies

$$e^{\delta(Tx, Ty)} \leq (e^{D(fy, Ty)})^{\frac{9}{10}} \leq (e^{R(x, y)})^{\frac{9}{10}}.$$

- (3) For $x \in (1, 2]$ and $y \in [0, 1]$, we have

$$\delta(Tx, Ty) = 1 \leq \frac{3}{2} \leq \frac{9}{10} D(fy, Ty),$$

this yields

$$e^{\delta(Tx, Ty)} \leq (e^{D(fx, Tx)})^{\frac{9}{10}} \leq (e^{R(x, y)})^{\frac{9}{10}}.$$

- (4) For $x, y \in (1, 2]$ we have

$$\delta(Tx, Ty) = 1 \leq \frac{3}{2} \leq \frac{9}{10} D(fx, Tx),$$

then

$$e^{\delta(Tx, Ty)} \leq (e^{D(fx, Tx)})^{\frac{9}{10}} \leq (e^{R(x, y)})^{\frac{9}{10}}.$$

hence f and T satisfy (1), therefore 1 is the unique common strict fixed point of f and S .

4. APPLICATION TO INTEGRAL INCLUSIONS

In this section, we apply the obtained results to assert the existence of solution for a system of integral inclusions.

Consider the following integral inclusions system's.

$$x_i(t) \in g(t) + \int_0^t K_i(t, s, x_i(s)) ds, i = 1, 2 \quad (2)$$

where g is a continuous function on $[0, 1]$, i.e., $f \in C([0, 1], \mathbb{R})$ and $K_i : [0, 1] \times [0, 1] \times \mathbb{R} \rightarrow CB(\mathbb{R})$ are a set valued functions.

Clearly $X = C([0, 1])$ with convergence uniform metric's $d_\infty(x, y) = \sup_{x \in X} |x(t) - y(t)|$ is a complete metric space. Define two set valued mappings:

$$Sx_1(t) = \{z \in X, z(t) \in f(t) + \int_0^t K_1(t, s, x_1(s)) ds\},$$

$$Tx_2(t) = \{z \in X, z(t) \in f(t) + \int_0^t K_2(t, s, x_2(s))ds\}.$$

Assume that;

A_1 : The function $K_i : (t, s) \mapsto K_i(t, s, x_i(s))$ are continuous on $[0, 1] \times (0, 1]$ for all $x \in C((0, 1])$;

A_2 : For all $x_i \in X$ and $k_i \in K_i$ ($i = 1, 2$), there exists a function $\varphi : [0, 1] \times [0, 1] \rightarrow [0, +\infty)$ such that

$$|k_1(t, s, x_1(s)) - k_2(t, s, x_2(s))| \leq \varphi(t, s)|x_1 - x_2|;$$

A_3 : There exists $\tau > 0$ such that

$$\sup_{t \in [0, 1]} \int_0^t \varphi(t, s)ds \leq e^{-\tau};$$

A_4 : There exist two sequences $\{x_n\}, \{y_n\}$ and two elements x, y in X such that

$$\lim_{n \rightarrow \infty} Sx_n = M \in B(X),$$

$$\lim_{n \rightarrow \infty} x_n = x \in M$$

and

$$\lim_{n \rightarrow \infty} Ty_n = N \in B(X),$$

$$\lim_{n \rightarrow \infty} y_n = y \in N.$$

Theorem 18. *Under assumptions $(A_1) - (A_4)$ the system of integral inclusions (2) has a solution in $C((0, 1]) \times C([0, 1])$.*

Proof. The system (2) has a solution if and only if S and T have a common fixed point.

Denote I_X the identity operator on X .

From condition (4), the two pairs (I_X, S) and (I_X, T) are subsequentially continuous as well as δ -compatible.

For the contractive condition (1), let $x_1, x_2 \in C([0, 1])$ and $z_1 \in Sx_1$, then there exists $k_1 \in K_1$ such that

$$z_1(t) = f(t) + \int_0^t k_1(t, s, x_1(s))ds.$$

Let $z_2 \in f(t) + \int_0^t K_2(t, s, x_2(t))ds$, i.e.,

$$z_2(t) = f(t) + \int_0^1 k_2(t, s, x_2(s))ds,$$

for some $k_2 \in K_2$, so we have

$$|z_1 - z_2| \leq \int_0^t |k_1(t, s, x_1(s)) - k_2(t, s, x_2(s))|ds$$

$$\leq \int_0^t |x_1 - x_2| \varphi(t, s) ds.$$

Since $K_i, i = 1, 2$ are bounded, so we have

$$\sup_{z_i \in X} |z_1 - z_2| \leq \|x_1 - x_2\|_\infty \int_0^t \varphi(t, s) ds,$$

which implies that

$$\begin{aligned} \delta(Sx_1, Tx_2) &\leq e^{-\tau} d(x_1, x_2) \\ &\leq e^{-\tau} \max\{d(x_1, x_2), d(x_1, Sx_1), d(x_2, Tx_2), \frac{1}{2}(d(x_1, Tx_2) + d(x_2, Sx_1))\}. \end{aligned}$$

Since θ is non decreasing we get

$$e^{\sqrt{\delta(Sx_1, Tx_2)}} \leq \left(e^{\frac{\sqrt{L(x_1, x_2)}}{2}} \right)^{e^{-\tau}},$$

where $L(x_1, x_2) = \max\{d(x_1, x_2), d(x_1, Sx_1), d(x_2, Tx_2), \frac{1}{2}(d(x_1, Tx_2) + d(x_2, Sx_1))\}$.

Hence all hypotheses of Theorem 14 are satisfied, with $\theta(t) = e^{\sqrt{t}}$, $k = e^{-\tau}$ and $f = g = I_X$, therefore the system (2) has a solution. \square

5. CONCLUSION

We have established common fixed point theorems for two hybrid pairs θ -contraction using δ -distance without exploiting the notion of continuity or reciprocal continuity, weak reciprocal continuity. Since θ -contraction is a proper generalization of ordinary contraction, our results generalize, extend and improve the results of Jleli and Samet [12] and others existing in literature, without using completeness of space or subspace, containment requirement of range space.

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